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ON SEVERAL INEQUALITIES RELATED TO CONVEX FUNCTIONS

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Abstract. In this paper, for a function $f:\mathscr{X}\to\mathbb{R}$, we introduce the following expression: $\Delta_\lambda(f)(x,y):=\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)$, where $x,y\in\mathscr{X}$ and $\lambda\in\mathbb{R}$. The purpose of this article is to characterize this expression, by finding various estimates of it. We also give some characterizations of $\Delta_\lambda(f)(x,y)$ when function f is convex, which prove refinements of Young's inequality. Finally, we give several inequalities in a normed space.

1. Introduction

In the literature related to the theory of inequalities, many of the published papers contain studies of certain inequalities which used convexity (see e.g. [5], [14], [16], [17]).

Let $\mathscr{X}$ be a convex subset of a real vector space and let $f:\mathscr{X}\to\mathbb{R}$ be a function such that

$$f(\lambda x+(1-\lambda)y)\leqslant\lambda f(x)+(1-\lambda)f(y) \tag{1}$$

for all $x,y\in\mathscr{X}$ and $\lambda\in[0,1]$. We say that function f is convex. For $\lambda=\frac{1}{2}$, we have

$$f\left(\frac{x+y}{2}\right)\leqslant\frac{f(x)+f(y)}{2} \tag{2}$$

for all $x,y\in\mathscr{X}$. If f is concave, then the above inequalities should be reversed. If I denote a nondegenerate interval of $\mathbb{R}$ and $\mathscr{X}=I$, then inequality (1) generates a series of inequalities, including Young's inequality, thus, for $f(t)=-\log t$, we obtain

$$\lambda x+(1-\lambda)y\geqslant x^\lambda y^{1-\lambda} \tag{3}$$

for every $x,y>0$ and $\lambda\in[0,1]$. In many papers, improvements, generalizations or reverse inequalities of Young's inequality have been studied (see e.g. [4], [5], [7], [11], [12], [13], [19], [20]).

Let $f:\mathscr{X}\to\mathbb{R}$ be a function. We introduce the following expression:

$$\Delta_\lambda(f)(x,y):=\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y),$$

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where $x,y \in \mathscr{X}$ and $\lambda \in \mathbb{R}$. If f is a convex function and $\lambda \in [0,1]$, then we obtain that $\Delta_\lambda(f)(x,y) \geqslant 0$, for all $x,y \in \mathscr{X}$. We find the following properties:

$$\Delta_\lambda(f)(x,x) = \Delta_0(f)(x,y) = \Delta_1(f)(x,y) = 0,$$

$$\Delta_\lambda(f)(x,y) = \Delta_{1-\lambda}(f)(y,x)$$

and

$$\Delta_{1/2}(f)(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y) + \frac{1}{2}(\Delta_\lambda(f)(x,y) + \Delta_\lambda(f)(y,x)) = \Delta_{1/2}(f)(x,y)$$

for every $x,y \in \mathscr{X}$.

In [13], we found a result which can be rewritten as

$$m\frac{\lambda(1-\lambda)}{2}(x-y)^2 \leqslant \Delta_\lambda(f)(x,y) \leqslant M\frac{\lambda(1-\lambda)}{2}(x-y)^2, \tag{4}$$

where $\lambda \in [0,1]$ and $f : [x,y] \to \mathbb{R}$ is a twice differentiable function such that there exist real constants m and M so that $m \leqslant f'' \leqslant M$. According to inequality (4) for $\lambda = \frac{1}{2}$ we obtain the following result, previously established in [3]:

$$\frac{m}{8}(x-y)^2 \leqslant \Delta_{1/2}(f)(x,y) \leqslant \frac{M}{8}(x-y)^2 \tag{5}$$

for every $x,y \in I$, where I is a nondegenerate interval of $\mathbb{R}$.

Let $f : [x,y] \to \mathbb{R}$ be a convex continuous function. Then, in terms of $\Delta_\lambda(f)(x,y)$, Hardy, Litlewood and Pólya [6] remark that

$$\Delta_{1/2}(f)(x,y) \geqslant \Delta_{1/2}(f)(z,t)$$

for every $x \leqslant z \leqslant t \leqslant y$.

In addition, some historical overview of studied problem are given below.

