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ON STRONG DEVIATION THEOREMS CONCERNING ARRAY OF RANDOM VARIABLES WITH APPLICATIONS

YA-HUI ZHU, WEI-CAI PENG*, YONG-JIN ZHANG AND ZHONG-ZHI WANG*

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Abstract. In this paper, the concept of generalized relative entropy is firstly introduced as the random measure between two probability measures μ and $\tilde{\mu}$, then a class of strong deviation theorem (small deviation theorem) for array of dependent random variables is established. Based on the strong deviation theorem and its corollaries, a kind of strong deviation theorems and strong law of large numbers for row-wise negatively dependent random variables are obtained finally.

1. Introduction

The classical strong laws of large numbers (SLLNs) mainly deal with independent random variables (r.v.'s). The investigation of limit theorems for dependent r.v.'s is extensive and episodic. The strong law of large numbers for various types of mixed or associated random variables can be found, *e.g.*, in Lu and Lin ([10]). Conventionally, some techniques such as measure theoretic techniques, moment inequality and martingale method are used in developing the limit theorem of random sequences ([4]). In the 1990s, a new proof of the SLLNs for Bernoulli sequences was given by Wen Liu ([8]) in the study of real number theory by constructing monotonically increasing functions, and using the famous Lebesgue's theorem on derivatives of monotonic functions exist almost everywhere. Since then, the new method of studying limit theorems had been developed by Liu, Yang and their collaborators. The main idea of this method is first to construct the likelihood ratio or martingale with a parameter then to use the likelihood ratio that converges almost everywhere or martingale convergence theorem to prove that some limits exist almost everywhere. Using this approach, Liu and his collaborators have successfully studied strong limit theorems for arbitrary random sequences, the SLLNs for nonhomogeneous Markov chains, nonhomogeneous Markov chains indexed by a tree and Shannon-McMillan theorem, while the strong deviation theorem (the strong limit theorem expressed by inequality) is the special result of this

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* Corresponding author.

new method. We refer to the book of Liu ([9]) that contains classical results as well as more interesting results on strong deviation theorems of dependent random variables. The main focus of this work is to obtain some strong deviation theorems for sums of array of row-wise negatively dependent random variables.

The rest of this paper is arranged as follows. In Section 2, we introduce notations and some basic definitions and state a few elementary lemmas to be used in the proofs if the main results. In Section 3 we give some strong deviation theorems for bounded random variables and list some simple consequences. In section 4 we establish some strong deviation theorems and strong law of large numbers for sub-Gaussian random variables.

2. Definitions and technical lemmas

In this section we present some definitions and lemmas firstly to be used in the proofs of our main results. Throughout this paper we deal with the fixed probability space $(\Omega, \mathscr{F}, \mu)$. We first give some notations. Let $\mathbb{N}^+$ be the set of positive integers and $\mathbb{R}$ the set of real numbers. In the sequel we use the conventions that the symbol $\mathbf{1}_A$ denotes the indicator function of set A, $\mathbb{E}_\mu$ denotes the expectation under probability measure μ and ω is a sample point.

Let $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ be a triangle array of absolutely continuous random variables with the joint density functions of $\xi^n(\omega)$

$$p_n(x_{n1}, \cdots, x_{nn}), \ \ n = 1, 2, \cdots.$$

Let $\tilde{\mu}$ be an another probability measure on $(\Omega, \mathscr{F})$ and assume that the joint density functions of $\xi^n(\omega)$ with respect to measure $\tilde{\mu}$ are

$$q_n(x_{n1}, \cdots, x_{nn}), \ \ n = 1, 2, \cdots.$$

We call $\tilde{\mu}$ the reference measure.

DEFINITION 1. Let $\xi(\omega)$ be a triangle array of absolutely continuous random variables. Define

$$\mathscr{R}_n(\omega) = \begin{cases} \frac{q_n(\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))}{p_n(\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))} & \text{if denominator } > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad n = 1, 2, \cdots \tag{1}$$

where ω is a sample point. In statistical terms, $\mathscr{R}_n(\omega)$ is called the likelihood ratio, which is of fundamental importance in the theory of testing the statistical hypotheses.

DEFINITION 2. Let $\mathscr{R}_n(\omega)$ be defined as in (1). Define

$$h_\mu^{\tilde{\mu}}(\omega) := -\liminf \frac{1}{n} \log \mathscr{R}_n(\omega) \tag{2}$$

with $\log 0 = -\infty$. $h_\mu^{\tilde{\mu}}(\omega)$ is called the general relative entropy of probability measure μ with respect to $\tilde{\mu}$.

It will be shown in (13) that $h_\mu^{\tilde{\mu}}(\omega) \geqslant 0$ $a.s.$ in general. Hence $h_\mu^{\tilde{\mu}}(\omega)$ can be used as a random measure of the deviation between the true joint density p_n and the reference density q_n. Roughly speaking, this deviation may be regarded as the difference between μ and $\tilde{\mu}$. The smaller $h_\mu^{\tilde{\mu}}(\omega)$ is, the smaller the deviation will be. The purpose of this paper is to establish some strong limit theorem represented by inequalities with random bounds for dependent random variables, by using the notion of general relative entropy and the Borel-Cantelli lemma, and to extend the analytic technique proposed by Liu ([8]) to the case of array of random variables.

REMARK 1. In general, the calculation of $h_\mu^{\tilde{\mu}}(\omega)$ is a difficult problem, the following two examples show that it is easier to calculate in some special cases.

