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FURTHER REFINEMENTS OF THE TAN–XIE INEQUALITY FOR SECTOR MATRICES AND ITS APPLICATIONS

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Abstract. In this paper, we present some further refinements of the Tan-Xie inequality for sector matrices and its applications due to Nasiri and Furuichi [J. Math. Inequal., 15 (2021), 1425–1434].

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n\times n$ complex matrices and A^* denote the conjugate transpose of A. The matrix $A\in\mathbb{M}_n(\mathbb{C})$ is called accretive if $\Re A$ is positive definite, and accretive-dissipative matrix if both $\Re A$ and $\Im A$ are positive definite, where $\Re A=\frac{1}{2}(A+A^*)$ and $\Im A=\frac{1}{2i}(A-A^*)$ are called the real part and imaginary part of A, respectively ([2, p. 6]). For two Hermitian matrices $A,B\in\mathbb{M}_n(\mathbb{C})$, $A\geqslant B$ means that $A-B$ is positive semi-definite. In addition, a norm $\|\cdot\|$ on $\mathbb{M}_n(\mathbb{C})$ is unitarily invariant if $\|UAV\|=\|A\|$ for any $A\in\mathbb{M}_n(\mathbb{C})$ and all unitarily matrices $U,V\in\mathbb{M}_n(\mathbb{C})$.

Recall that the numerical range of $A\in\mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A)=\{x^*Ax : x\in\mathbb{C}^n,\ x^*x=1\}.$$

S_α denotes the sector region in the complex plane as follows

$$S_\alpha=\{z\in\mathbb{C} : \Re z>0,\ |\Im z|\leqslant(\Re z)\tan\alpha\}$$

for $\alpha\in[0,\frac{\pi}{2})$. It is clearly that $W(A)\subseteq S_0$ means A is positive definite. And if $W(A),W(B)\subseteq S_\alpha$ for some $\alpha\in[0,\frac{\pi}{2})$, then $W(A+B)\subseteq S_\alpha$. Very recently, Nasiri and Furuichi [7] showed that $W(A)\subseteq S_\alpha$ implies $W(A^{-1})\subseteq S_\alpha$. We denote $A\in\mathbb{M}_n(\mathbb{C})$ with $W(A)\subset S_\alpha$ for $\alpha\in[0,\frac{\pi}{2})$ by $A\in S_\alpha$ for our convenience.

If $A,B\in\mathbb{M}_n(\mathbb{C})$ are positive definite, then the weighted Arithmetic-Geometric-Harmonic (AM-GM-HM) means are defined as

$$A\nabla_\nu B=(1-\nu)A+\nu B,\quad A\sharp_\nu B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}$$

and

$$A!_\nu B=((1-\nu)A^{-1}+\nu B^{-1})^{-1}$$

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for $v \in [0,1]$, denoted by $A\nabla B$, $A\sharp B$ and $A!B$ for brevity when $v = \frac{1}{2}$, respectively. Besides, we default the Kantorovich constant to $K(h) = \frac{(h+1)^2}{4h}$ for $h := \frac{M}{m} \geqslant 1$ with $0 < m \leqslant M$ if there is no special explanation in the rest of this paper.

A linear map $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is called positive if it maps positive definite matrices into positive difinite matrices and is said to be unital if it maps identity matrices to identity matrices. Recently, Tan and Chen [9] proved that for any positive linear map Φ, $A \in S_\alpha$ implies $\Phi(A) \in S_\alpha$ and $\Re\Phi(A) = \Phi(\Re A)$.

The famous Choi's inequality [1, p. 41] involving a positive unital linear map Φ and a positive definite $A \in \mathbb{M}_n(\mathbb{C})$ reads

$$\Phi^{-1}(A) \leqslant \Phi(A^{-1}). \tag{1.1}$$

In 2020, Tan and Xie [10] obtained the following AM-GM-HM means inequalities:

$$\cos^2(\alpha)\Re(A!_vB) \leqslant \Re(A\sharp_vB) \leqslant \sec^2(\alpha)\Re(A\nabla_vB), \tag{1.2}$$

where $A, B \in S_\alpha$ and $v \in [0,1]$.