The Jensen inequality can be rewritten in the form of the corresponding functional, i.e.

$$\mathscr{J}_n(f,\mathbf{x},\mathbf{p}) := \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right), \tag{6}$$

where the function $f : I \to \mathbb{R}$ is convex on the interval I and $\mathbf{x} = (x_1,\ldots,x_n) \in I^n$, $\mathbf{p} = (p_1,\ldots,p_n) \in \mathbb{R}_+^n$, $P_n = \sum_{i=1}^n p_i > 0$. Dragomir et al. [2] noticed that the Jensen functional is superadditive, that is,

$$\mathscr{J}_n(f,\mathbf{x},\mathbf{p}+\mathbf{q}) \geqslant \mathscr{J}_n(f,\mathbf{x},\mathbf{p}) + \mathscr{J}_n(f,\mathbf{x},\mathbf{q}), \tag{7}$$

where $\mathbf{p},\mathbf{q} \in \mathbb{R}_+^n$.

In the following years, this relation will become the starting point for improving the Jensen–type inequalities since it implies the so called monotonicity of the Jensen functional, i.e.

$$\mathscr{J}_n(f,\mathbf{x},\mathbf{p}) \geqslant \mathscr{J}_n(f,\mathbf{x},\mathbf{q}) \geqslant 0, \tag{8}$$

whenever $\mathbf{p} \geqslant \mathbf{q}$, i.e. $p_i \geqslant q_i$, $i = 1, \dots, n$ (see also [14], p.717). Note that $\sum_{i=1}^{n} p_i$ need not be equal to 1, otherwise we would not be able to prove these relations.

By virtue of (8), Krnić et al. [8], established the mutual bounds for the Jensen functional expressed in terms of the corresponding non–weighted functional. More precisely, they proved that

$$n \min_{1\leqslant i\leqslant n} \{p_i\} \mathscr{I}_n(f,\mathbf{x}) \leqslant \mathscr{J}_n(f,\mathbf{x},\mathbf{p}) \leqslant n \max_{1\leqslant i\leqslant n} \{p_i\} \mathscr{I}_n(f,\mathbf{x}), \tag{9}$$

where $\mathscr{I}_n(f,\mathbf{x})$ stands for the associated non–weighted functional, i.e.

$$\mathscr{I}_n(f,\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f(\frac{1}{n}\sum_{i=1}^{n} x_i).$$

Also inequality (9) was also proved by Dragomir [1] or by Mitroi [15].

The lower bound in (9) represents the refinement, while the upper one is the reverse of the Jensen inequality. Based on this property, numerous inequalities such the Young inequality, the Hölder inequality, power mean inequalities, etc. have been refined (see, e.g. [8, 9] and the references cited therein). These new results about the Jensen inequality are collected in monograph [10]. For $n = 2$, in inequalities (6), (7), (8) and (9), we obtain some characterizations of $\Delta_\lambda(f)(x,y)$ when $\lambda \geqslant 0$.

The purpose of this article is to characterize expression $\Delta_\lambda(f)(x,y)$, by finding various estimates of it. We also give some characterizations of $\Delta_\lambda(f)(x,y)$ when function f is convex, which prove refinements of Young's inequality. Finally, choosing a particular case for a convex function f we give several inequalities in a normed space.

2. Main results

Next, we give some relations related to $\Delta_{\cdot}(\cdot)(\cdot,\cdot)$, relations necessary to prove some inequalities of the Young type. Let $\mathscr{X}$ be a convex subset of a real vector space.

LEMMA 1. *Let $f : \mathscr{X} \to \mathbb{R}$ be a function and $x,y \in \mathscr{X}$. If $\lambda \in \mathbb{R}$, then the following equalities hold:*

$$\Delta_\lambda(f)(x,y) = \Delta_{2\lambda}(f)\left(\frac{1}{2}(x+y),y\right) + 2\lambda\Delta_{1/2}(f)(x,y) \tag{10}$$

and

$$\Delta_\lambda(f)(x,y) = \Delta_{2\lambda-1}(f)\left(x,\frac{1}{2}(x+y)\right) + 2(1-\lambda)\Delta_{1/2}(f)(x,y). \tag{11}$$