EXAMPLE 1. Let $\xi(\omega) = \{\xi_n(\omega)\}_{n\in\mathbb{N}^+}$ be a sequence of $i.i.d.r.v.'s$ whose probability density functions, under the probability measures μ and $\tilde{\mu}$ respectively, are given by $f_1(x)$ and $f_2(x)$, then we have

$$h_\mu^{\tilde{\mu}}(\omega) = D(\mu\|\tilde{\mu})$$

where $D(\mu\|\tilde{\mu})$ denotes the relative entropy between μ and $\tilde{\mu}$. In fact, according to the definitions of $h_\mu^{\tilde{\mu}}(\omega)$ and $D(\mu\|\tilde{\mu})$ we have

$$\begin{aligned} h_\mu^{\tilde{\mu}}(\omega) &= -\liminf_n \frac{1}{n} \log \frac{\prod_{k=1}^{n} f_2(\xi_k(\omega))}{\prod_{k=1}^{n} f_1(\xi_k(\omega))} \\ &= \limsup_n \frac{1}{n} \sum_{k=1}^{n} \log \left[\frac{f_1(\xi_k(\omega))}{f_2(\xi_k(\omega))}\right] \\ &= D(\mu\|\tilde{\mu}) \ \ \mu - a.s. \ \text{(By the classcial SLLN).} \end{aligned}$$

EXAMPLE 2. Recall that a sequence of random variables $\xi_1(\omega),\cdots,\xi_n(\omega)$ are said to be negatively dependent if, for every n,

$$P\{\cap_{k=1}^n[\xi_k(\omega) \leqslant x_k]\} \leqslant \Pi_{k=1}^n P[\xi_k(\omega) \leqslant x_k]$$

and

$$P\{\cap_{k=1}^n[\xi_k(\omega) > x_k]\} \leqslant \Pi_{k=1}^n P[\xi_k(\omega) > x_k].$$

The notion of negatively dependent random variables was introduced by Lehmann ([7]) and developed by Joag-Dev and Proschan ([11]).

Let $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega),\cdots,\xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ be defined on probability space $(\Omega,\mathscr{F},\mu)$ with joint density functions

$$p_n(x_{n1},\cdots,x_{nn}) \ \ n \in \mathbb{N}^+$$

and the marginal density functions

$$p(x_{nk}) \ \ 1 \leqslant k \leqslant n, \ \ n \in \mathbb{N}^+.$$

By the Kolomogrov's measure extension theorem, there exists a measure $\tilde{\mu}$ on $(\Omega, \mathscr{F})$ such that the joint density function of $\xi^n(\omega)$ is

$$\Pi_{k=1}^n p(x_{nk})$$

i.e. $\xi(\omega)$ is rowwise independent under measure $\tilde{\mu}$. Hence, if $\xi(\omega)$ is rowwise negatively dependent under measure μ then, for every n, we have

$$p_n(x_{n1}, \cdots, x_{nn}) \leqslant \Pi_{k=1}^n p(x_{nk}).$$

We immediately deduce from the definition 2 that

$$h_{\mu}^{\tilde{\mu}}(\omega) = 0 \;\; \mu - a.s.$$

The following lemma 1 is an important technical tool in the proof of our main results.

LEMMA 1. *Let $\{\xi_n(\omega)\}_{n\in\mathbb{N}^+}$ be a sequence of nonnegative r.v.'s with $\mathbb{E}_\mu \xi_n(\omega) \leqslant 1$, and let $\{\sigma_n(\omega)\}_{n\in\mathbb{N}^+}$ be a sequence of positive nondecreasing r.v.'s such that $\frac{1}{\sigma_n(\omega)} = o(\frac{1}{\log n})$ $\mu - a.s.$ (as $n \to \infty$). Then*

$$\limsup_n \frac{1}{\sigma_n(\omega)} \log \xi_n(\omega) \leqslant 0 \quad \mu - a.s. \tag{3}$$

Proof. By Markov's inequality, we have for any $t_n > 0$, $n = 1, 2, \cdots$

$$\mu\{\omega : \xi_n(\omega) \geqslant t_n\} \leqslant \frac{1}{t_n}$$

i.e.

$$\mu\left\{\omega : \frac{1}{\sigma_n(\omega)} \log \xi_n(\omega) \geqslant \frac{1}{\sigma_n(\omega)} \log t_n\right\} \leqslant \frac{1}{t_n}.$$

Put $t_n = n^{1+\varepsilon}$ $(\varepsilon > 0)$ and noting that $\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} < \infty$, we have by the Borel-Cantelli lemma that the event

$$\left\{\omega : \frac{1}{\sigma_n(\omega)} \log \xi_n(\omega) \geqslant \frac{1}{\sigma_n(\omega)} \log t_n\right\}$$

occurs only finitely often with probability 1. Thus (3) follows since $\frac{1}{\sigma_n(\omega)} \log t_n = \frac{(1+\varepsilon)\log n}{\sigma_n(\omega)} \to 0$ $\mu - a.s.$ (as $n \to \infty$) □

LEMMA 2. *For $|x| \leqslant 1$, we have*
In case of $0 < \lambda < 1$,

$$\lambda^x \leqslant 1 + x \log \lambda + \frac{1}{2\lambda} \log^2 \lambda. \tag{4}$$

In case of $1 < \lambda < M$,

$$\lambda^x \leqslant 1 + x \log \lambda + \frac{M}{2} \log^2 \lambda. \tag{5}$$

Proof. Let $t = \log\lambda^x$ in the well known inequality $e^t \leqslant 1+t+\frac{1}{2}t^2e^{|t|}$ $t \in \mathbb{R}$, we have

$$\lambda^x \leqslant 1 + x\log\lambda + \frac{1}{2}(\log\lambda)^2 x^2 e^{|x\log\lambda|},$$

then (4) and (5) follows. □

LEMMA 3. *For* $0 \leqslant x \leqslant 1$, *and* $0 < \lambda < \infty$, *we have*

$$\lambda^x \leqslant 1 + (\lambda - 1)x. \tag{6}$$

Proof. Apply the Jensen's inequality, it follows that

$$\log\lambda^x = x\log\lambda + (1-x)\log 1 \leqslant \log(\lambda x + 1 - x) = \log[1+(\lambda-1)x],$$

which implies (6) holds. □

LEMMA 4. (see [2]) *Let* $\xi_1(\omega), \xi_2(\omega), \cdots, \xi_n(\omega)$ *be negatively dependent random variables.*