In 2021, Nasiri and Furuichi [7] present a reverse of the double inequality (1.2) involving positive linear maps as follows:

THEOREM 1. *Let $A, B \in S_\alpha$ and $v \in [0,1]$. Then for every positive unital linear map Φ, we have the following*

(i) if $0 < mI_n \leqslant \Re(A),\ \Re(B) \leqslant MI_n$, then

$$K^{-2}(h)\cos^8(\alpha)\Phi^2(\Re(A\nabla_vB)) \leqslant \Phi^2(\Re(A\sharp_vB)). \tag{1.3}$$

(ii) if $0 < mI_n \leqslant \Re(A^{-1}), \Re(B^{-1}) \leqslant MI_n$, then

$$\Phi^2(\Re(A\sharp_vB)) \leqslant \sec^8(\alpha)K^2(h)\Phi^2(\Re(A!_vB)). \tag{1.4}$$

In this paper, we try to give some generalizations and further refinements of Theorem 1. As applications, we obtain some inequalities for determinant, singular and unitarily invariant norm.

2. Main results

Firstly, we give further refinements of inequality (1.3).

LEMMA 1. ([3]) *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive definite. Then*

$$\|AB\| \leqslant \frac{1}{4}\|A+B\|^2.$$

LEMMA 2. ([8]) *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be accretive and let $v \in [0,1]$. Then*

$$\Re A\sharp_v\Re B \leqslant \Re(A\sharp_vB).$$

THEOREM 2. *If $A,B \in S_\alpha$, $0 < mI_n \leqslant \Re(A)$, $\Re(B) \leqslant MI_n$ and $v \in [0,1]$, then for every positive unital linear map Φ, we have*

$$\Phi^2\left(\Re\left(A\nabla_v B\right)\right) \leqslant K^2(h)\Phi^2(\Re(A\sharp_v B)). \tag{2.1}$$

Proof. Under the conditions, we have

$$(MI_n - \Re(A))(mI_n - \Re(A))\Re^{-1}(A) \leqslant 0,$$

and

$$(MI_n - \Re(B))(mI_n - \Re(B))\Re^{-1}(B) \leqslant 0,$$

we obtain

$$\Re(A\nabla_v B) + Mm(\Re^{-1}(A)\nabla_v \Re^{-1}(B)) \leqslant (M+m)I_n. \tag{2.2}$$

Inequality (2.1) is equivalent to

$$\left\|\Phi\left(\Re\left(A\nabla_v B\right)\right)\Phi^{-1}(\Re(A\sharp_v B))\right\| \leqslant K(h).$$

By computations, we have

$$\begin{aligned}
&\left\|Mm\Phi\left(\Re\left(A\nabla_v B\right)\right)\Phi^{-1}(\Re(A\sharp_v B))\right\| \\
&\leqslant \frac{1}{4}\left\|\Phi\left(\Re\left(A\nabla_v B\right)\right) + Mm\Phi^{-1}(\Re(A\sharp_v B))\right\|^2 \quad \text{(by Lemma 1)} \\
&\leqslant \frac{1}{4}\left\|\Phi\left(\Re\left(A\nabla_v B\right)\right) + Mm\Phi(\Re^{-1}(A\sharp_v B))\right\|^2 \quad \text{(by (1.1))} \\
&\leqslant \frac{1}{4}\left\|\Phi\left(\Re\left(A\nabla_v B\right)\right) + Mm\Phi((\Re(A)\sharp_v \Re(B))^{-1})\right\|^2 \quad \text{(by Lemma 2)} \\
&= \frac{1}{4}\left\|\Phi\left(\Re\left(A\nabla_v B\right)\right) + Mm\Phi(\Re^{-1}(A)\sharp_v \Re^{-1}(B))\right\|^2 \\
&\leqslant \frac{1}{4}\left\|\Phi\left(\Re\left(A\nabla_v B\right)\right) + Mm\Phi\left(\Re^{-1}(A)\nabla_v \Re^{-1}(B)\right)\right\|^2 \quad \text{(by AM} - \text{GM inequality)} \\
&\leqslant \frac{1}{4}(M+m)^2. \quad \text{(by (2.2))}
\end{aligned}$$

This completes the proof. □

Next, we give a generalization of Theorem 2.