Proof. Using the definition of $\Delta_\lambda(f)(x,y)$, by regrouping the terms, we obtain

$$\begin{aligned}
\Delta_{2\lambda}(f)\left(\frac{1}{2}(x+y),y\right) &= 2\lambda f\left(\frac{x+y}{2}\right) + (1-2\lambda)f(y) - f\left(2\lambda\frac{x+y}{2} + (1-2\lambda)y\right)\\
&= \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) - \lambda\left(f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right)\\
&= \Delta_\lambda(f)(x,y) - 2\lambda\Delta_{1/2}(f)(x,y),
\end{aligned}$$

which implies the first relation of the statement. In the same way, we have

$$\begin{aligned}
&\Delta_{2\lambda-1}(f)\left(x,\frac{1}{2}(x+y)\right)=(2\lambda-1)f(x)+(2-2\lambda)f\left(\frac{x+y}{2}\right)\\
&\qquad -f\left((2\lambda-1)x+(2-2\lambda)\frac{x+y}{2}\right)\\
&=\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)-(1-\lambda)\left(f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right)\\
&=\Delta_{\lambda}(f)(x,y)-2(1-\lambda)\Delta_{1/2}(f)(x,y),
\end{aligned}$$

which implies the second relation of the statement. □

If $\lambda\in[0,1]$ and $f:\mathscr{X}\to\mathbb{R}$ is a convex function, then it is easy to see that $\Delta_{\lambda}(f)(x,y)\geqslant 0$, for every $x,y\in\mathscr{X}$. Next, we study de case when $\lambda\notin(0,1)$.

LEMMA 2. *Let $f:\mathscr{X}\to\mathbb{R}$ be a convex function. If $\lambda\in\mathbb{R}-(0,1)$, then the following inequality holds:*

$$\Delta_{\lambda}(f)(x,y)\leqslant 0 \tag{12}$$

for all $x,y\in\mathscr{X}$.

Proof. We study two cases:
I) If $\lambda\leqslant 0$, then we obtain

$$\begin{aligned}
\Delta_{\lambda}(f)(x,y)&=\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)\\
&=-(-\lambda f(x)+f(\lambda x+(1-\lambda)y))+(1-\lambda)f(y)\\
&=-(1-\lambda)\left(\frac{-\lambda}{1-\lambda}f(x)+\frac{1}{1-\lambda}f(\lambda x+(1-\lambda)y)-f(y)\right)\leqslant 0.
\end{aligned}$$

II) If $\lambda\geqslant 1$, then, using the triangle inequality, we deduce

$$\begin{aligned}
\Delta_{\lambda}(f)(x,y)&=\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)t\\
&=-(-(1-\lambda)f(y)+f(\lambda x+(1-\lambda)y)-\lambda f(x))\\
&=-\lambda\left(\frac{-(1-\lambda)}{\lambda}f(y)+\frac{1}{\lambda}f(\lambda x+(1-\lambda)y)-f(x)\right)\leqslant 0.
\end{aligned}$$

Therefore, the inequality of the statement is true. □

PROPOSITION 1. *Let $f:\mathscr{X}\to\mathbb{R}$ be a convex function. If $\lambda\in[0,1]$, then the following inequality holds:*

$$2\min\{\lambda,1-\lambda\}\Delta_{1/2}(f)(x,y)\leqslant\Delta_{\lambda}(f)(x,y)\leqslant 2\max\{\lambda,1-\lambda\}\Delta_{1/2}(f)(x,y) \tag{13}$$

for all $x,y\in\mathscr{X}$.

Proof. For $\lambda \in [0, \frac{1}{2}]$, we have $2\lambda \in [0,1]$ and $2\lambda - 1 \in [-1,0]$, so, we show that $\Delta_{2\lambda}(f)\left(\frac{1}{2}(x+y), y\right) \geqslant 0$ and using Lemma 2 we have inequality $\Delta_{2\lambda-1}(f)\left(x, \frac{1}{2}(x+y)\right) \leqslant 0$. From equalities (10) and (11), we obtain

$$2\lambda \Delta_{1/2}(f)(x,y) \leqslant \Delta_\lambda(f)(x,y) \leqslant 2(1-\lambda)\Delta_{1/2}(f)(x,y). \tag{14}$$

For $\lambda \in [\frac{1}{2}, 1]$, in the same way, we prove that

$$2(1-\lambda)\Delta_{1/2}(f)(x,y) \leqslant \Delta_\lambda(f)(x,y) \leqslant 2\lambda \Delta_{1/2}(f)(x,y). \tag{15}$$

Consequently, combining inequalities (14) and (15) the inequality of the statement is true. □

REMARK 1. Therefore, Proposition (1) in this paper is relation (9) for $n = 2$, which has been established in [8]. But, the proof is different and we keep it as alternative proof.