(1) If $f_1, f_2, \cdots, f_n$ *is a sequence measurable functions which are all monotone increasing (or all are monotone decreasing). Then* $f_1(\xi_1(\omega)), f_2(\xi_2(\omega)), \cdots, f_n(\xi_n(\omega))$ *are negatively dependent random variables, too.*

(2) $\mathbb{E}_\mu(\xi_1(\omega)\cdots\xi_n(\omega)) \leqslant \mathbb{E}_\mu(\xi_1(\omega))\cdots\mathbb{E}_\mu(\xi_n(\omega))$, *provided the expectations exist.*

LEMMA 5. (see [1]) *Let* $\xi(\omega)$ *be a centered random variable such that* $\xi(\omega) \leqslant 1$ $\mu - a.s.$ *and* $Var(\xi(\omega)) \leqslant v$ *for some positive constant* v. *Then, for any positive* λ,

$$\log\mathbb{E}_\mu[\exp(\lambda\xi(\omega))] \leqslant \frac{1}{4}\varphi(v)\lambda^2 \tag{7}$$

where

$$\varphi(v) = \begin{cases} \frac{1-v^2}{|\log v|} & if\ v < 1, \\ 2v & if\ v \geqslant 1. \end{cases}$$

3. Strong deviation theorems

With the preliminary preparation, we can now state and prove the main conclusions of this paper.

THEOREM 1. *Let* $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ *be a triangle array of random variables, and* $h_\mu^{\tilde{\mu}}(\omega)$ *be defined as in (2). Let*

$$\{f_{nk}(x_{n1}, \cdots, x_{nk}), 1 \leqslant k \leqslant n\}_{n\in\mathbb{N}^+}$$

be an array of bounded ($|f_{nk}| \leqslant c$), real valued, multivariate Borel functions. Denote $\tilde{\xi}_{nk}(\omega) = f_{nk}(\xi_{n1}(\omega), \cdots, \xi_{nk}(\omega))$, $k = 1, \cdots, n$, $n \in \mathbb{N}^+$, $S_n(\omega) := \sum_{k=1}^{n} \tilde{\xi}_{nk}(\omega)$ *and* $T_n(\omega) := \sum_{k=1}^{n} \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega), \cdots, \xi_{n(k-1)}(\omega))$. *Then*

$$\limsup_n \frac{S_n(\omega) - T_n(\omega)}{n} \leqslant c\sqrt{2h_\mu^{\tilde{\mu}}(\omega)} \quad \mu - a.s. \tag{8}$$

$$\liminf_n \frac{S_n(\omega) - T_n(\omega)}{n} \geqslant c\tau(h_\mu^{\tilde{\mu}}(\omega)) \qquad \mu - a.s. \tag{9}$$

where $\tau(x) = \sup_{0<\lambda<1}\left\{\frac{\log\lambda}{2\lambda} + \frac{x}{\log\lambda}, x \geqslant 0\right\}$.

Proof. We assume without lose of generality that $|f_{nk}| \leqslant 1$ (as long as the range is bounded the restriction to $[-1, 1]$ is immaterial, as one can always rescala). Let $1 < \lambda < M$ be a constant and let, for every $n \in \mathbb{N}^+$, $k = 1, 2, \cdots, n$,

$$h_{nk}(x_{n1}, \cdots, x_{nk}) = \frac{\lambda^{f_{nk}(x_{n1}, \cdots, x_{nk})} \cdot q_k(x_{nk}|x_{n1}, \cdots, x_{n(k-1)})}{1 + \log\lambda \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|x_{n1}, \cdots, x_{n(k-1)}) + \frac{M}{2}\log^2\lambda}.$$

Let

$$\Lambda_n^{(1)}(\lambda, \omega) = \begin{cases} \frac{\prod_{k=1}^{n} h_{nk}(\xi_{n1}(\omega), \cdots, \xi_{nk}(\omega))}{p_n(\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))} & \text{if denominator } > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Note that

$$\begin{aligned}
&\mathbb{E}_\mu \Lambda_n^{(1)}(\lambda, \omega) \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\prod_{k=1}^{n} h_{nk}(x_{n1}, \cdots, x_{nk})}{p_n(x_{n1}, \cdots, x_{nn})} \cdot p_n(x_{n1}, \cdots, x_{nn}) dx_{n1} \cdots dx_{nn} \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{n-1} h_{nk}(x_{n1}, \cdots, x_{nk}) dx_{n1} \cdots dx_{n(n-1)} \\
&\quad \cdot \frac{\int \lambda^{f_{nn}(x_{n1}, \cdots, x_{nn})} \cdot q_n(x_{nn}|x_{n1}, \cdots, x_{n(n-1)}) dx_{nn}}{1 + \log\lambda \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nn}(\omega)|x_{n1}, \cdots, x_{n(n-1)}) + \frac{M}{2}\log^2\lambda} \\
&= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{n-1} h_{nk}(x_{n1}, \cdots, x_{nk}) dx_{n1} \cdots dx_{n(n-1)} \\
&\quad \cdot \frac{\mathbb{E}_{\tilde{\mu}}(\lambda^{\tilde{\xi}_{nn}(\omega)}|(x_{n1}, \cdots, x_{n(n-1)})}{1 + \log\lambda \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nn}(\omega)|x_{n1}, \cdots, x_{n(n-1)}) + \frac{M}{2}\log^2\lambda} \\
&\leqslant \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{n-1} h_{nk}(x_{n1}, \cdots, x_{nk}) dx_{n1} \cdots dx_{n(n-1)} \quad \text{(by (5))} \\
&\leqslant 1.
\end{aligned}$$

According to lemma 1, we have

$$\limsup_n \frac{1}{n} \log \Lambda_n^{(1)}(\lambda, \omega) \leqslant 0 \qquad \mu - a.s. \tag{11}$$

Putting $\lambda \to 1^+$ in (11), we have

$$\limsup_n \frac{1}{n} \log \mathscr{R}_n(\omega) \leqslant 0 \qquad \mu - a.s. \tag{12}$$

