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PARAMETERIZED MORE ACCURATE HARDY–HILBERT–TYPE INEQUALITIES AND APPLICATIONS

YONG HONG, YANRU ZHONG* AND BICHENG YANG

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Abstract. By means of the weight coefficients, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate Hardy-Hilbert-type inequality with the general homogeneous kernel and the discrete intermediate variables is given. The equivalent form and a few equivalent statements of the best possible constant factor related to some parameters are obtained. As applications, the operator expressions, a few particular cases and some examples are considered.

1. Introduction

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\geqslant 0$, $0<\sum_{m=1}^{\infty}a_m^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$, we have a more accurate Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ as follows (cf. [3], Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

Since $\frac{1}{m+n}<\frac{1}{m+n-1}$ $(m,n\in\mathbf{N}=\{1,2,\cdots\})$, inequality (1) reduces to the following well known Hardy-Hilbert's inequality with the sane best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

Stting $\mu_i,\nu_i>0$ $(i,j\in\mathbf{N})$ and

$$U_m:=\sum_{i=1}^{m}\mu_i, V_n:=\sum_{j=1}^{n}\nu_j\ (m,n\in\mathbf{N}), \tag{3}$$

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* Corresponding author.

we still obtain the following Hardy-Hilbert-type inequality (cf. [1], Theorem 320):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\nu_{nj}^{q-1}}\right)^{\frac{1}{q}}. \tag{4}$$

For $\mu_i=\nu_i=1\ (i,j\in\mathbf{N})$, inequality (4) reduces to (2).

If $f(x),g(y)\geqslant 0$, $0<\int_0^\infty f^p(x)dx<\infty$ and $0<\int_0^\infty g^q(y)dy<\infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{5}$$

with the best constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], Theorem 316). In 1998, by introducing an independent parameter $\lambda>0$, Yang [23, 24] gave an extension of (2) (for $p=q=2$) with the best possible constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ as follows:

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x+y)^\lambda}dxdy$$
$$<B(\frac{\lambda}{2},\frac{\lambda}{2})\left(\int_0^\infty x^{1-\lambda}f^2(x)dx\int_0^\infty y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}}, \tag{6}$$

where, $B(u,v):=\int_0^\infty\frac{t^{u-1}}{(1+t)^{u+v}}dt\ \ (u,v>0)$ is the beta function.

Inequalities (1), (2) and (4)–(6) with their extensions are important in analysis and its applications (cf. [1, 2, 4, 5, 12, 15, 19–21, 25, 26, 30]).

The following half-discrete Hilbert-type inequality was provided (cf. [3], Theorem 351): if $K(x)\ \ (x>0)$ is a decreasing function, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\phi(s)=\int_0^\infty K(t)t^{s-1}dt<\infty$, then for $a_n\geqslant 0$, $0<\sum_{n=1}^\infty a_n^p<\infty$,

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^{\infty}K(nx)a_n\right)^p dx<\phi^p(\frac{1}{q})\sum_{n=1}^{\infty}a_n^p. \tag{7}$$

Some new extensions of (7) with their applications were provided by [6, 16–18, 27, 28].

In 2016, by the use of the technique of real analysis, Hong et al. [7] considered some equivalent statements of the extensions of (2) with the best possible constant factor related to a few parameters. The other similar works about the extensions of (5) and (6) were given by [8–11, 22].

In this paper, according to the way of [7], by means of the weight functions, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate Hardy-Hilbert-type inequality with the general homogeneous kernel and the discrete intermediate variables is given, which is a more accurate extension of (4). The equivalent form and the equivalent statements of the best possible constant factor related to some parameters are considered. As applications, the operator expressions, a few particular cases and some examples are obtained.

2. Some lemmas

In what follows, we suppose that $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ 0 < \alpha, \beta \leqslant 1,\ \lambda \in R_+ = (0, \infty),\ \lambda_2,\ \lambda - \lambda_1 \leqslant \frac{1}{\alpha},\ \lambda_1,\ \lambda - \lambda_2 \leqslant \frac{1}{\beta}$, both $\{\mu_m\}_{m=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ are positive decreasing sequences, with $\widehat{\mu} \in [0, \frac{\mu_1}{2}]$ and $\widehat{\nu} \in [0, \frac{\nu_1}{2}]$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, satisfying for any $u, x, y > 0$,

$$k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x,y).$$

Also, $k_\lambda(x,y)$ is a strictly decreasing and strictly convex function with respect to $x, y > 0$, such that

$$(-1)^i \frac{\partial}{\partial x^i} k_\lambda(x,y) > 0,\ \ (-1)^i \frac{\partial}{\partial y^i} k_\lambda(x,y) > 0\ \ (i = 1, 2),$$

and for $\gamma = \lambda_1,\ \lambda - \lambda_2$,

$$k_\lambda(\gamma) := \int_0^\infty k_\lambda(u, 1) u^{\gamma - 1} du \in \mathbf{R}_+. \tag{8}$$