In inequality (13) for the convex function $f : [x,y] \to \mathbb{R}$ with $f(t) = -\log t$, we deduce the following inequality [12]:

$$1 \leqslant \left(\frac{x+y}{2\sqrt{xy}}\right)^{2\min\{\lambda,1-\lambda\}} \leqslant \frac{\lambda x + (1-\lambda)y}{x^\lambda y^{1-\lambda}} \leqslant \left(\frac{x+y}{2\sqrt{xy}}\right)^{2\max\{\lambda,1-\lambda\}} \tag{16}$$

for all $x, y > 0$ and $\lambda \in [0,1]$ (see also [8]). This inequality represents a refinement of Young's inequality. Inequality (16) can be presented with Kantorovich constant, thus

$$K^{\min\{\lambda,1-\lambda\}}(h,2)x^\lambda y^{1-\lambda} \leqslant \lambda x + (1-\lambda)y \leqslant K^{\max\{\lambda,1-\lambda\}}(h,2)x^\lambda y^{1-\lambda}, \tag{17}$$

where $x, y > 0$, $\lambda \in [0,1]$, $K(h,2) = \dfrac{(h+1)^2}{4h}$ and $h = \dfrac{y}{x}$. Notice that the first inequality in (17) was obtained by Zou et al. in [20] while the second was obtained by Liao et al. [11].

If $\lambda \in (0,1)$, then inequality (14) can be written, for a nondegenerate interval I, as

$$0 \leqslant \frac{\Delta_\lambda(f)(x,y)}{2\max\{\lambda, 1-\lambda\}} \leqslant \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \leqslant \frac{\Delta_\lambda(f)(x,y)}{2\min\{\lambda,1-\lambda\}} \tag{18}$$

for all $x, y \in I$. For a convex function $f : I \to \mathbb{R}_+$, with $\lambda = \frac{f(y)}{f(x)+f(y)}$, inequality (18) becomes:

$$\begin{aligned} 0 &\leqslant \min\{f(x), f(y)\}\left(1 - \frac{f(x)+f(y)}{2f(x)f(y)} f\left(\frac{yf(x)+xf(y)}{f(x)+f(y)}\right)\right) \\ &\leqslant \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \\ &\leqslant \max\{f(x), f(y)\}\left(1 - \frac{f(x)+f(y)}{2f(x)f(y)} f\left(\frac{yf(x)+xf(y)}{f(x)+f(y)}\right)\right) \end{aligned} \tag{19}$$

for all $x,y \in I$ with $f(x) \neq 0$ and $f(y) \neq 0$.

Inequality (19) can be rewritten as

$$\frac{2f\left(\frac{x+y}{2}\right) - |f(x)-f(y)|}{\min\{f(x),f(y)\}}$$

$$\leqslant \frac{f(x)+f(y)}{f(x)f(y)} f\left(\frac{yf(x)+xf(y)}{f(x)+f(y)}\right) \leqslant \frac{2f\left(\frac{x+y}{2}\right) + |f(x)-f(y)|}{\max\{f(x),f(y)\}} \tag{20}$$

for every $x,y \in I$ with $f(x) \neq 0$ and $f(y) \neq 0$.

In general, for $a,b > 0$ and using relation (13), for $\lambda = \frac{a}{a+b}$, we find the following inequality:

$$\frac{2\min\{a,b\}}{a+b}\Delta_{1/2}(f)(x,y) \leqslant \Delta_{\frac{a}{a+b}}(f)(x,y) \leqslant \frac{2\max\{a,b\}}{a+b}\Delta_{1/2}(f)(x,y) \tag{21}$$

for all $x,y \in \mathscr{X}$, which can be rewritten as

$$2\min\{a,b\}\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right) \leqslant af(x)+bf(y)-(a+b)f\left(\frac{ax+by}{a+b}\right)$$

$$\leqslant 2\max\{a,b\}\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right) \tag{22}$$

for all numbers x and y in $\mathscr{X}$ and $a,b \in \mathbb{R}_+$. In addition, (22) are also proved in [8].