Thus, we have

$$h_\mu^{\tilde{\mu}}(\omega) \geqslant 0 \qquad \mu - a.s. \tag{13}$$

Noticing that

$$\begin{aligned}
&\prod_{k=1}^n h_{nk}(\xi_{n1}(\omega), \cdots, \xi_{nk}(\omega)) \\
&= \prod_{k=1}^n \frac{\lambda^{\tilde{\xi}_{nk}(\omega)} \cdot q_k(\xi_{nk}(\omega)|\xi_{n1}(\omega), \cdots, \xi_{n(k-1)}(\omega))}{1 + \log \lambda \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega), \cdots, \xi_{n(k-1)}(\omega)) + \frac{M}{2} \log^2 \lambda} \\
&= \lambda^{S_n(\omega)} \cdot \prod_{k=1}^n \frac{q_k(\xi_{nk}(\omega)|\xi_{n1}(\omega), \cdots, \xi_{n(k-1)}(\omega))}{1 + \log \lambda \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega), \cdots, \xi_{n(k-1)}(\omega)) + \frac{M}{2} \log^2 \lambda}.
\end{aligned} \tag{14}$$

It follows from (10) and (14) that

$$\begin{aligned}
\log \Lambda_n^{(1)}(\lambda, \omega) = \log \lambda \cdot S_n(\omega) - \sum_{k=1}^n \log \Big[1 + \log \lambda \cdot \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega), \cdots, \xi_{n(k-1)}(\omega)) \\
+ \frac{M}{2} \log^2 \lambda \Big] + \log \mathscr{R}_n(\omega).
\end{aligned} \tag{15}$$

We have by (11) and (15)

$$\begin{aligned}
&\limsup_n \frac{1}{n} \{ S_n(\omega) \log \lambda - \sum_{k=1}^n \log \Big[1 + \log \lambda \cdot \mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega), \cdots, \xi_{n(k-1)}(\omega)) \\
&+ \frac{M}{2} \log^2 \lambda \Big] + \log \mathscr{R}_n(\omega) \} \leqslant 0 \;\; \mu - a.s.
\end{aligned} \tag{16}$$

Let $1<\lambda<M$ and M sufficiently large. Dividing the both sides of (16) by $\log\lambda$, we obtain by the property of the superior limit and the inequality $0\leqslant\log(1+x)\leqslant x\ (x\geqslant 0)$

$$\begin{aligned}
&\limsup_n \frac{S_n(\omega)-T_n(\omega)}{n} \\
&\leqslant \limsup_n \frac{1}{n}\Bigg[\sum_{k=1}^{n} \frac{\log[1+\log\lambda\cdot\mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega))+\frac{M}{2}\log^2\lambda]}{\log\lambda} \\
&\quad -\mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega))\Bigg]+\frac{h_\mu^{\tilde{\mu}}(\omega)}{\log\lambda} \\
&\leqslant \limsup_n \frac{1}{n}\sum_{k=1}^{n}\Bigg[\frac{\log\lambda\cdot\mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega))+\frac{M}{2}\log^2\lambda}{\log\lambda} \\
&\quad -\mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega))\Bigg]+\frac{h_\mu^{\tilde{\mu}}(\omega)}{\log\lambda} \\
&\leqslant \frac{M}{2}\log\lambda+\frac{h_\mu^{\tilde{\mu}}(\omega)}{\log\lambda} \quad \mu-a.s. \qquad (17)
\end{aligned}$$

i.e.

$$\limsup_n \frac{S_n(\omega)-T_n(\omega)}{n}\leqslant\frac{M}{2}\log\lambda+\frac{h_\mu^{\tilde{\mu}}(\omega)}{\log\lambda}:=g(\lambda,h_\mu^{\tilde{\mu}}) \quad \mu-a.s. \qquad (18)$$

where $g(\lambda,x)=\frac{M}{2}\log\lambda+\frac{x}{\log\lambda}$. It is easy to see that if $x>0$, then $g(\lambda,x)$ as a function of λ attains its smallest value $g_{\min}=\sqrt{2Mx}$ on the interval $(1,+\infty)$, and $g(\lambda,0)$ is increasing on the interval $(1,+\infty)$ and $\lim_{\lambda\to1^+}g(\lambda,0)=0$. We have by the continuity of g with respect to λ,

$$\inf_{1<\lambda<\infty} g(\lambda,h_\mu^{\tilde{\mu}}(\omega))=\sqrt{2Mh_\mu^{\tilde{\mu}}(\omega)}. \qquad (19)$$

Putting $M\to1^+$, we obtain from (18) and (19)

$$\limsup_n \frac{S_n(\omega)-T_n(\omega)}{n}\leqslant\sqrt{2h_\mu^{\tilde{\mu}}(\omega)} \quad \mu-a.s. \qquad (20)$$

Let $0<\lambda<1$ be a constant and let, for every $n\in\mathbb{N}^+$, $k=1,2,\cdots,n$,

$$h'_{nk}(x_{n1},\cdots,x_{nk})=\frac{\lambda^{f_{nk}(x_{n1},\cdots,x_{nk})}\cdot q_k(x_{nk}|x_{n1},\cdots,x_{n(k-1)})}{1+\log\lambda\,\mathbb{E}_{\tilde{\mu}}(\tilde{\xi}_{nk}(\omega)|x_{n1},\cdots,x_{n(k-1)})+\frac{1}{2\lambda}\log^2\lambda}.$$

Let

$$\Lambda_n^{(2)}(\lambda,\omega)=\begin{cases}\frac{\prod_{k=1}^n h'_{nk}(\xi_{n1}(\omega),\cdots,\xi_{nk}(\omega))}{p_n(\xi_{n1}(\omega),\cdots,\xi_{nn}(\omega))} & \text{if denominator } >0,\\ 0 & \text{otherwise.}\end{cases}$$

