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ANISOTROPIC NONLINEAR ELLIPTIC SYSTEM WITH DEGENERATE COERCIVITY AND L^m DATA

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Abstract. In a bounded open subset $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$, we study nonlinear degenerate anisotropic elliptic systems with L^m data, where m being small. Therefore, we prove the existence of a weak solution $u : \Omega \to \mathbb{R}^N$ with $N \geqslant 2$, under various hypotheses on the data f.

1. Introduction

Let us consider the Dirichlet elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} D_i\left(a_i(x,u(x),D_i u(x))\right) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$, $f,u : \Omega \to \mathbb{R}^N$, $N \geqslant 2$, $D_i u = \dfrac{\partial u}{\partial x_i}$, and the vector fields $a_i : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ $i = 1,\dots,n$ are Carathéodory functions.

Hence, we aim at proving the existence of a bounded weak solution u for problem (1.1) under suitable assumptions on the vector fields a_i for all $i = 1,\dots,n$, and the right hand side $f \in L^m(\Omega;\mathbb{R}^N)$, we have studied the following states: the first is when $f \in \left(W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)\right)'$, and the last case f is not in $\left(W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)\right)'$ therefore the solution u doesn't belong to $W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$ but is in $W_0^{1,(q_i)}(\Omega;\mathbb{R}^N)$ such that $q_i \in (1,p_i)$ for all $i = 1,\dots,n$, where q_i as in (3.12) and p_i as in (3.11). Our regularity results given in Theorems 3.3 and 3.4 are new and have not been proven before neither in the isotropic nor in the anisotropic case. An important feature is the fact that, due to (3.1), the operator $A(u) = -\sum_{i=1}^{n} D_i(a_i(x,u,D_i u))$ is well defined between $W_0^{1,(p_i)}(\Omega)$ and its dual space $(W_0^{1,(p_i)}(\Omega))'$ but, from (3.2), it fails to be coercive if u is large. This shows that the classical methods for elliptic operators can't be applied. To overcome this problem, we will proceed by approximation by means of truncations in a_i to get a coercive differential operator. Next, we prove some anisotropic uniform estimates on

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the sequence of approximate solutions. Finally, we pass to the limit in the approximate system to obtain the existence of a weak solution for problem (1.1).

Existence of weak solutions u has been profoundly examined in [21, 35, 8, 33], while uniqueness seems to be a delicate matter, see [13]. For the scalar case with lower order term we refer the reader to [9, 10, 11, 7, 20]. For some recent developments on anisotropic elliptic equations and systems, see [4, 28, 5, 2, 3, 15, 18, 19]. The case of p-Laplacian operator is treated in [17, 12], and the anisotropic case, in which each component of the gradient $D_i u$ may have a possibly different exponent p_i, is dealt in [22] and [23].

In addition, it is important to mention that for the given system we've to add a basic condition, as in some previous works treated, the isotropic case when $p = 2$: sometimes use assumptions on the support off-diagonal coefficients when we deal with elliptic systems: in [32] off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ must disappear when y^α is large, while [36] the system is supposed to be tridiagonal, i.e., $a_{i,j}^{\alpha,\beta} = 0$ for $\beta > \alpha$; [24] and [25] require that the support of $a_{i,j}^{\alpha,\beta}(x,y)$ is confined in square along the $y^\alpha = \pm y^\beta$ diagonals, in [26] different conditions on the support are given. In the case of the anisotropic elliptic system, in [14] and [16] the authors use the following structure condition

$$\forall x \in \Omega,\ \ \forall \xi \in \mathbb{R}^N,\ \ \forall s \in \mathbb{R}^N \quad \text{with} \quad |s| \leqslant 1,$$
$$a_i(x,\xi).((I - s\otimes s)\xi) \geqslant 0,\ \ i = 1,\dots,n,$$

where $(I - s\otimes s)$ is the rank $N-1$ orthogonal projector onto the space orthogonal to the unit vector $s \in \mathbb{R}^N$, see also [1, 6, 29].

This paper is organized in the following way: The section 2 is devoted to mathematical preliminaries. Following that, assumptions and results are highlighted in section 3. Next, we approximate problem (1.1) with some non degenerate problems in section 4, and we show some a priori estimates on the solutions of these problems in section 5. Finally, we pass to the limit in the approximate problem in section 6.

2. Mathematical preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $p_i > 1$, $i = 1,\dots,n$, $n \geqslant 2$. We introduce the anisotropic Sobolev spaces $W^{1,(p_i)}(\Omega)$ and $W_0^{1,(p_i)}(\Omega)$ which are defined respectively by

$$W^{1,(p_i)}(\Omega) = \left\{ g \in W^{1,1}(\Omega) : D_i g \in L^{p_i}(\Omega),\ \forall i = 1,\dots,n \right\},$$

and

$$W_0^{1,(p_i)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,(p_i)}(\Omega).$$

which is a Banach space under the norm

$$\|g\|_{W_0^{1,(p_i)}(\Omega)} = \|g\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|D_i g\|_{L^{p_i}(\Omega)}.$$

We need the anisotropic Sobolev embedding Theorem.

THEOREM 2.1. [34] *Suppose* $g \in W_0^{1,(p_i)}(\Omega)$. *Then*

$$\|g\|_{L^q(\Omega)} \leqslant C \prod_{i=1}^{n} \|D_i g\|_{L^{p_i}(\Omega)}^{\frac{1}{n}}. \tag{2.1}$$

where

$$\begin{cases} q = \overline{p}^* = \dfrac{n\overline{p}}{n-\overline{p}} & \text{if } \overline{p} < n, \\ q \in [1,\infty) & \text{if } \overline{p} \geqslant n, \end{cases} \qquad \frac{1}{\overline{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}.$$

The constant C, *depends on* $p_1,\dots,p_n$, n *if* $\overline{p} < n$. *Furthermore, if* $\overline{p} \geqslant n$, *the inequality* (2.1) *is true for all* $q \geqslant 1$ *and* C *depends on* q *and* $|\Omega|$.

One has the following lemma.