Let I be a nondegenerate interval of $\mathbb{R}$. For $a+b=1$ and $\mathscr{X} = I$, this inequality is given by Mitroi [15], as a particular case of the Dragomir inequality [1].

Therefore, our interest is to refine inequality (22), which can be obtained from inequality (13).

THEOREM 1. *Suppose that $f : \mathscr{X} \to \mathbb{R}$ is a convex function. If $\lambda \in [0, \frac{1}{2}]$, then the following inequality holds:*

$$2\lambda\Delta_{1/2}(f)(x,y) + 2\min\{2\lambda, 1-2\lambda\}\Delta_{1/2}(f)\left(\frac{1}{2}(x+y), y\right) \leqslant \Delta_\lambda(f)(x,y)$$

$$\leqslant 2\lambda\Delta_{1/2}(f)(x,y) + 2\max\{2\lambda, 1-2\lambda\}\Delta_{1/2}(f)\left(\frac{1}{2}(x+y), y\right) \tag{23}$$

and if $\lambda \in [\frac{1}{2}, 1]$, then the inequality

$$2(1-\lambda)\Delta_{1/2}(f)(x,y) + 2\min\{2\lambda-1, 2-2\lambda\}\Delta_{1/2}(f)\left(x, \frac{1}{2}(x+y)\right) \leqslant \Delta_\lambda(f)(x,y)$$

$$\leqslant 2(1-\lambda)\Delta_{1/2}(f)(x,y) + 2\max\{2\lambda-1, 2-2\lambda\}\Delta_{1/2}(f)\left(x, \frac{1}{2}(x+y)\right) \tag{24}$$

holds.

Proof. For $\lambda \in [0,\frac{1}{2}]$, we have $2\lambda \in [0,1]$ and replacing x by $\frac{1}{2}(x+y)$ in inequality (13), we deduce

$$2\min\{2\lambda,1-2\lambda\}\Delta_{1/2}(f)\left(\frac{1}{2}(x+y),y\right) \leqslant \Delta_{2\lambda}(f)\left(\frac{1}{2}(x+y),y\right)$$
$$\leqslant 2\max\{2\lambda,1-2\lambda\}\Delta_{1/2}(f)\left(\frac{1}{2}(x+y),y\right). \tag{25}$$

Consequently, combining equality (10) with inequality (25), we show the first inequality of the statement. For $\lambda \in [\frac{1}{2},1]$, we have $2\lambda-1 \in [0,1]$ and replacing y by $\frac{1}{2}(x+y)$ in inequality (13), we deduce

$$2\min\{2\lambda-1,2-2\lambda\}\Delta_{1/2}(f)\left(x,\frac{1}{2}(x+y)\right) \leqslant \Delta_{2\lambda-1}(f)\left(x,\frac{1}{2}(x+y)\right)$$
$$\leqslant 2\max\{2\lambda-1,2-2\lambda\}\Delta_{1/2}(f)\left(x,\frac{1}{2}(x+y)\right). \tag{26}$$

Consequently, combining equality (11) with inequality (26) we prove the second inequality of the statement. $\square$

REMARK 2. In inequality (23) for the convex function $f:(0,\infty)\to\mathbb{R}$ with $f(t)=t^p$, where $p\in(-\infty,0]\cup[1,\infty)$, we obtain the following inequality:

$$\min\{\lambda,1-\lambda\}\left(x^p+y^p-2^{1-p}(x+y)^p\right) \leqslant \lambda x^p+(1-\lambda)y^p-(\lambda x+(1-\lambda)y)^p$$
$$\leqslant \max\{\lambda,1-\lambda\}\left(x^p+y^p-2^{1-p}(x+y)^p\right) \tag{27}$$

for all $x,y>0$ and $\lambda\in[0,1]$.

In inequality (23) for the convex function $f:(0,\infty)\to\mathbb{R}$ with $f(t)=-\log t$, we deduce the following inequality:

$$1 \leqslant \left(\frac{x+y}{2\sqrt{xy}}\right)^{2\lambda}\left(\frac{x+3y}{2\sqrt{2y(x+y)}}\right)^{2\min\{2\lambda,1-2\lambda\}} \leqslant \frac{\lambda x+(1-\lambda)y}{x^\lambda y^{1-\lambda}}$$
$$\leqslant \left(\frac{x+y}{2\sqrt{xy}}\right)^{2\lambda}\left(\frac{x+3y}{2\sqrt{2y(x+y)}}\right)^{2\max\{2\lambda,1-2\lambda\}} \tag{28}$$

for all $x,y>0$ and $\lambda\in[0,\frac{1}{2}]$. This inequality represents an improvement of Young's inequality, which refines inequality (16).