It is not difficult to check that $\mathbb{E}_\mu \Lambda_n^{(2)}(\lambda,\omega) \leqslant 1$, hence we have $\limsup_n \frac{1}{n}\log\Lambda_n^{(2)}(\lambda,\omega) \leqslant 0 \;\; \mu - a.s.$

Note that

$$\log\Lambda_n^{(2)}(\lambda,\omega) = \log\lambda S_n(\omega) - \sum_{k=1}^{n}\log\left[1+\log\lambda\mathbb{E}_{\tilde\mu}(\tilde\xi_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega)) + \frac{1}{2\lambda}\log^2\lambda\right] + \log\mathscr{R}_n(\omega).$$

Thus

$$\limsup_n \frac{1}{n}\{S_n(\omega)\log\lambda - \sum_{k=1}^{n}\log\left[1+\log\lambda\mathbb{E}_{\tilde\mu}(\tilde\xi_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega)) + \frac{1}{2\lambda}\log^2\lambda\right] + \log\mathscr{R}_n(\omega)\} \leqslant 0 \;\; \mu - a.s. \tag{21}$$

Dividing the both sides of (21) by $\log\lambda$, we obtain by the property of the inferior limit and the inequality $\log(1+x) \leqslant x \;\; (-1 < x < 0)$

$$\begin{aligned}
&\liminf_n \frac{S_n(\omega)-T_n(\omega)}{n} \\
&\geqslant \liminf_n \frac{1}{n}\sum_{k=1}^{n}[\frac{\log[1+\log\lambda\mathbb{E}_{\tilde\mu}(\tilde\xi_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega)) + \frac{1}{2\lambda}\log^2\lambda]}{\log\lambda} \\
&\quad - \mathbb{E}_{\tilde\mu}(\tilde\xi_{nk}(\omega)|\xi_{n1}(\omega),\cdots,\xi_{n(k-1)}(\omega))] + \frac{h_\mu^{\tilde\mu}(\omega)}{\log\lambda} \\
&\geqslant \frac{1}{2\lambda}\log\lambda + \frac{h_\mu^{\tilde\mu}(\omega)}{\log\lambda} \\
&:= h(\lambda, h_\mu^{\tilde\mu}(\omega)) \quad \mu - a.s.
\end{aligned}$$

where $h(\lambda,x) = \frac{1}{2\lambda}\log\lambda + \frac{x}{\log\lambda}$. Let $\tau(x) = \sup_{0<\lambda<1}\{h(\lambda,x), x \geqslant 0\}$, it is obvious that $\tau(x) \leqslant 0$ and is monotonically decreasing with respect to x. Hence

$$\liminf_n \frac{S_n(\omega)-T_n(\omega)}{n} \geqslant \tau(h_\mu^{\tilde\mu}(\omega)) \quad \mu - a.s. \tag{22}$$

The proof is completed. □

REMARK 2. Replacing the density function by probability mass function, Theorem 1 also holds for discrete random variables.

As direct consequence and applications of Theorem 1, we have the following interesting corollaries.

COROLLARY 1. *Under the conditions of Theorem 1, if* $\mu = \tilde{\mu}$ *or* $h_{\mu}^{\tilde{\mu}}(\omega) = 0$ $\mu - a.s.$*, then*

$$\lim_n \frac{S_n(\omega) - T_n(\omega)}{n} = 0 \ \ \mu - a.s. \tag{23}$$

Proof. This result is the direct consequence of Theorem 1 with $\mu = \tilde{\mu}$ or $h_{\mu}^{\tilde{\mu}}(\omega) = 0$ $\mu - a.s.$. □

COROLLARY 2. (SLLN) *Let* $\xi(\omega) = \{\xi_n(\omega)\}_{n\in\mathbb{N}^+}$ *be a sequence of uniformly bounded negatively dependent r.v.'s defined on* $(\Omega, \mathscr{F}, \mu)$*, then*

$$\lim_n \frac{\Sigma_{k=1}^n \xi_k(\omega)}{n} = \mathbb{E}_{\mu}\xi_1(\omega) \ \ \mu - a.s.$$

Proof. Without loss of generality, we may assume that $|\xi_n(\omega)| \leqslant 1$, $n \in \mathbb{N}^+$, and let $f_{nk} = \xi_k(\omega)$, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}^+$. Let $\tilde{\mu}$ be the reference measure on $(\Omega, \mathscr{F})$ and assume the $\xi(\omega) = \{\xi_n(\omega)\}_{n\in\mathbb{N}^+}$ is independent under $\tilde{\mu}$. From example 2 we know that $h_{\mu}^{\tilde{\mu}}(\omega) = 0$ $a.s.$, the proof is trivially follows. □

COROLLARY 3. *Let* $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ *be an array of rowwise negatively dependent random variables with the common distribution function* $F(x)$*, and let* $f_{nk}(x) = \mathbf{1}_{(-\infty,x]}(\xi_{nk}(\omega))$, $1 \leqslant k \leqslant n$*. Denote* $\widehat{F}_n(x) = \frac{1}{n}\sum_{k=1}^n f_{nk}(x) = \frac{1}{n}\sum_{k=1}^n \mathbf{1}_{(-\infty,x]}(\xi_{nk}(\omega))$*, that is the empirical distribution function (edf) of* $\xi^n(\omega)$*. Then*

$$\lim_n \widehat{F}_n(x) = F(x) \ \ \mu - a.s.$$

Proof. It follows from application of corollary 2 since $\mathbb{E}_{\mu} f_{nk}(x) = F(x)$. □

COROLLARY 4. *Let* $\{A_{nk}, 1 \leqslant k \leqslant n\}_{n\in\mathbb{N}^+}$ *be an array of rowwise independent events, if* $\frac{1}{\sum_{k=1}^n \mu(A_{nk})} = o(\frac{1}{\log n})$ *as* $n \to \infty$*. Then*

$$\lim_n \frac{\sum_{k=1}^n \mathbf{1}_{A_{nk}}(\omega)}{\sum_{k=1}^n \mu(A_{nk})} = 1 \ \ \mu - a.s.$$