LEMMA 3. ([1]) *Let $A,B \in \mathbb{M}_n(\mathbb{C})$ be positive definite. Then for $1 \leqslant r < +\infty$,*

$$\|A^r + B^r\| \leqslant \|(A+B)^r\|.$$

THEOREM 3. *If $A,B\in S_\alpha$, $0<mI_n\leqslant \Re(A)$, $\Re(B)\leqslant MI_n$, $1<\beta\leqslant 2$, $p\geqslant 2\beta$ and $v\in[0,1]$, then for every positive unital linear map Φ, we have*

$$\Phi^p\left(\Re\left(A\nabla_v B\right)\right)\leqslant \frac{\left(K^{\frac{\beta}{2}}(h)(M^\beta+m^\beta)\right)^{\frac{2p}{\beta}}}{16M^pm^p}\Phi^p\left(\Re(A\sharp_v B)\right). \tag{2.3}$$

Proof. Since

$$mI_n\leqslant\Phi\left((1-v)\Re(A)+v\Re(B)\right)=\Phi\left(\Re\left(A\nabla_v B\right)\right)\leqslant MI_n,$$

we have

$$M^\beta m^\beta\Phi^{-\beta}\left(\Re\left(A\nabla_v B\right)\right)+\Phi^\beta\left(\Re\left(A\nabla_v B\right)\right)\leqslant\left(M^\beta+m^\beta\right)I_n. \tag{2.4}$$

By (2.1) and the famous L-H inequality, we get

$$\Phi^{-\beta}\left(\Re(A\sharp_v B)\right)\leqslant K^\beta(h)\Phi^{-\beta}\left(\Re\left(A\nabla_v B\right)\right). \tag{2.5}$$

By computation, we have

$$\begin{aligned}
&\left\|M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{\frac{p}{2}}\left(\Re\left(A\nabla_v B\right)\right)\Phi^{-\frac{p}{2}}\left(\Re(A\sharp_v B)\right)\right\|\\
&\leqslant\frac{1}{4}\left\|K^{\frac{p}{4}}(h)\Phi^{\frac{p}{2}}\left(\Re\left(A\nabla_v B\right)\right)+\left(\frac{M^2m^2}{K(h)}\right)^{\frac{p}{4}}\Phi^{-\frac{p}{2}}\left(\Re(A\sharp_v B)\right)\right\|^2 \quad \text{(by Lemma 1)}\\
&\leqslant\frac{1}{4}\left\|K^{\frac{\beta}{2}}(h)\Phi^{\beta}\left(\Re\left(A\nabla_v B\right)\right)+\left(\frac{M^2m^2}{K(h)}\right)^{\frac{\beta}{2}}\Phi^{-\beta}\left(\Re(A\sharp_v B)\right)\right\|^{\frac{p}{\beta}} \quad \text{(by Lemma 3)}\\
&\leqslant\frac{1}{4}\left\|K^{\frac{\beta}{2}}(h)\Phi^{\beta}\left(\Re\left(A\nabla_v B\right)\right)+K^{\frac{\beta}{2}}(h)M^\beta m^\beta\Phi^{-\beta}\left(\Re\left(A\nabla_v B\right)\right)\right\|^{\frac{p}{\beta}} \quad \text{(by (2.5))}\\
&=\frac{1}{4}\left\|K^{\frac{\beta}{2}}(h)\left(\Phi^{\beta}\left(\Re\left(A\nabla_v B\right)\right)+M^\beta m^\beta\Phi^{-\beta}\left(\Re\left(A\nabla_v B\right)\right)\right)\right\|^{\frac{p}{\beta}}\\
&\leqslant\frac{1}{4}\left(K^{\frac{\beta}{2}}(h)(M^\beta+m^\beta)\right)^{\frac{p}{\beta}}. \quad \text{(by (2.4))}
\end{aligned}$$