We are studying the problem of comparing the upper bound from inequality (13) with the upper bounds from inequalities (23) and (24) to see which is better. For $\lambda\in[0,\frac{1}{4}]$, by simple calculations, we prove the inequality

$$2\lambda\Delta_{1/2}(f)(x,y)+2\max\{2\lambda,1-2\lambda\}\Delta_{1/2}(f)\left(\frac{1}{2}(x+y),y\right) \leqslant 2(1-\lambda)\Delta_{1/2}(f)(x,y). \tag{29}$$

Therefore, for $\lambda \in [0, \frac{1}{4}]$ the upper bound from inequality (23) is better. But, for $\lambda \in [\frac{1}{4}, \frac{1}{2}]$ inequality (24) becomes

$$2(1-\lambda)f\left(\frac{x+y}{2}\right) + (4\lambda-1)f(y) \leqslant (1-2\lambda)f(x) + 4\lambda f\left(\frac{x+3y}{4}\right),$$

which is true for $x=-y$ and false for $x=-3y$, when $0 \in \mathscr{X}$ and $f(0)=0$. For $\lambda \in [\frac{3}{4}, 1]$, by simple calculations, we prove the inequality

$$2(1-\lambda)\Delta_{1/2}(x,y) + 2\max\{2\lambda-1, 2-2\lambda\}\Delta_{1/2}\left(x, \frac{1}{2}(x+y)\right) \leqslant 2\lambda\Delta_{1/2}(x,y). \quad (30)$$

Consequently, for $\lambda \in [\frac{3}{4}, 1]$ the upper bound from inequality (24) is better. But, for $\lambda \in [\frac{1}{2}, \frac{3}{4}]$ inequality (30) becomes

$$2\lambda f\left(\frac{x+y}{2}\right) + (3-4\lambda)f(x) \leqslant (2\lambda-1)f(y) + 4(1-\lambda)f\left(\frac{3x+y}{4}\right),$$

which is true for $y=-x$ and false for $y=-3x$, when $0 \in \mathscr{X}$ and $f(0)=0$.

We choose two real numbers a and b such that $0 < a \leqslant b$, if we use relation (23), for $\lambda = \dfrac{a}{a+b} \leqslant \frac{1}{2}$, then we obtain the following inequality:

$$\begin{aligned}
2\frac{a}{a+b}\Delta_{1/2}(f)(x,y) + \frac{2\min\{2a, b-a\}}{a+b}\Delta_{1/2}(f)\left(\frac{1}{2}(x+y), y\right) &\leqslant \Delta_{\frac{a}{a+b}}(f)(x,y) \\
\leqslant 2\frac{a}{a+b}\Delta_{1/2}(f)(x,y) + \frac{2\max\{2a, b-a\}}{a+b}\Delta_{1/2}(f)\left(\frac{1}{2}(x+y), y\right) & \qquad (31)
\end{aligned}$$

for all $x, y \in \mathscr{X}$, which can be rewritten as

$$\begin{aligned}
a\left(f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right) + \min\{2a, b-a\}\left(f\left(\frac{x+y}{2}\right)+f(y)-2f\left(\frac{x+3y}{4}\right)\right) \\
\leqslant af(x)+bf(y)-(a+b)f\left(\frac{ax+by}{a+b}\right) \\
\leqslant a\left(f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right) \\
+\max\{2a, b-a\}\left(f\left(\frac{x+y}{2}\right)+f(y)-2f\left(\frac{x+3y}{4}\right)\right) \qquad (32)
\end{aligned}$$

for every x and y in $\mathscr{X}$ and $a, b \in \mathbb{R}_+$, $a \leqslant b$. This inequality refined the first part of inequality (22).