Proof. Let $\lambda > 0$ be a constant, and let

$$\Lambda_n^{(3)}(\lambda, \omega) = \prod_{k=1}^n \frac{\lambda^{\mathbf{1}_{A_{nk}}(\omega)}}{1 + (\lambda - 1)\mu(A_{nk})}, \ \ n = 1, 2, \cdots.$$

From lemma 3, it is easy to verify that $\mathbb{E}_{\mu}\Lambda_n^{(3)}(\lambda, \omega) \leqslant 1$. Let $\sigma_n := \sum_{k=1}^n \mu(A_{nk})$, we have by lemma 1 that $\frac{1}{\sigma_n}\log \Lambda_n^{(3)}(\lambda, \omega) \leqslant 0$ $\mu - a.s.$ i.e.

$$\limsup_n \frac{1}{\sigma_n}\sum_{k=1}^n \mathbf{1}_{A_{nk}}(\omega)\log\lambda \leqslant \frac{1}{\sigma_n}\sum_{k=1}^n \log[1 + (\lambda - 1)\mu(A_{nk})] \leqslant (\lambda - 1) \ \ \mu - a.s. \tag{24}$$

Let $\lambda > 1$. Dividing the both sides of (24) by $\log\lambda$, we have

$$\limsup_{n} \frac{1}{\sigma_n} \sum_{k=1}^{n} \mathbf{1}_{A_{nk}}(\omega) \leqslant \frac{\lambda - 1}{\log\lambda} \quad \mu - a.s.$$

Let $\lambda \to 1^+$, by Hospital's rule we have

$$\limsup_{n} \frac{1}{\sigma_n} \sum_{k=1}^{n} \mathbf{1}_{A_{nk}}(\omega) \leqslant 1 \quad \mu - a.s.$$

The remainder of the argument is analogous to that in Theorem 1, hence the statement is proved. □

4. Sub-gaussian random variables

In probability, Gaussian random variables are the easiest and most commonly used distribution encountered. We shall now focus our attention on strong deviation theorem and law of large numbers for sub-gaussian random variables.

Sub-gaussianity properties of random variables and random processes (see [3]) are important features, since they allow us to derive results concerning, for instance, strong law of large numbers (see [12]), large deviations inequalities, asymptotic behaviour of particular processes of the behaviour of their supremum.

A random variables $\xi(\omega)$ is called Sub-Gaussian of its moment generating function is majorized by the moment generating function of a centered Gaussian random variables with variance σ^2, *i.e.*

$$\mathbb{E}_\mu[\exp(\lambda\xi(\omega))] \leqslant \mathbb{E}_\mu[\exp(\eta(\omega)\lambda)] = \exp\left(\frac{\sigma^2\lambda^2}{2}\right) \tag{25}$$

where $\eta(\omega) \sim \mathbb{N}(0, \sigma^2)$. In terms of cumulant generating function, this condition takes the form $\log\mathbb{E}_\mu \exp(\lambda\xi(\omega)) \leqslant \sigma^2\lambda^2/2$. The sub-Gaussian standard (norm) $\tau_\mu(\xi(\omega))$ is defined as follows:

$$\tau_\mu(\xi(\omega)) = \inf\left\{\sigma \geqslant 0 : \forall\lambda \in \mathbb{R} \ \ \log\mathbb{E}_\mu[\exp(\lambda\xi(\omega))] \leqslant \frac{\sigma^2\lambda^2}{2}\right\}.$$

PROPOSITION 1. (see [5]) If $\xi(\omega)$ is sub-gaussian, then for $\lambda > 0$

$$\mu(\xi(\omega) > \lambda) \leqslant \exp\left\{-\frac{\lambda^2}{2[\tau(\xi(\omega))]^2}\right\}.$$

PROPOSITION 2. (see [5]) If $\xi_1(\omega), \cdots, \xi_n(\omega)$ are negatively dependent (or acceptable), sub-gaussian random variables, then $\Sigma_{i=1}^n \xi_i(\omega)$ is sub-gaussian with $\tau(\Sigma_{i=1}^n \xi_i(\omega)) = \{\Sigma_{i=1}^n [\tau(\xi_i(\omega))]^2\}^{1/2}$.

PROPOSITION 3. (see [5]) If $\xi(\omega)$ is bounded ($|\xi(\omega)| \leqslant C$) and $\mathbb{E}_\mu\xi(\omega) = 0$, then $\xi(\omega)$ is sub-gaussian with $\tau(\xi(\omega)) = \sqrt{2}C$.

THEOREM 2. *Let $\xi(\omega)=\{\xi^n(\omega)=(\xi_{n1}(\omega),\cdots,\xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ be a triangle array of arbitrarily dependent random variables. Further, assume that $\xi(\omega)$ be a triangle array of rowwise negatively dependent (see [6]) random variables under probability measure $\tilde{\mu}$, and $h_\mu^{\tilde{\mu}}(\omega)$ be defined as in (2). If, in addition, $\sum_{k=1}^n[\tau_{\tilde{\mu}}(\xi_{nk}(\omega))]^2\leqslant Cn$ for all n where C is any positive constant. Then*

$$-\sqrt{2Ch_\mu^{\tilde{\mu}}(\omega)}\leqslant\liminf_n\frac{S_n(\omega)}{n}\leqslant\limsup_n\frac{S_n(\omega)}{n}\leqslant\sqrt{2Ch_\mu^{\tilde{\mu}}(\omega)}\quad \mu-a.s. \tag{26}$$

here and in the following $S_n(\omega)$ denotes $\sum_{k=1}^n\xi_{nk}(\omega)$.