LEMMA 2.2. [31] *Let* $v \in W_0^{1,(p_i)}(\Omega)$. *Then there exists a constant* $C > 0$ *such that*

$$\|v\|_{L^{p_i}(\Omega)} \leqslant C\|\partial_i v\|_{L^{p_i}(\Omega)}.$$

We use standard notation for the vector and matrix-valued versions of the space/norm introduced above, as an example the $\mathbb{R}^N$-valued of $W_0^{1,(p_i)}(\Omega)$ is denoted by $W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$.

3. Assumptions and main results

We assume that the vector fields $a_i : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $i = 1,\dots,n$, are Carathédory functions, satisfying the following conditions, there exist $c_1 > 0$, $\tau > 0$, such that for a.e. $x \in \Omega$, all $u \in \mathbb{R}^N$, and all $\xi, \xi' \in \mathbb{R}^N$, $i = 1,\dots,n$, we have

$$|a_i(x,u,\xi)| \leqslant c_1 \left(|h| + |u|^{\overline{p}} + \sum_{j=1}^{n} |\xi_j|^{p_j} \right)^{1-\frac{1}{p_i}}, \quad |h| \in L^1(\Omega), \tag{3.1}$$

$$\tau \frac{|\xi|^{p_i}}{(1+|u|)^{\theta}} \leqslant a_i(x,u,\xi).\xi, \tag{3.2}$$

$$\left(a_i(x,u,\xi) - a_i(x,u,\xi')\right).\left(\xi - \xi'\right) \geqslant 0, \tag{3.3}$$

$$0 < \theta < \overline{p} - 1, \tag{3.4}$$

where $p_i > 1$ is real number and $1/\overline{p} = 1/n \sum_{i=1}^{n} 1/p_i$. Defining $p_+ = \max_{1\leqslant i\leqslant n} p_i$ and $p_- = \min_{1\leqslant i\leqslant n} p_i$.

The fundamental problem presented by extending the results of an equation to a system is to obtain as estimation of the truncation, since the truncation is different for both scalar and vector cases, therefore as additional structure condition is needed in order to prove the existence of a solution for the elliptic system with L^m data. Here we use the following condition

$$a_{i,l}(x,u,\xi)\xi_l \geqslant 0, \ \forall (x,\xi) \in \Omega \times \mathbb{R}^N, \ i = 1,\dots,n, \ l = 1,\dots,N, \tag{3.5}$$

see set assumption in [37].

Here $a_{i,l}$ and ξ_l, are the l-th row of vectors a_i and ξ, respectively.

We make the further restriction

$$m \leqslant \frac{n\overline{p}}{n\overline{p} + \overline{p} - np_-}. \tag{3.6}$$

As prototype examples, we consider the following models

$$\begin{cases} -\sum_{i=1}^{n} D_i \left(\dfrac{|D_i u|^{p_i-2} D_i u}{(1+|u|)^\theta} \right) = f & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\sum_{i=1}^{n} D_i \left(\dfrac{a(u)|D_i u|^{\frac{p_i-2}{2}}}{(1+|u|)^\theta} D_i u \right) = f & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a : \mathbb{R}^N \to (0,\infty)$ is a bounded continuous function.

Our main results are the following

THEOREM 3.1. *Let $f \in L^m(\Omega;\mathbb{R}^N)$ with $m > \frac{n}{\overline{p}}$ satisfying (3.6), $n \geqslant 3$, $N \geqslant 2$, and assume that $\overline{p} \in [2,n)$. Then every solution $u \in W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$ to problem (1.1) is such that $u \in L^\infty(\Omega;\mathbb{R}^N)$, which is weak solution of (1.1) in the sense that*

$$\int_\Omega \sum_{i=1}^{n} a_i(x,u,D_i u).D_i\varphi \, dx = \int_\Omega f.\varphi \, dx, \ \forall \varphi \in W_0^{1,(p_i)}(\Omega;\mathbb{R}^N). \tag{3.7}$$

THEOREM 3.2. *Let $f \in L^m(\Omega;\mathbb{R}^N)$ with $m = \frac{n}{\overline{p}}$ satisfying (3.6), $n \geqslant 3$, $N \geqslant 2$, assume that $\overline{p} \in [2,n)$. Then every solution $u \in W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$ to problem (1.1) is such that $e^{\lambda |u|^{1-\frac{\theta}{\overline{p}-1}}} \in L^1(\Omega)$ for every $\lambda > 0$, which is weak solution of (1.1) in the sense (3.7).*

REMARK 1.

- If we take $N = 1$, the results of Theorems 3.1–3.2 are the same as the results of Theorems 1.1–1.2 found in [18].

THEOREM 3.3. *Let $f \in L^m(\Omega;\mathbb{R}^N)$. Assume that θ as in (3.4) and $\overline{p} < n$, such that*

$$\frac{n\overline{p}}{n\overline{p}-(1+\theta)(n-\overline{p})} \leqslant m < \frac{n}{\overline{p}}. \tag{3.8}$$

Then every weak solution $u \in W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$ to problem (1.1) in the sense of (3.7), is such that $u \in L^r(\Omega;\mathbb{R}^N)$, with

$$r = \frac{nm\overline{p}(p_+ - \theta - 1)}{mp_+(n-\overline{p}) - n\overline{p}(m-1)}, \quad p_+ = \max_{1\leqslant i\leqslant n} p_i. \tag{3.9}$$

REMARK 2.

- Hypothesis $m \geqslant \dfrac{n\overline{p}}{n\overline{p}-(1+\theta)(n-\overline{p})}$ guarantees that $r \geqslant \overline{p}^*$. Since we have the continuous embedding $W_0^{1,(p_i)}(\Omega) \hookrightarrow L^{\overline{p}^*}(\Omega)$ we deduce that $u \in L^r(\Omega)$ for all $r \in [1,\infty)$.
- If $p_i = 2$, and $N = 1$, $f \in L^m(\Omega)$, with $\dfrac{2n}{n+2-\theta(n-2)} \leqslant m < \dfrac{n}{2}$, then $u \in H_0^1(\Omega) \cap L^r(\Omega)$, such that $r = \dfrac{nm(1-\theta)}{n-2m}$, which is the same result as in Theorem 1.3 in [8].