A generalization of the equalities from Lemma 1 is given below:

THEOREM 2. *Let $f : \mathscr{X} \to \mathbb{R}$ be a function and the natural number $n \geqslant 1$. If $\lambda \in \mathbb{R}$, then the following equalities hold:*

$$\Delta_\lambda(f)(x,y) = \lambda \sum_{k=1}^{n} 2^k \Delta_{1/2}(f)\left(\frac{1}{2^{k-1}}x + \left(1 - \frac{1}{2^{k-1}}\right)y, y\right) + \Delta_{2^n\lambda}(f)\left(\frac{1}{2^n}x + \left(1 - \frac{1}{2^n}\right)y, y\right) \tag{33}$$

and

$$\Delta_\lambda(f)(x,y) = (1-\lambda) \sum_{k=1}^{n} 2^k \Delta_{1/2}(f)\left(x, \left(1 - \frac{1}{2^{k-1}}\right)x + \frac{1}{2^{k-1}}y\right) + \Delta_{2^n\lambda+1-2^n}(f)\left(x, \left(1 - \frac{1}{2^n}\right)x + \frac{1}{2^n}y\right) \tag{34}$$

for all $x, y \in \mathscr{X}$.

Proof. Using Lemma 1 for $\lambda \in \mathbb{R}$, we have

$$\Delta_\lambda(f)(x,y) = \Delta_{2\lambda}(f)\left(\frac{1}{2}(x+y), y\right) + 2\lambda\Delta_{1/2}(f)(x,y).$$

We replace λ by 2λ and x by $\frac{1}{2}(x+y)$, in the above equality and we get

$$\Delta_{2\lambda}(f)\left(\frac{1}{2}(x+y), y\right) = \Delta_{2^2\lambda}(f)\left(\frac{1}{2}\left(\frac{1}{2}(x+y)+y\right), y\right) + 2^2\lambda\Delta_{1/2}(f)\left(\frac{1}{2}(x+y), y\right).$$

If we inductively repeat the above substitutions, for $k \geqslant 1$, then we have

$$\Delta_{2^{k-1}\lambda}(f)\left(\frac{1}{2^{k-1}}x + \left(1 - \frac{1}{2^{k-1}}\right)y, y\right) = \Delta_{2^k\lambda}(f)\left(\frac{1}{2^k}x + \left(1 - \frac{1}{2^k}\right)y, y\right)$$

$$+ 2^k\lambda\Delta_{1/2}(f)\left(\frac{1}{2^{k-1}}x + \left(1 - \frac{1}{2^{k-1}}\right)y, y\right).$$

Therefore, summarizing the above relations for $k \in \{1, \ldots, n\}$, we obtain the relation of the statement. Applying equality (33) and taking into account that $\Delta_\lambda(f)(x,y) = \Delta_{1-\lambda}(f)(y,x)$, we deduce equality (34). $\square$

These equalities offer the possibility to refine inequalities (23) and (24), thus

THEOREM 3. *Let $f:\mathscr{X}\to\mathbb{R}$ be a convex function and n a natural number, $n\geqslant 1$. If $\lambda\in\left[0,\frac{1}{2^n}\right]$, then the following inequality holds:*

$$\begin{aligned}
&\lambda\sum_{k=1}^{n}2^k\Delta_{1/2}(f)\left(\frac{1}{2^{k-1}}x+\left(1-\frac{1}{2^{k-1}}\right)y,y\right)\\
&+2\min\{2^n\lambda,1-2^n\lambda\}\Delta_{1/2}(f)\left(\frac{1}{2^n}x+\left(1-\frac{1}{2^n}\right)y,y\right)\leqslant\Delta_\lambda(f)(x,y)\\
&\leqslant\lambda\sum_{k=1}^{n}2^k\Delta_{1/2}(f)\left(\frac{1}{2^{k-1}}x+\left(1-\frac{1}{2^{k-1}}\right)y,y\right)\\
&+2\max\{2^n\lambda,1-2^n\lambda\}\Delta_{1/2}(f)\left(\frac{1}{2^n}x+\left(1-\frac{1}{2^n}\right)y,y\right)
\end{aligned}\tag{35}$$

and if $\lambda\in\left[1-\frac{1}{2^n},1\right]$, then the following inequality holds:

$$\begin{aligned}
&(1-\lambda)\sum_{k=1}^{n}2^k\Delta_{1/2}(f)\left(x,\left(1-\frac{1}{2^{k-1}}\right)x+\frac{1}{2^{k-1}}y\right)\\
&+2\min\{\lambda',1-\lambda'\}\Delta_{1/2}(f)\left(x,\left(1-\frac{1}{2^n}\right)x+\frac{1}{2^n}y\right)\leqslant\Delta_\lambda(f)(x,y)\\
&\leqslant(1-\lambda)\sum_{k=1}^{n}2^k\Delta_{1/2}(f)\left(x,\left(1-\frac{1}{2^{k-1}}\right)x+\frac{1}{2^{k-1}}y\right)\\
&+2\max\{\lambda',1-\lambda'\}\Delta_{1/2}(f)\left(x,\left(1-\frac{1}{2^n}\right)x+\frac{1}{2^n}y\right),
\end{aligned}\tag{36}$$

where $\lambda'=2^n\lambda+1-2^n$ and $x,y\in\mathscr{X}$.