Proof. Let $\lambda\in\mathbb{R}$ be a real constant and let, for every $n\in\mathbb{N}^+$, $k=1,2,\cdots,n$,

$$h_n(x_{n1},\cdots,x_{nn})=\frac{\exp[\lambda(x_{n1}+\cdots+x_{nn})]\cdot q_n(x_{n1},\cdots,x_{nn})}{\prod_{k=1}^n\mathbb{E}_{\tilde{\mu}}e^{\lambda\xi_{nk}(\omega)}}.$$

Let

$$\Lambda_n^{(4)}(\lambda,\omega)=\begin{cases}\frac{h_n(\xi_{n1}(\omega),\cdots,\xi_{nn}(\omega))}{p_n(\xi_{n1}(\omega),\cdots,\xi_{nn}(\omega))} & \text{if denominator } >0,\\ 0 & \text{otherwise.}\end{cases} \tag{27}$$

Note that

$$\begin{aligned}&\mathbb{E}_\mu\Lambda_n^{(4)}(\lambda,\omega)\\ &=\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\frac{h_n(x_{n1},\cdots,x_{nn})}{p_n(x_{n1},\cdots,x_{nn})}\cdot p_n(x_{n1},\cdots,x_{nn})dx_{n1}\cdots dx_{nn}\\ &=\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}h_n(x_{n1},\cdots,x_{n(n-1)})dx_{n1}\cdots dx_{n(n-1)}\\ &=\frac{\mathbb{E}_{\tilde{\mu}}[e^{\lambda S_n(\omega)}]}{\prod_{k=1}^n\mathbb{E}_{\tilde{\mu}}e^{\lambda\xi_{nk}(\omega)}}\\ &\leqslant 1 \text{ (by negative dependence)}\end{aligned}$$

which implies

$$\limsup_n\frac{1}{n}\log\Lambda_n^{(4)}(\lambda,\omega)\leqslant 0\qquad \mu-a.s. \tag{28}$$

Noticing that

$$\log\Lambda_n^{(4)}(\lambda,\omega)=\lambda S_n(\omega)-\sum_{k=1}^n\log\mathbb{E}_{\tilde{\mu}}(e^{\lambda\xi_{nk}(\omega)})+\log\mathscr{R}_n(\omega). \tag{29}$$

We have by (28) and (29)

$$\limsup_n\frac{1}{n}\{\lambda S_n(\omega)-\sum_{k=1}^n\log\mathbb{E}_{\tilde{\mu}}(e^{\lambda\xi_{nk}(\omega)})+\log\mathscr{R}_n(\omega)\}\leqslant 0\ \ \mu-a.s. \tag{30}$$

Put $\lambda > 0$, we have by (30) that

$$\limsup_n \frac{S_n(\omega)}{n} \leqslant \limsup_n \frac{\lambda}{2n} \sum_{k=1}^{n} \left[\tau_{\tilde{\mu}}(\xi_{nk}(\omega))\right]^2 + \frac{h_\mu^{\tilde{\mu}}(\omega)}{\lambda}$$
$$\leqslant \frac{C}{2}\lambda + \frac{h_\mu^{\tilde{\mu}}(\omega)}{\lambda} \quad \mu - a.s.$$

i.e.

$$\limsup_n \frac{S_n(\omega)}{n} \leqslant \frac{C}{2}\lambda + \frac{h_\mu^{\tilde{\mu}}(\omega)}{\lambda} := g(\lambda, h_\mu^{\tilde{\mu}}) \quad \mu - a.s.$$

where $g(\lambda,x) = \frac{C}{2}\lambda + \frac{x}{\lambda}$. It is easy to see that if $x > 0$, then $g(\lambda,x)$ as a function of λ attains its smallest value $g_{\min} = \sqrt{2Cx}$ on the interval $(0,+\infty)$, and $\lim_{\lambda\to 0^+} g(\lambda,0) = 0$. Therefore

$$\limsup_n \frac{S_n(\omega)}{n} \leqslant \sqrt{2Ch_\mu^{\tilde{\mu}}(\omega)} \quad \mu - a.s.$$

Putting $\lambda < 0$, we have by the same way that

$$\liminf_n \frac{S_n(\omega)}{n} \geqslant -\sqrt{2Ch_\mu^{\tilde{\mu}}(\omega)} \quad \mu - a.s.$$

concluding the proof of the theorem. □

COROLLARY 5. *Let* $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ *be a triangle array of arbitrarily dependent random variables. Furthermore, assume that* $\xi(\omega)$ *is a triangle array of rowwise negatively dependent (see* [6]*) centered random variables under probability measure* $\tilde{\mu}$ *such that* $\xi_{nk}(\omega) \leqslant 1$ *and* $Var(\xi_{nk}(\omega)) \leqslant v$, $1 \leqslant k \leqslant n$, $n \in \mathbb{N}^+$ *for some positive constant* v, *and* $h_\mu^{\tilde{\mu}}(\omega)$ *is defined as in (2). Then*

$$\limsup_n \frac{S_n(\omega)}{n} \leqslant \sqrt{\varphi(v)h_\mu^{\tilde{\mu}}(\omega)} \quad \mu - a.s.$$

where $S_n(\omega) := \sum_{k=1}^n \xi_{nk}(\omega)$ *and*

$$\varphi(v) = \begin{cases} \frac{1-v^2}{|\log v|} & if\ v < 1, \\ 2v & if\ v \geqslant 1. \end{cases}$$

The proof of this corollary can be completed by the method analogous to that used above.