Using the expressions (3), we still assume that $a_m, b_n \geqslant 0$, such that

$$0 < \sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p[1 - \alpha(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q})] - 1} \frac{a_m^p}{\mu_{m+1}^{p-1}} < \infty$$

and

$$0 < \sum_{n=1}^{\infty} (V_n - \widehat{\nu})^{q[1 - \beta(\frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p})] - 1} \frac{b_n^q}{\nu_{n+1}^{q-1}} < \infty.$$

We set $\mu(t) := \mu_m,\ t \in (m-1, m],\ \ \nu(t) := \nu_n,\ t \in (n-1, n]\ \ (m, n \in \mathbf{N})$, and

$$U(x) := \int_0^x \mu(t) dt,\ \ V(y) := \int_0^y \nu(t) dt\ \ (x, y \geqslant 0).$$

It follows that $U(m) = U_m,\ V(n) = V_n,\ U(\frac{1}{2}) = \frac{\mu_1}{2}, V(\frac{1}{2}) = \frac{\nu_1}{2}$ and

$$U'(x) =: \mu(x) = \mu_m,\ \ x \in (m-1, m],$$
$$V'(y) =: \nu(y) = \nu_n,\ \ y \in (n-1, n]\ \ (m, n \in \mathbf{N}).$$

LEMMA 1. *For fixed $m \in \mathbf{N}$, the following continuous function*

$$f_m(y) := k_\lambda((U_m - \widehat{\mu})^\alpha, (V(y) - \widehat{\upsilon})^\beta)(V(y) - \widehat{\upsilon})^{\lambda_2 \beta - 1}$$

is strictly decreasing and strictly convex with respect to $y \in (n - \frac{1}{2}, n + \frac{1}{2})\ \ (n \in \mathbf{N})$.

Proof. For $y \in (n-\frac{1}{2}, n)$, $0 < \beta \leqslant 1$, $\lambda_2\beta - 1 \leqslant 0$, we find

$$\begin{aligned} f_m'(y) = {} & [\beta k_\lambda'((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\nu})^\beta)(V(y)-\widehat{\upsilon})^{\beta-1} \\ & +(\lambda_2\beta-1)k_\lambda((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\upsilon})^\beta)(V(y)-\widehat{\upsilon})^{-1}] \\ & \times (V(y)-\widehat{\upsilon})^{\lambda_2\beta-1}\nu_n < 0. \end{aligned}$$

Replacing ν_n by ν_{n+1} in the above expression, we have $f_m'(y) < 0$ $(y \in (n, n+\frac{1}{2}))$. In view of $f_m(y)$ is continuous in $(n-\frac{1}{2}, n+\frac{1}{2})$, it follows that $f_m(y)$ is a strictly decreasing function with respect to $(n-\frac{1}{2}, n+\frac{1}{2})$ $(n \in \mathbf{N})$.

For $y \in (n-\frac{1}{2}, n)$, we find

$$\begin{aligned} f_m''(y) = {} & [\beta^2 k_\lambda''((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\nu})^\beta)(V(y)-\widehat{\upsilon})^{2\beta-2} \\ & +\beta(\beta-1)k_\lambda'((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\nu})^\beta)(V(y)-\widehat{\upsilon})^{\beta-2} \\ & +(\lambda_2\beta-1)\beta k_\lambda'((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\upsilon})^\beta)(V(y)-\widehat{\upsilon})^{\beta-2} \\ & +(1-\lambda_2\beta)k_\lambda((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\nu})^\beta)(V(y)-\widehat{\upsilon})^{-2}] \\ & \times (V(y)-\widehat{\upsilon})^{\lambda_2\beta-1}\nu_n^2 + [\beta k_\lambda'((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\nu})^\beta)(V(y)-\widehat{\upsilon})^{\beta-1} \\ & +(\lambda_2\beta-1)k_\lambda((U_m-\widehat{\mu})^\alpha, (V(y)-\widehat{\upsilon})^\beta)(V(y)-\widehat{\upsilon})^{-1}] \\ & \times (V(y)-\widehat{\upsilon})^{\lambda_2\beta-2}\nu_n^2 > 0. \end{aligned}$$

Replacing ν_n by ν_{n+1} in the above expression, it follows that $f_m''(y) > 0$ $(y \in (n, n+\frac{1}{2})$.