THEOREM 3.4. *Under the assumptions (3.1)–(3.5), let $f \in L^m(\Omega;\mathbb{R}^N)$, with $m > 1$ and $\overline{p} < n$, such that for $i = 1,\dots,n$*

$$\frac{n\overline{p}}{\overline{p}(1+\theta)+2n(\overline{p}-1-\theta)} < m < \frac{n\overline{p}}{n\overline{p}-(1+\theta)(n-\overline{p})} \tag{3.10}$$

$$\frac{n\overline{p}-m\overline{p}(1+\theta)}{nm(\overline{p}-1-\theta)} < p_i < \frac{n\overline{p}-m\overline{p}(1+\theta)}{n\overline{p}-m\overline{p}(1+\theta)-nm(\overline{p}-1-\theta)}. \tag{3.11}$$

Then there exists a function $u \in W_0^{1,(q_i)}(\Omega;\mathbb{R}^N)$, with

$$q_i = \frac{p_i nm(\overline{p}-1-\theta)}{n\overline{p}-m\overline{p}(1+\theta)} < p_i, \quad i = 1,\dots,n, \tag{3.12}$$

which allows to solve (1.1) in the sense

$$\int_\Omega \sum_{i=1}^n a_i(x,u,D_i u).D_i\varphi\,dx = \int_\Omega f.\varphi\,dx, \quad \forall \varphi \in C_0^\infty(\Omega;\mathbb{R}^N). \tag{3.13}$$

REMARK 3.

- If $\theta > \frac{n\overline{p}+\overline{p}-2n}{2n-\overline{p}}$ then the lower bound for m in (3.10) is smaller than 1, and if $\theta < \frac{n\overline{p}+\overline{p}-2n}{2n-\overline{p}}$ then the lower bound for m in (3.10) is greater than 1.
- The lower bound of m in (3.10) guarantees that (3.11) is well defined.

- We note that $q_i < p_i$ for all $i = 1, \dots, n$, due to the upper bound for m in (3.10).
- Since we have $\theta < \overline{p} - 1$ the inequalities (3.8) and (3.10) are well defined.
- If we take $p_i = 2$, and $N = 1$, with $f \in L^m(\Omega)$, $\frac{n}{n+1-\theta(n-1)} < m < \frac{2n}{n+2-\theta(n-2)}$, then $u \in W_0^{1,q}(\Omega)$, with $q = \frac{nm(1-\theta)}{n-m(1+\theta)} < 2$, which is the same result as in Theorem 1.8 in [8].

4. Approximate solutions

Let T_ε be the standard scalar truncation defined as

$$\begin{aligned} T_\varepsilon : \mathbb{R} &\to \mathbb{R} \\ x \ &\to T_\varepsilon(x) = \begin{cases} x & \text{if } \ |x| \leqslant \varepsilon, \\ \varepsilon\,\mathrm{sign(x)} & \text{if } \ |x| > \varepsilon. \end{cases} \end{aligned}$$

We introduce the following cubic truncation function

$$\begin{aligned} \mathscr{T}_\varepsilon(y) &= (T_\varepsilon(y_1), \dots, T_\varepsilon(y_N)) \\ &= (\max(-\varepsilon, \min(\varepsilon, y_1)), \dots, \max(-\varepsilon, \min(\varepsilon, y_N))), \end{aligned}$$

which satisfies

$$|\mathscr{T}_\varepsilon(y)| \leqslant |y|, \quad |\mathscr{T}_\varepsilon(y)| \leqslant N\varepsilon. \tag{4.1}$$

We consider the following family of approximate problems

$$\begin{cases} -\sum_{i=1}^{n} D_i\left(a_i(x, \mathscr{T}_\varepsilon(u_\varepsilon), D_i u_\varepsilon)\right) = f_\varepsilon & x \in \Omega, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

where $f_\varepsilon = \mathscr{T}_\varepsilon(f)$, such that

$$f_\varepsilon \to f \,\text{strongly in}\, L^m(\Omega; \mathbb{R}^N), \tag{4.3}$$

and

$$\|f_\varepsilon\|_{L^m(\Omega;\mathbb{R}^N)} \leqslant \|f\|_{L^m(\Omega;\mathbb{R}^N)}. \tag{4.4}$$

We are going to prove the existence of weak solution for problem (4.2) that is a function $u_\varepsilon \in W_0^{1,(p_i)}(\Omega; \mathbb{R}^N)$ such that for all $\varphi \in W_0^{1,(p_i)}(\Omega; \mathbb{R}^N)$

$$\int_\Omega \sum_{i=1}^{n} a_i(x, \mathscr{T}_\varepsilon(u_\varepsilon), D_i u_\varepsilon).D_i\varphi\, dx = \int_\Omega f_\varepsilon.\varphi\, dx. \tag{4.5}$$

For $u_\varepsilon, v_\varepsilon \in W_0^{1,(p_i)}(\Omega; \mathbb{R}^N)$, we denote by A the operator

$$A : u_\varepsilon \to \left(v_\varepsilon \to \int_\Omega \sum_{i=1}^{n} a_i(x, \mathscr{T}_\varepsilon(u_\varepsilon), D_i u_\varepsilon).D_i v_\varepsilon\, dx\right).$$

Clearly, A is well-defined and monotone. This function satisfies condition (3.2), we deduce from the coercivity condition that A is coercive.

The growth condition (3.1) implies that A is hemicontinuous. By (3.1) and by using Hölder's inequality we obtain

$$|\langle Au_\varepsilon, v_\varepsilon\rangle| \leqslant c_1 \sum_{i=1}^{n} \left\{ \int_\Omega \left(|h| + |u_\varepsilon|^{\overline{p}} + \sum_{j=1}^{n} |D_j u_\varepsilon|^{p_j} \right) dx \right\}^{\frac{1}{p_i'}}.$$

Since the solution $|u_\varepsilon|$ is in $L^{\overline{p}}(\Omega)$, this because, there exists $j \in \{1,\dots,n\}$ so that $p_j \geqslant \overline{p}$, then $|u_\varepsilon| \in L^{\overline{p}}(\Omega)$, by using Lemma 2.2, the boundedness of A. According to (3.3) and that $f_\varepsilon \in \left(W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)\right)'$, we can apply the surjectivity result given in [27]. This gives the existence of a weak solution $u_\varepsilon \in W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$ for problem (4.2) each of them satisfying the weak formulation (4.5).