Proof. Using the inequalities from Proposition 1 and combining with equalities (33) and (34), we deduce that the inequalities of the statement are true. □

For a real normed space $\mathscr{X}=(\mathscr{X},\|\cdot\|)$, function $f(x)=\|x\|^r$ ($x\in\mathscr{X}$ and $1\leqslant r<\infty$) is a convex function. Therefore, we obtain

$$\Delta_\lambda(f)(x,y)=\lambda\|x\|^r+(1-\lambda)\|y\|^r-\|\lambda x+(1-\lambda)\|^r,$$

where $x,y\in\mathscr{X}$, $r\geqslant 1$ and $0\leqslant\lambda\leqslant 1$.

For $r=1$ in the above equality, we find $\Delta_\lambda(f)(x,y)=\lambda\|x\|+(1-\lambda)\|y\|-\|\lambda x+(1-\lambda)y\|$, where $x,y\in\mathscr{X}$ and $0\leqslant\lambda\leqslant 1$, which in fact is the expression of $d_\lambda(x,y)$ from [18].

In inequality (23) for the convex function $f(x)=\|x\|^r$, where $r\in[1,\infty)$, we obtain the following inequality:

$$\begin{aligned}
\min\{\lambda,1-\lambda\}\left(\|x\|^r+\|y\|^r-2^{1-r}\|x+y\|^r\right)&\leqslant\lambda\|x\|^r+(1-\lambda)\|y\|^r-\|\lambda x+(1-\lambda)y\|^r\\
&\leqslant\max\{\lambda,1-\lambda\}\left(\|x\|^r+\|y\|^r-2^{1-r}\|x+y\|^r\right)
\end{aligned}\tag{37}$$

for all $x,y \in \mathscr{X}$ and $\lambda \in [0,1]$. If we replace y by $-y$ in inequality (2) we find inequality

$$\min\{\lambda,1-\lambda\}\left(\|x\|^r+\|y\|^r-2^{1-r}\|x-y\|^r\right) \leqslant \lambda\|x\|^r+(1-\lambda)\|y\|^r-\|\lambda x-(1-\lambda)y\|^r$$
$$\leqslant \max\{\lambda,1-\lambda\}\left(\|x\|^r+\|y\|^r-2^{1-r}\|x-y\|^r\right) \tag{38}$$

for all $x,y \in \mathscr{X}$ and $\lambda \in [0,1]$.

Let p be a real number such that $p>0$ and we take $\lambda = \frac{\|x\|^{p-1}}{\|x\|^{p-1}+\|y\|^{p-1}}$ in inequality (38), then we have the following inequality:

$$\min{}^{p-1}\{\|x\|,\|y\|\}\left(\|x\|^r+\|y\|^r-2^{1-r}\|x-y\|^r\right)$$
$$\leqslant \|x\|^{p+r-1}+\|y\|^{p+r-1}-\left(\|x\|^{p-1}+\|y\|^{p-1}\right)^{1-r}\left\|\|x\|^{p-1}x-\|y\|^{p-1}y\right\|^r$$
$$\leqslant \max{}^{p-1}\{\|x\|,\|y\|\}\left(\|x\|^r+\|y\|^r-2^{1-r}\|x-y\|^r\right) \tag{39}$$

for all nonzero vectors $x,y \in \mathscr{X}$, $r \geqslant 1$ and $p>0$.

By replacing parameter λ with various values or choosing various particular cases of the convex function f, we obtain other applications for $\Delta_\lambda(f)(x,y)$, where $x,y \in \mathscr{X}$. It remains for the reader to find other estimates of the expression $\Delta_\lambda(f)(x,y)$, where $x,y \in \mathscr{X}$ and $\lambda \in \mathbb{R}$.

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