That is,

$$\left\|\Phi^{\frac{p}{2}}\left(\Re\left(A\nabla_v B\right)\right)\Phi^{-\frac{p}{2}}\left(\Re(A\sharp_v B)\right)\right\|\leqslant\frac{\left(K^{\frac{\beta}{2}}(h)(M^\beta+m^\beta)\right)^{\frac{p}{\beta}}}{4M^{\frac{p}{2}}m^{\frac{p}{2}}},$$

which is equivalent to inequality (2.3). □

The following theorem explains that the factor in inequality (1.4) could be $\sec^4(\alpha)K^2(h)$ under some conditions.

LEMMA 4. ([8]) *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be accretive and $v\in[0,1]$. Then*

$$(\Re A)!_v(\Re B)\leqslant\Re(A!_vB).$$

THEOREM 4. *Let* $v \in [0,1]$, $A, B \in S_\alpha$ *and* $0 < mI_n \leqslant \mathfrak{R}^{-1}(A)$, $\mathfrak{R}^{-1}(B) \leqslant MI_n$. *Then for every positive unital linear map* Φ, *we have*

$$\Phi^2(\mathfrak{R}(A\sharp_v B)) \leqslant (\sec^2(\alpha)K(h))^2\Phi^2\left(\mathfrak{R}\left(A!_v B\right)\right). \tag{2.6}$$

Proof. Under the conditions, we can get

$$\mathfrak{R}^{-1}(A)\nabla_v\mathfrak{R}^{-1}(B) + Mm\mathfrak{R}\left(A\nabla_v B\right) \leqslant (M+m)I_n. \tag{2.7}$$

By computation, we have

$$\begin{aligned}
&\left\| \sec^2(\alpha)Mm\Phi(\mathfrak{R}(A\sharp_v B))\Phi^{-1}\left(\mathfrak{R}\left(A!_v B\right)\right)\right\| \\
&\leqslant \frac{1}{4}\left\| Mm\Phi(\mathfrak{R}(A\sharp_v B)) + \sec^2(\alpha)\Phi^{-1}\left(\mathfrak{R}\left(A!_v B\right)\right)\right\|^2 \quad \text{(by Lemma 1)} \\
&\leqslant \frac{1}{4}\left\| Mm\Phi(\mathfrak{R}(A\sharp_v B)) + \sec^2(\alpha)\Phi\left(\mathfrak{R}^{-1}\left(A!_v B\right)\right)\right\|^2 \quad \text{(by (1.1))} \\
&\leqslant \frac{1}{4}\left\| Mm\Phi(\mathfrak{R}(A\sharp_v B)) + \sec^2(\alpha)\Phi\left(\mathfrak{R}^{-1}(A)\nabla_v\mathfrak{R}^{-1}(B)\right)\right\|^2 \quad \text{(by Lemma 4)} \\
&\leqslant \frac{1}{4}\left\| \sec^2(\alpha)Mm\Phi\left(\mathfrak{R}\left(A\nabla_v B\right)\right) + \sec^2(\alpha)\Phi\left(\mathfrak{R}^{-1}(A)\nabla_v\mathfrak{R}^{-1}(B)\right)\right\|^2 \quad \text{(by (1.2))} \\
&= \frac{1}{4}\left\| \sec^2(\alpha)\Phi\left(Mm\mathfrak{R}\left(A\nabla_v B\right) + \mathfrak{R}^{-1}(A)\nabla_v\mathfrak{R}^{-1}(B)\right)\right\|^2 \\
&\leqslant \frac{1}{4}\sec^4(\alpha)(M+m)^2. \quad \text{(by (2.7))}
\end{aligned}$$

That is,

$$\left\|\Phi(\mathfrak{R}(A\sharp_v B))\Phi^{-1}\left(\mathfrak{R}\left(A!_v B\right)\right)\right\| \leqslant \sec^2(\alpha)K(h).$$

This complete the proof. □

It is natural to ask whether (2.6) can be generalized following the line of (2.3). However, we don't have a satisfactory answer to these questions for the time being.