5. A priori estimates

Throughout the paper, we will denote by c or C the positive constants depending only on the data of the problem, but not on ε.

LEMMA 5.1. *Assume that $m > \frac{n}{\overline{p}}$, and let $f_\varepsilon \in L^m(\Omega;\mathbb{R}^N)$, $\overline{p} \geqslant 2$, and let u_ε be a solution of (4.2) in the sense of (4.5). Then, the sequence (u_ε) is bounded in $W_0^{1,(p_i)}(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$.*

LEMMA 5.2. *Assume that $m = \frac{n}{\overline{p}}$, and let $f_\varepsilon \in L^m(\Omega;\mathbb{R}^N), \overline{p} \geqslant 2$ and let u_ε be a solution of (4.2) in the sense of (4.5). Then, there exists $\lambda > 0$ such that $e^{\lambda |u_\varepsilon|^{1-\frac{\theta}{\overline{p}-1}}} \in L^1(\Omega)$.*

Proof of Lemmas 5.1–5.2. We define the function

$$\begin{aligned} G_k : \mathbb{R}^N &\to \mathbb{R}^N \\ y &\to G_k(y) = y - \mathscr{T}_k(y) = (y_1 - T_k(y_1), \dots, y_N - T_k(y_N)). \end{aligned}$$

For $k > 0$, if we take $G_k(u_\varepsilon)$ as test function in (4.5) yields

$$\sum_{l=1}^{N} \int_{\{|u_\varepsilon^l| > k\}} \sum_{i=1}^{n} a_{i,l}(x, \mathscr{T}_\varepsilon(u_\varepsilon), D_i u_\varepsilon) D_i u_\varepsilon^l \, dx = \int_\Omega f_\varepsilon . G_k(u_\varepsilon)\, dx. \tag{5.1}$$

Using (3.2), (3.5), and (5.1) we can deduce that

$$\begin{aligned}
&\int_{\{\sum_{l=1}^{N}|u_\varepsilon^l|>k\}}\sum_{i=1}^{n}\frac{|D_iu_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^\theta}\,dx\\
&\leqslant\frac{1}{\tau}\int_{\{\sum_{l=1}^{N}|u_\varepsilon^l|>k\}}\sum_{i=1}^{n}a_i(x,\mathscr{T}_\varepsilon(u_\varepsilon),D_iu_\varepsilon).D_iu_\varepsilon\,dx\\
&\leqslant\frac{1}{\tau}\sum_{l=1}^{N}\int_{\{|u_\varepsilon^l|>k\}}\sum_{i=1}^{n}a_{i,l}(x,\mathscr{T}_\varepsilon(u_\varepsilon),D_iu_\varepsilon)D_iu_\varepsilon^l\,dx\\
&\leqslant\frac{1}{\tau}\int_\Omega f_\varepsilon.G_k(u_\varepsilon)\,dx.
\end{aligned}$$

Therefore, Hölder's inequality implies

$$\int_{\{|u_\varepsilon|>k\}}\sum_{i=1}^{n}\frac{|D_iu_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^\theta}\,dx\leqslant c\|f_\varepsilon\|_{L^m(\Omega;\mathbb{R}^N)}\left(\int_\Omega|G_k(u_\varepsilon)|^{m'}\,dx\right)^{\frac{1}{m'}},\tag{5.2}$$

such that $|u_\varepsilon|=\sum_{l=1}^{N}|u_\varepsilon^l|$ (1-norm of vector $u_\varepsilon=(u_\varepsilon^1,\ldots,u_\varepsilon^N)$). In fact (5.2) is exactly (3.2) in [18]. So the rest of the proof that (u_ε) is bounded in $L^\infty(\Omega)$ and $e^{\lambda|u_\varepsilon|^{1-\frac{\theta}{\overline{p}-1}}}$ is bounded in $L^1(\Omega)$ is similar to scalar case in [18].

We are going to prove that (u_ε) is bounded in $W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$: taking u_ε as test function in (4.5), we obtain

$$\int_\Omega\sum_{i=1}^{n}a_i(x,u_\varepsilon,D_iu_\varepsilon).D_iu_\varepsilon\,dx\leqslant\|u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^N)}\|f_\varepsilon\|_{L^1(\Omega;\mathbb{R}^N)}.$$

By assumption (3.2) and (4.1), we get

$$\tau\int_\Omega\sum_{i=1}^{n}\frac{|D_iu_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^\theta}\,dx\leqslant\|u_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^N)}\|f\|_{L^1(\Omega;\mathbb{R}^N)},$$

so,

$$\begin{aligned}
\int_\Omega\sum_{i=1}^{n}|D_iu_\varepsilon|^{p_i}\,dx&\leqslant\frac{\|u_\varepsilon\|_{L^\infty}}{\tau}(1+\|u_\varepsilon\|_{L^\infty})^\theta\,\|f\|_{L^1}\\
&\leqslant C.
\end{aligned}$$

Therefore, the proof of Lemmas 5.1–5.2 is concluded. □

LEMMA 5.3. *Let $p_i\in[1,+\infty)$, $i=1,\ldots,n$. We have for all $k>0$*

$$\int_{\{k\leqslant|u_\varepsilon|<k+1\}}|D_iu_\varepsilon|^{p_i}\,dx\leqslant c(2+k)^\theta\int_{\{|u_\varepsilon|\geqslant k\}}|f_\varepsilon|\,dx.\tag{5.3}$$

Proof. We take $\mathscr{T}_1\left(G_k(u_\varepsilon)\right)$ as a test function in (4.5), such that

$$\mathscr{T}_1(G_k(u_\varepsilon)) = \mathscr{T}_1(u_\varepsilon - \mathscr{T}_k(u_\varepsilon)) = \left(T_1(u_\varepsilon^1 - T_k(u_\varepsilon^1)), \ldots, T_1(u_\varepsilon^N - T_k(u_\varepsilon^N))\right)$$

where for all $x \in \mathbb{R}$

$$T_1(x - T_k(x)) = \begin{cases} x - k\,\mathrm{sign}(x); & k \leqslant |x| \leqslant k+1, \\ \mathrm{sign}(x); & |x| \geqslant k+1, \\ 0; & |x| \leqslant k. \end{cases}$$