REMARK 3. It is obvious that if the condition of $\xi_{nk}(\omega) \leqslant 1$ is replaced by $\xi_{nk}(\omega) \geqslant -1$ in Corollary 5, we can easily establish the following inequality

$$\liminf_n \frac{S_n(\omega)}{n} \geqslant -\sqrt{\varphi(v)h_\mu^{\tilde{\mu}}(\omega)} \quad \mu - a.s.$$

THEOREM 3. *Let* $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ *be a triangle array of rowwise negatively dependent (or acceptable r.v.'s). If* $\sum_{i=1}^n [\tau(\xi_{ni}(\omega))]^2 \leqslant Cn^{2-d}$ *for all* n *where* C *and* d *are any positive constants, then*

$$\lim_n \frac{S_n(\omega)}{n} = 0 \;\; \mu - a.s.$$

Proof. Let $\varepsilon > 0$ be given. By Proposition 1,

$$\begin{aligned}\mu(|S_n(\omega)| > n\varepsilon) &= \mu(S_n(\omega) > n\varepsilon) + \mu(-S_n(\omega) > n\varepsilon)\\ &\leqslant 2\exp\left\{-\frac{\varepsilon^2 n^2}{2\sum_{i=1}^n [\tau(\xi_{ni}(\omega))]^2}\right\}\\ &\leqslant 2\exp\left[-\varepsilon^2 n^d/2C\right]\end{aligned}$$

for each n. Thus

$$\sum_{n=1}^{\infty} \mu(|S_n(\omega)| > n\varepsilon) \leqslant 2\sum_{n=1}^{\infty} \exp(-\varepsilon^2 n^d/2C) < \infty.$$

The first Borel-Cantelli lemma completes the proof. □

COROLLARY 6. *Let* $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ *be a triangle array of rowwise negatively dependent (or acceptable r.v.'s) which are bounded by a constant* C. *Then for* $\delta > \frac{1}{2}$,

$$\lim_n \frac{1}{n}\sum_{i=1}^n (\xi_{ni}(\omega) - \mathbb{E}_\mu \xi_{ni}(\omega)) = 0 \;\; \mu - a.s.$$

Proof. It is obvious $|\xi_{ni}(\omega) - \mathbb{E}_\mu \xi_{ni}(\omega)| \leqslant 2C$, and Proposition 3 provides the sub-gaussian property. Note that, for any $\varepsilon > 0$,

$$\mu(|\sum_{i=1}^n (\xi_{ni}(\omega) - \mathbb{E}_\mu \xi_{ni}(\omega))| > n^\delta \varepsilon) \leqslant 2\exp\left[-\frac{\varepsilon^2 n^{2\delta-1}}{16C^2}\right]$$

for each n, and which completes the proof. □

THEOREM 4. *Let* $\xi(\omega) = \{\xi^n(\omega) = (\xi_{n1}(\omega), \cdots, \xi_{nn}(\omega))\}_{n\in\mathbb{N}^+}$ *be a triangle array of rowwise negatively dependent random variables such that* $\mathbb{E}_\mu \xi_{ni}(\omega) = 0$ *for all* $1 \leqslant i \leqslant 1, n \in \mathbb{N}^+$. *If*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{\mathbb{E}_\mu |\xi_{ni}(\omega)|^{2pq}}{i^q} < \infty \tag{31}$$

where q *is some positive constant and* p *is some constant,* $1 < p \leqslant 2$. *Then,*

$$\lim_n \frac{1}{n}\sum_{i=1}^n \xi_{ni}(\omega) = 0 \qquad \mu - a.s. \tag{32}$$

Proof. First, let $\eta_{ni}(\omega)=\xi_{ni}(\omega)\mathbf{1}_{[|\xi_{ni}(\omega)|\leqslant i^{1/2p}]}$, $1\leqslant i\leqslant n$, $n\in\mathbb{N}^+$. Note that

$$\begin{aligned}\sum_{n=1}^{\infty}\mu(\sum_{i=1}^{n}\xi_{ni}(\omega)\neq\sum_{i=1}^{n}\eta_{ni}(\omega)) &\leqslant \sum_{n=1}^{\infty}\sum_{i=1}^{n}\mu(\xi_{ni}(\omega)\neq\eta_{ni}(\omega))\\ &=\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mu(|\xi_{ni}(\omega)|>i^{\frac{1}{2p}})\\ &\leqslant\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\mathbb{E}_\mu|\xi_{ni}(\omega)|^{2pq}}{i^q}<\infty.\end{aligned}$$

Hence,

$$\mu\left\{\lim_n\frac{1}{n}\sum_{i=1}^{n}\xi_{ni}(\omega)=\lim_n\frac{1}{n}\sum_{i=1}^{n}\eta_{ni}(\omega)\right\}=1.$$

Noticing that $\mathbb{E}_\mu\xi_{ni}(\omega)=0$ for all $1\leqslant i\leqslant n$, $n\in\mathbb{N}^+$, we have by (31) that

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\mu\eta_{ni}(\omega)\to 0\ \ (as\ n\to\infty). \tag{33}$$

Since $\eta_{ni}(\omega)-\mathbb{E}_\mu\eta_{ni}(\omega)$ is sub-gaussian with $\tau(\eta_{ni}(\omega)-\mathbb{E}_\mu\eta_{ni}(\omega))\leqslant 2\sqrt{2}i^{\frac{1}{2p}}$, we have for any $\varepsilon>0$

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\mu[|\sum_{i=1}^{n}(\eta_{ni}(\omega)-\mathbb{E}_\mu\eta_{ni}(\omega))|>n\varepsilon] &\leqslant\sum_{n=1}^{\infty}\exp\left[-\frac{n^2\varepsilon^2}{16\Sigma_{i=1}^{n}i^{\frac{1}{p}}}\right]\\ &\leqslant\sum_{n=1}^{\infty}\exp\left[-\frac{n^{1-\frac{1}{p}}\varepsilon^2}{16}\right]<\infty.\end{aligned} \tag{34}$$

(32) follows immediately from (33) and (34). □

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Ya-hui Zhu
School of Microelectronics and Data Science
Anhui University of Technology
Ma'anshan, 243002, China
and
School of Mathematics and Big Data
ChaoHu University
Heifei, 238000, China

Wei-cai Peng
School of Mathematics and Big Data
ChaoHu University
Heifei, 238000, China

Yong-jin Zhang
School of Microelectronics and Data Science
Anhui University of Technology
Ma'anshan, 243002, China

Zhong-zhi Wang
School of Microelectronics and Data Science
Anhui University of Technology
Ma'anshan, 243002, China
e-mail: zhongzhiw@126.com