Next, we give some inequalities for determinant, singular and unitarily invariant norm by Theorem 2 and Theorem 4.

LEMMA 5. ([6]) *Let* $A \in S_\alpha$. *Then*

$$|\det A| \leqslant \sec^n(\alpha)\det(\mathfrak{R}A).$$

LEMMA 6. ([5]) *If* $A \in \mathbb{M}_n(\mathbb{C})$ *has positive definite real part, then*

$$\det\left(\mathfrak{R}A\right) \leqslant |\det A|.$$

LEMMA 7. ([4]) *Let* $A \in S_\alpha$. *Then*

$$s_j(A) \leqslant \sec^2(\alpha)\lambda_j(\mathfrak{R}A).$$

LEMMA 8. ([11]) *Let* $A \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$\lambda_j(\Re A) \leqslant s_j(A),$$

where $\lambda_j(A)$ *and* $s_j(A)$ *is the* j*-th largest eigenvalue and singular value of* A.

LEMMA 9. ([12]) *Let* $A \in S_\alpha$. *Then for any unitarily invariant norm* $\|\cdot\|$,

$$\|A\| \leqslant \sec(\alpha)\|\Re A\|.$$

LEMMA 10. ([2]) *Let* $A \in \mathbb{M}_n(\mathbb{C})$. *Then for any unitarily invariant norm* $\|\cdot\|$,

$$\|\Re A\| \leqslant \|A\|.$$

THEOREM 5. *Let* $A, B \in S_\alpha$ *and* $v \in [0,1]$. *If* $0 < mI \leqslant \Re(A)$, $\Re(B) \leqslant MI$, *then*

$$\cos^n(\alpha)|\det(A\nabla_v B)| \leqslant K^n(h)|\det(A\sharp_v B)|;$$

$$\cos^2(\alpha)s_j(A\nabla_v B) \leqslant K(h)s_j(A\sharp_v B);$$

$$\cos(\alpha)||A\nabla_v B|| \leqslant K(h)||A\sharp_v B||;$$

Proof. By computations, we have

$$\begin{aligned}\cos^n(\alpha)|\det(A\nabla_v B)| &\leqslant \det(\Re(A\nabla_v B))\\ &\leqslant K^n(h)\det(\Re(A\sharp_v B))\\ &\leqslant K^n(h)|\det(A\sharp_v B)|,\end{aligned}$$

where the first inequality is by Lemma 5, the second one is by (2.1), and the last inequality is by Lemma 6.

$$\begin{aligned}\cos^2(\alpha)s_j(A\nabla_v B) &\leqslant \lambda_j(\Re(A\nabla_v B))\\ &= s_j(\Re(A\nabla_v B))\\ &\leqslant K(h)s_j(\Re(A\sharp_v B))\\ &\leqslant K(h)s_j(A\sharp_v B),\end{aligned}$$

where the first inequality is by Lemma 7, the second one is by (2.1), and the last inequality is by Lemma 8.

$$\begin{aligned}\cos(\alpha)||A\nabla_v B|| &\leqslant ||\Re(A\nabla_v B)||\\ &\leqslant K(h)||\Re(A\sharp_v B)||\\ &\leqslant K(h)||A\sharp_v B||,\end{aligned}$$

where the first inequality is by Lemma 9, the second one is by (2.1), and the last inequality is by Lemma 10. □

THEOREM 6. *Let* $A,B\in S_\alpha$ *and* $v\in[0,1]$. *If* $0<mI_n\leqslant \Re^{-1}(A),\ \Re^{-1}(B)\leqslant MI_n$, *then*

$$\cos^{3n}(\alpha)|\det(A\sharp_v B)|\leqslant K^n(h)|\det(A!_v B)|;$$

$$\cos^{4}(\alpha)s_j(A\sharp_v B)\leqslant K(h)s_j(A!_v B);$$

$$\cos^{3}(\alpha)||A\sharp_v B||\leqslant K(h)||A!_v B||;$$

Proof. By replacing (2.1) by (2.6) in the proof of Theorem 5, we can get the proof of Theorem 6 similarly, so we omit the details. □

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