So

$$\sum_{l=1}^{N} \int_{\{k \leqslant |u_\varepsilon^l| < k+1\}} \sum_{i=1}^{n} a_{i,l}(x, u_\varepsilon, D_i u_\varepsilon) D_i u_\varepsilon^l \, dx = \int_\Omega f_\varepsilon . \mathscr{T}_1(G_k(u_\varepsilon))\, dx. \tag{5.4}$$

According to (3.2) and (5.4), we have

$$\begin{aligned} &\int_{\{k \leqslant \sum_{l=1}^{N} |u_\varepsilon^l| < k+1\}} \sum_{i=1}^{n} \frac{|D_i u_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^\theta}\, dx \\ &\leqslant \int_{\left\{k \leqslant \sum_{l=1}^{N} |u_\varepsilon^l| < k+1\right\}} \sum_{i=1}^{n} a_i(x, u_\varepsilon, D_i u_\varepsilon) . D_i u_\varepsilon \, dx \\ &\leqslant \sum_{l=1}^{N} \int_{\{k \leqslant |u_\varepsilon^l| < k+1\}} \sum_{i=1}^{n} a_{i,l}(x, u_\varepsilon, D_i u_\varepsilon) D_i u_\varepsilon^l \, dx \\ &= \int_\Omega f_\varepsilon . \mathscr{T}_1(G_k(u_\varepsilon))\, dx. \end{aligned}$$

which implies

$$\sum_{i=1}^{n} \int_{\{k \leqslant |u_\varepsilon| < k+1\}} |D_i u_\varepsilon|^{p_i}\, dx \leqslant \frac{1}{\tau}(2+k)^\theta \int_{\{|u_\varepsilon| \geqslant k\}} |f_\varepsilon|\, dx.$$

Consequently

$$\int_{\{k \leqslant |u_\varepsilon| < k+1\}} |D_i u_\varepsilon|^{p_i}\, dx \leqslant c(2+k)^\theta \int_{\{|u_\varepsilon| \geqslant k\}} |f_\varepsilon|\, dx. \quad \square$$

LEMMA 5.4. *Assume that m satisfies (3.8), and let u_ε be a solution of (4.2) in the sense of (4.5). Let r as in (3.9), then u_ε is bounded in $W_0^{1,(p_i)}(\Omega;\mathbb{R}^N) \cap L^r(\Omega;\mathbb{R}^N)$.*

Proof. We set

$$A_k = \{x \in \Omega : |u_\varepsilon| \geqslant k\}, \qquad B_k = \{x \in \Omega : k \leqslant |u_\varepsilon| < k+1\}.$$

Let $\gamma \geqslant 1$, using the fact that

$$\left(\sum_{i=1}^{N} \alpha_i\right)^p \leqslant N^p \sum_{i=1}^{N} (\alpha_i)^p, \quad \alpha_i \geqslant 0, \tag{5.5}$$

and due to the anisotropic Sobolev inequality (2.1), we have

$$\begin{aligned}\left(\int_\Omega |u_\varepsilon|^{\gamma\overline{p}^*}\,dx\right)^{\frac{1}{\overline{p}^*}} &\leqslant c\left(\int_\Omega \sum_{l=1}^{N} |u_\varepsilon^l|^{\gamma\overline{p}^*}\,dx\right)^{\frac{1}{\overline{p}^*}}\\ &\leqslant C\prod_{j=1}^{n}\left(\sum_{k=0}^{+\infty}\int_{B_k}|u_\varepsilon|^{p_j(\gamma-1)}|D_ju_\varepsilon|^{p_j}\,dx\right)^{\frac{1}{np_j}} \qquad (5.6)\\ &\leqslant \prod_{j=1}^{n}\left(C\sum_{k=0}^{+\infty}(1+k)^{p_j(\gamma-1)}\int_{B_k}|D_ju_\varepsilon|^{p_j}\,dx\right)^{\frac{1}{np_j}},\end{aligned}$$

from Lemma 5.3, we get

$$\begin{aligned}\left(\int_\Omega |u_\varepsilon|^{\gamma\overline{p}^*}\,dx\right)^{\frac{1}{\overline{p}^*}} &\leqslant \prod_{j=1}^{n}\left(C\sum_{k=0}^{+\infty}(1+k)^{p_j(\gamma-1)}(2+k)^{\theta}\int_{A_k}|f_\varepsilon|\,dx\right)^{\frac{1}{np_j}}\\ &\leqslant \prod_{j=1}^{n}\left(C\sum_{k=0}^{+\infty}(2+k)^{p_j(\gamma-1)+\theta}\sum_{h=k}^{\infty}\int_{A_h}|f_\varepsilon|\,dx\right)^{\frac{1}{np_j}}.\end{aligned}$$

Therefore, changing the order of summation, and using the fact that

$$\sum_{k=0}^{h} k^{\rho} \leqslant c(h+1)^{\rho+1}, \qquad (5.7)$$

where $c = c(\rho)$, one obtains

$$\begin{aligned}\left(\int_\Omega |u_\varepsilon|^{\gamma\overline{p}^*}\,dx\right)^{\frac{1}{\overline{p}^*}} &\leqslant \prod_{j=1}^{n}\left(C\sum_{h=0}^{\infty}\int_{B_h}|f_\varepsilon|\sum_{k=0}^{h}(2+k)^{p_j(\gamma-1)+\theta}\,dx\right)^{\frac{1}{np_j}}\\ &\leqslant \prod_{j=1}^{n}\left(C\int_\Omega |f_\varepsilon|\,(3+|u_\varepsilon|)^{p_j(\gamma-1)+\theta+1}\,dx\right)^{\frac{1}{np_j}}\\ &\leqslant \prod_{j=1}^{n}\left(\sum_{i=1}^{n} C\int_\Omega |f_\varepsilon|\,(3+|u_\varepsilon|)^{p_i(\gamma-1)+\theta+1}\,dx\right)^{\frac{1}{np_j}}\\ &\leqslant C\left(\sum_{i=1}^{n}\int_\Omega |f|\,(3+|u_\varepsilon|)^{p_i(\gamma-1)+\theta+1}\,dx\right)^{\frac{1}{\overline{p}}}.\end{aligned}$$

Since $m > 1$, by Hölder's inequality and this last inequality, we obtain

$$\left(\int_\Omega |u_\varepsilon|^{\gamma\overline{p}^*}\,dx\right)^{\frac{\overline{p}}{\overline{p}^*}} \leqslant C\sum_{i=1}^{n}\left(\int_\Omega (3+|u_\varepsilon|)^{(p_i(\gamma-1)+\theta+1)m'}\,dx\right)^{\frac{1}{m'}}.$$

Knowing that $\min\limits_{1\leqslant i\leqslant n} p_i = p_- \leqslant p_i \leqslant \max\limits_{1\leqslant i\leqslant n} p_i = p_+$, we have

$$\begin{aligned}\left(\int_\Omega |u_\varepsilon|^{\gamma\overline{p}^*}\,dx\right)^{\frac{\overline{p}}{\overline{p}^*}} &\leqslant C\left(1+\sum_{i=1}^n \int_\Omega |u_\varepsilon|^{(p_+(\gamma-1)+\theta+1)m'}\,dx\right)^{\frac{1}{m'}} \\ &\leqslant C'\left(1+\int_\Omega |u_\varepsilon|^{(p_+(\gamma-1)+\theta+1)m'}\,dx\right)^{\frac{1}{m'}}.\end{aligned} \tag{5.8}$$

Now choose γ so that

$$\overline{p}^*\gamma = (1+p_+(\gamma-1)+\theta)m',$$

thus

$$\gamma = \frac{(n-\overline{p})m(1+\theta-p_+)}{n\overline{p}(m-1)-mp_+(n-\overline{p})}.$$

The lower bound of m in (3.8) is equivalent to $\gamma \geqslant 1$. It follows that

$$r = \gamma\overline{p}^* = \frac{nm\overline{p}(1+\theta-p_+)}{n\overline{p}(m-1)-mp_+(n-\overline{p})} \geqslant \overline{p}^*. \tag{5.9}$$

Since $m < \dfrac{n}{\overline{p}}$ implies $\dfrac{\overline{p}}{\overline{p}^*} \geqslant \dfrac{1}{m'}$, (5.8) yields

$$\int_\Omega |u_\varepsilon|^r\,dx \leqslant c. \tag{5.10}$$

Combining (5.6)–(5.10), we deduce that

$$\begin{aligned}\int_\Omega |u_\varepsilon|^{p_j(\gamma-1)}|D_ju_\varepsilon|^{p_j}\,dx &\leqslant C\left(1+\int_\Omega |u_\varepsilon|^r\,dx\right)^{\frac{1}{m'}} \\ &\leqslant C.\end{aligned} \tag{5.11}$$

Therefore, by Lemma 5.3, that $\gamma \geqslant 1$, and (5.11), we can write for all $i = 1,\dots,n$

$$\begin{aligned}\int_\Omega |D_iu_\varepsilon|^{p_i}\,dx &= \int_{\{|u_\varepsilon|\leqslant 1\}} |D_iu_\varepsilon|^{p_i}\,dx + \int_{\{|u_\varepsilon|\geqslant 1\}} |D_iu_\varepsilon|^{p_i}\,dx \\ &\leqslant C + \int_{\{|u_\varepsilon|\geqslant 1\}} |u_\varepsilon|^{p_i^*}(\gamma-1)|D_iu_\varepsilon|^{p_i}\,dx \\ &\leqslant C.\end{aligned}$$

This finishes the proof of Lemma 5.4. □

LEMMA 5.5. *Assume that m satisfies* (3.10)*, and let u_ε be a solution of* (4.2) *in the sense of* (4.5)*. Then, the sequence (u_ε) is bounded in $W_0^{1,(q_i)}(\Omega;\mathbb{R}^N)\cap L^s(\Omega;\mathbb{R}^N)$ such that*

$$s = \overline{q}^* = \frac{n\overline{q}}{n-\overline{q}} = \frac{nm(\overline{p}-1-\theta)}{n-m\overline{p}}, \tag{5.12}$$

and

$$q_i = \frac{p_inm(\overline{p}-1-\theta)}{n\overline{p}-m\overline{p}(1+\theta)}. \tag{5.13}$$

Proof. Let λ be a positive number to be determined later. Due to Lemma 5.3, we get

$$\begin{aligned}\int_\Omega \frac{|D_i u_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^\lambda}\,dx &= \sum_{k=0}^{\infty}\int_{B_k}\frac{|D_i u_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^\lambda}\,dx\\ &\leqslant \sum_{k=0}^{\infty}\frac{1}{(1+k)^\lambda}\int_{B_k}|D_i u_\varepsilon|^{p_i}\,dx\\ &\leqslant c\sum_{k=0}^{\infty}\frac{1}{(1+k)^\lambda}(2+k)^\theta\int_{A_k}|f_\varepsilon|\,dx\\ &\leqslant c\sum_{k=0}^{\infty}(1+k)^{\theta-\lambda}\sum_{h=k}^{\infty}\int_{B_h}|f_\varepsilon|\,dx.\end{aligned}$$

On the other, changing the order of summation, and using (5.7), and by Hölder's inequality we conclude that

$$\begin{aligned}\int_\Omega \frac{|D_i u_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^\lambda}\,dx &\leqslant c\sum_{h=0}^{\infty}\int_{B_h}|f_\varepsilon|\sum_{k=0}^{h}(1+k)^{\theta-\lambda}\,dx\\ &\leqslant c\sum_{h=0}^{\infty}(2+h)^{1+\theta-\lambda}\int_{B_h}|f_\varepsilon|\,dx\\ &\leqslant c\sum_{k=0}^{\infty}\int_{B_k}|f_\varepsilon|\,(2+|u_\varepsilon|)^{\theta-\lambda+1}\,dx\\ &= c\int_\Omega |f_\varepsilon|\,(2+|u_\varepsilon|)^{\theta-\lambda+1}\,dx\\ &\leqslant c\left(\|f_\varepsilon\|_{L^m}\left(\int_\Omega (2+|u_\varepsilon|)^{m'(\theta-\lambda+1)}\,dx\right)^{\frac{1}{m'}}\right)\\ &\leqslant C\left(1+\left(\int_\Omega |u_\varepsilon|^{m'(\theta-\lambda+1)}\,dx\right)^{\frac{1}{m'}}\right).\end{aligned}$$

Let s and q_i be as in (5.12) and (5.13). Next, it can be checked that $q_i < p_i$. Using again (5.5) and the anisotropic Sobolev inequality (2.1), we see that

$$\begin{aligned}\int_\Omega |u_\varepsilon|^{\overline{q}^*}\,dx = \int_\Omega\left(\sum_{l=1}^{N}|u_\varepsilon^l|\right)^{\overline{q}^*}dx &\leqslant c\sum_{l=1}^{N}\int_\Omega |u_\varepsilon^l|^{\overline{q}^*}\,dx\\ &\leqslant c\sum_{l=1}^{N}\prod_{j=1}^{n}\left(\int_\Omega |D_j u_\varepsilon^l|^{q_j}\,dx\right)^{\frac{\overline{q}^*}{nq_j}}\\ &\leqslant C\sum_{l=1}^{N}\prod_{j=1}^{n}\left(\int_\Omega |D_j u_\varepsilon|^{q_j}\,dx\right)^{\frac{\overline{q}^*}{nq_j}}\\ &= NC\prod_{j=1}^{n}\left(\int_\Omega |D_j u_\varepsilon|^{q_j}\,dx\right)^{\frac{\overline{q}^*}{nq_j}}\end{aligned}$$

$$\leqslant C\prod_{j=1}^{n}\left(\sum_{i=1}^{n}\int_{\Omega}|D_iu_\varepsilon|^{q_i}\,dx\right)^{\frac{\overline{q}^*}{nq_j}}$$
$$\leqslant C\left(\sum_{i=1}^{n}\int_{\Omega}|D_iu_\varepsilon|^{q_i}\,dx\right)^{\frac{\overline{q}^*}{\overline{q}}},$$

thus

$$\left(\int_{\Omega}|u_\varepsilon|^{\overline{q}^*}\,dx\right)^{\frac{\overline{q}}{\overline{q}^*}}\leqslant C\sum_{i=1}^{n}\int_{\Omega}|D_iu_\varepsilon|^{q_i}\,dx. \tag{5.14}$$

We can assume that

$$\frac{q_i}{p_i}=\frac{\overline{q}}{\overline{p}}.$$

If not, we set

$$\eta=\max_{1\leqslant i\leqslant n}\frac{q_i}{p_i},$$

and replace q_i by ηp_i. Notice that since $\eta p_i\geqslant q_i$, then D_iu_ε remains in a bounded set of $L^{\eta p_i}(\Omega)$, implies that D_iu_ε bounded in $L^{q_i}(\Omega;\mathbb{R}^N)$ for all $i=1,\dots,n$ and this involves the results.

In the sequel, we set $q_i=\eta p_i$, $\eta=\dfrac{\overline{q}}{\overline{p}}\in(0,1)$.

We choose λ such that

$$\lambda=\overline{q}^*\frac{1-\eta}{\eta},$$

so

$$\lambda=\frac{n\overline{p}-m\overline{p}(1+\theta)-nm(\overline{p}-1-\theta)}{n-m\overline{p}}.$$

It is easy to verify that this involves

$$(1+\theta-\lambda)m'\leqslant\overline{q}^*.$$

Using Hölder's inequality and below calculations, we get

$$\begin{aligned}
\int_{\Omega}|D_iu_\varepsilon|^{q_i}\,dx &= \int_{\Omega}\frac{|D_iu_\varepsilon|^{q_i}}{(1+|u_\varepsilon|)^{\frac{\lambda q_i}{p_i}}}(1+|u_\varepsilon|)^{\frac{\lambda q_i}{p_i}}\,dx\\
&\leqslant\left(\int_{\Omega}\frac{|D_iu_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^{\lambda}}\,dx\right)^{\frac{q_i}{p_i}}\left(\int_{\Omega}(1+|u_\varepsilon|)^{\frac{\lambda q_i}{p_i-q_i}}\,dx\right)^{1-\frac{q_i}{p_i}}\\
&\leqslant\left(\int_{\Omega}\frac{|D_iu_\varepsilon|^{p_i}}{(1+|u_\varepsilon|)^{\lambda}}\,dx\right)^{\eta}\left(\int_{\Omega}(1+|u_\varepsilon|)^{\frac{\lambda\eta}{1-\eta}}\,dx\right)^{1-\eta}\\
&\leqslant C\left\{1+\left(\int_{\Omega}|u_\varepsilon|^{m'(\theta-\lambda+1)}\,dx\right)^{\frac{\eta}{m'}}\right\}\left\{1+\left(\int_{\Omega}|u_\varepsilon|^{\frac{\lambda\eta}{1-\eta}}\,dx\right)^{1-\eta}\right\}\\
&\leqslant C\left\{1+\left(\int_{\Omega}|u_\varepsilon|^{\overline{q}^*}\,dx\right)^{\frac{\eta(1+\theta-\lambda)}{\overline{q}^*}}\right\}\left\{1+\left(\int_{\Omega}|u_\varepsilon|^{\overline{q}^*}\,dx\right)^{1-\eta}\right\},
\end{aligned}$$

it follows that

$$\sum_{i=1}^{N} \int_{\Omega} |D_i u_\varepsilon|^{q_i}\, dx \leqslant C' \left\{ 1 + \left(\int_{\Omega} |u_\varepsilon|^{\overline{q}^*}\, dx \right)^{\frac{\eta(1+\theta-\lambda)}{\overline{q}^*}+1-\eta} \right\}. \tag{5.15}$$

This inequality together with (5.14), implies

$$\begin{aligned} \left(\int_{\Omega} |u_\varepsilon|^{\overline{q}^*}\, dx \right)^{\frac{\overline{q}}{\overline{q}^*}} &\leqslant c \sum_{i=1}^{n} \int_{\Omega} |D_i u_\varepsilon|^{q_i}\, dx \\ &\leqslant C \left\{ 1 + \left(\int_{\Omega} |u_\varepsilon|^{\overline{q}^*}\, dx \right)^{\frac{\eta(1+\theta-\lambda)}{\overline{q}^*}+1-\eta} \right\}. \end{aligned}$$

From (3.4), we get

$$\frac{\overline{q}}{\overline{q}^*} > \frac{\eta(1+\theta-\lambda)}{\overline{q}^*} + 1 - \eta.$$

Consequently u_ε is bounded in $L^{\overline{q}^*}(\Omega;\mathbb{R}^N)$, so, the inequality (5.15) implies

$$\int_{\Omega} |D_i u_\varepsilon|^{q_i}\, dx \leqslant C. \tag{5.16}$$

This finishes the proof of Lemma 5.5. $\square$

6. Proof of main results

In this section, we prove Theorems 3.1, 3.2, 3.3, and 3.4 using the estimates of Section 5.

6.1. Proof of Theorem 3.4

By Lemma 5.5 the sequence u_ε is bounded in $W_0^{1,(q_i)}(\Omega;\mathbb{R}^N)$, where q_i is defined as (5.13), without losing generality, we can therefore assume that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad W_0^{1,(q_i)}(\Omega;\mathbb{R}^N).$$

The sequence u_ε remains in a bounded set of the space

$$W_0^{1,q_-}(\Omega;\mathbb{R}^N), \quad q_- = \min_{1\leqslant i\leqslant n} q_i.$$

Thanks to the Rellich embedding Theorem, we can extract a subsequence denoted again u_ε so that

$$u_\varepsilon \to u \quad \text{strongly in} \quad L^{q_0}(\Omega;\mathbb{R}^N) \tag{6.1}$$

and

$$u_\varepsilon \to u \text{ a.e. in} \,\Omega. \tag{6.2}$$

Using the same method as [30], we can prove

$$D_i u_\varepsilon \to D_i u \ \text{ a.e., in } \ \Omega, \ \ i = 1, \dots, n. \tag{6.3}$$

Since a_i are Carathéodory functions, by (6.2) and (6.3), we get

$$a_i(x, \mathscr{T}_\varepsilon(u_\varepsilon), D_i u_\varepsilon) \to a_i(x, u, D_i u), \ \text{ a.e., in } \ \Omega. \tag{6.4}$$

Now we prove that

$$a_i(x, u_\varepsilon, D_i u_\varepsilon) \ \text{ is uniformly bounded in } \ L^{\frac{q_i}{p_i - 1}}(\Omega; \mathbb{R}^N), \ \ \forall i = 1, \dots, n.$$

Where q_i satisfy (5.13), then we have for all $i = 1, \dots, n$.

$$1 < \frac{q_i}{p_i - 1} = \frac{p_i}{p_i - 1} \frac{nm(\overline{p} - 1 - \theta)}{n\overline{p} - m\overline{p}(1 + \theta)},$$

the choice of $\dfrac{q_i}{p_i - 1} > 1$ is possible since we have (3.11).

Now we let $\beta \in (0, 1)$ is a constant such that, for all $i = 1, \dots, n$

$$\frac{q_i}{p_i} = \beta = \frac{nm(\overline{p} - 1 - \theta)}{n\overline{p} - m\overline{p}(1 + \theta)} \tag{6.5}$$

and $\beta < 1$ since (3.10). Moreover

$$0 < p_i \beta = q_i. \tag{6.6}$$

Using the assumption (3.1), we get for all $i = 1, \dots, n$

$$\begin{aligned} |a_i(x, u_\varepsilon, D_i u_\varepsilon)|^{\frac{q_i}{p_i - 1}} &\leqslant c_1 \left(|h|^\beta + |u_\varepsilon|^{\overline{p}\beta} + \sum_{i=1}^{n} |D_j u_\varepsilon|^{p_j \beta} \right)^{\frac{q_i}{p_i \beta}} \\ &\leqslant c_1 \left(|h|^\beta + |u_\varepsilon|^{\overline{p}\beta} + \sum_{i=1}^{n} |D_i u_\varepsilon|^{q_i} \right). \end{aligned}$$

Therefore by (6.6) and since u_ε is in $L^{\overline{p}}(\Omega; \mathbb{R}^N)$, we find that, for all $i = 1, \dots, n$, $a_i(x, u_\varepsilon, D_i u_\varepsilon)$ uniformly bounded in $L^{\frac{q_i}{p_i - 1}}(\Omega; \mathbb{R}^N)$.

By (6.4) and Vitali's Theorem, we derive for all $i = 1, \dots, n$

$$a_i(x, \mathscr{T}_\varepsilon(u_\varepsilon), D_i u_\varepsilon) \to a_i(x, u, D_i u) \,\text{strongly in}\, L^1(\Omega; \mathbb{R}^N), \tag{6.7}$$

by (4.3), and (6.7), so that

$$\lim_{\varepsilon \to 0} \int_\Omega \sum_{i=1}^{n} a_i(x, \mathscr{T}_\varepsilon(u_\varepsilon), D_i u_\varepsilon).D_i \varphi \, dx = \int_\Omega \sum_{i=1}^{n} a_i(x, u, D_i u).D_i \varphi \, dx, \ \ \forall \varphi \in C_0^\infty(\Omega; \mathbb{R}^N).$$

6.2. Proof of Theorems 3.1, 3.2, and 3.3

The proof of Theorems 3.1, 3.2 and 3.3 is similar to the proof of Theorem 3.4.

From Lemma 5.1, the sequence u_ε is bounded in $W_0^{1,(p_i)}(\Omega;\mathbb{R}^N)$. This shows that we can extract a subsequence (denoted again by u_ε), such that

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \ \text{ weakly in } \ W_0^{1,p_i}(\Omega;\mathbb{R}^N),\\ &u_\varepsilon \to u \ \text{ strongly in } L^{p_-}(\Omega;\mathbb{R}^N), \ \ p_- = \min_{1\leqslant i\leqslant n} p_i,\\ &u_\varepsilon \to u \ \text{ a.e., in } \ \Omega. \end{aligned}$$

Arguing as the proof of Theorem 3.4, by using Lemma 5.1, we find that

$$a_i(x,\mathscr{T}_\varepsilon(u_\varepsilon),D_i u_\varepsilon) \rightharpoonup a_i(x,u,D_i u)$$

weakly in $L^{p_i'}(\Omega;\mathbb{R}^N)$, $\forall i = 1,\ldots,n$, with $p_i' = \frac{p_i}{p_i-1}$.

The proof of Theorems 3.1–3.3 are finished.

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