For $n \in \mathbf{N}$, since $\nu_n \geqslant \nu_{n+1} > 0$, we find that $f_m'(n-0) \geqslant f_m'(n+0)$, in the above expressions. In view of $f_m''(y) > 0$ $(y \in (n-\frac{1}{2}, n+\frac{1}{2}))$, it follows that $f_m'(y)$ is a strictly increasing function in $(n-\frac{1}{2}, n+\frac{1}{2})$, and then $f_m(y)$ is a strictly convex function with respect to $y \in (n-\frac{1}{2}, n+\frac{1}{2})$ $(n \in \mathbf{N})$.

The lemma is proved. □

DEFINITION 1. The following weight coefficients are defined: for $m, n \in \mathbf{N}$,

$$\omega(\lambda_2, m) := \sum_{n=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^\alpha, (V(n)-\widehat{\nu})^\beta)\frac{(V(n)-\widehat{\nu})^{\beta\lambda_2-1}\nu_{n+1}}{(U_m-\widehat{\mu})^{\alpha(\lambda_2-\lambda)}}. \tag{9}$$

$$\varpi(\lambda_1, n) := \sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^\alpha, (V(n)-\widehat{\nu})^\beta)\frac{(U_m-\widehat{\mu})^{\alpha\lambda_1-1}\mu_{m+1}}{(V(n)-\widehat{\nu})^{\beta(\lambda_1-\lambda)}}. \tag{10}$$

LEMMA 2. *The following inequalities are valid:*

$$\omega(\lambda_2, m) < \frac{1}{\beta}k_\lambda(\lambda-\lambda_2) \ \ (m \in \mathbf{N}), \tag{11}$$

$$\varpi(\lambda_1, n) < \frac{1}{\alpha}k_\lambda(\lambda_1) \ \ (n \in \mathbf{N}). \tag{12}$$

Proof. According to Lemma 1, by Hermite-Hadamard's inequality (cf. [13]), we find

$$\begin{aligned}\omega(\lambda_2,m) &\leqslant \sum_{n=1}^{\infty}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n-\widehat{\nu})^\beta)\frac{(V_n-\widehat{\nu})^{\beta\lambda_2-1}V'(y)}{(U_m-\widehat{\mu})^{\alpha(\lambda_2-\lambda)}}dy\\ &\leqslant \sum_{n=1}^{\infty}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} k_\lambda((U_m-\widehat{\mu})^\alpha,(V(y)-\widehat{\nu})^\beta)\frac{(V(y)-\widehat{\nu})^{\beta\lambda_2-1}V'(y)}{(U_m-\widehat{\mu})^{\alpha(\lambda_2-\lambda)}}dy\\ &= \int_{\frac{1}{2}}^{\infty} k_\lambda((U_m-\widehat{\mu})^\alpha,(V(y)-\widehat{\nu})^\beta)\frac{(V(y)-\widehat{\nu})^{\beta\lambda_2-1}}{(U_m-\widehat{\mu})^{\alpha(\lambda_2-\lambda)}}d(V(y)-\widehat{\nu}).\end{aligned}$$

Setting $t=\frac{(U_m-\widehat{\mu})^\alpha}{(V(y)-\widehat{\nu})^\beta}$, since $\widehat{\nu}\in[0,\frac{\upsilon_1}{2}]$, we find

$$\begin{aligned}\omega(\lambda_2,m) &< \frac{1}{\beta}\int_{\frac{(U_m-\widehat{\mu})^\alpha}{V(\infty)-\widehat{\nu})^\beta}}^{\frac{(U_m-\widehat{\mu})^\alpha}{V(\frac{1}{2})-\widehat{\nu})^\beta}} k_\lambda(t,1)t^{(\lambda-\lambda_2)-1}dt\\ &\leqslant \frac{1}{\beta}\int_0^\infty k_\lambda(t,1)t^{(\lambda-\lambda_2)-1}dt=\frac{1}{\beta}k_\lambda(\lambda-\lambda_2).\end{aligned}$$

Hence, we have (11).

In view of Lemma 1 and in the same way, for fixed $n\in\mathbf{N}$ and $0<\alpha\leqslant 1$, $\alpha\lambda_1-1\leqslant 0$, we can conclude that the following continuous function

$$g_n(x):=k_\lambda((U(x)-\widehat{\mu})^\alpha,(V_n-\widehat{\nu})^\beta)(U(x)-\widehat{\mu})^{\alpha\lambda_1-1}$$

is also a strictly decreasing and strictly convex function with respect to $x\in(m-\frac{1}{2},m+\frac{1}{2})$ $(m\in\mathbf{N})$. Setting, $u=\frac{(U(x)-\widehat{\mu})^\alpha}{(V_n-\widehat{\nu})^\beta}$, sine $\widehat{\mu}\in[0,\frac{\mu_1}{2}]$, we find

$$\begin{aligned}\varpi(\lambda_1,n) &\leqslant \sum_{m=1}^{\infty}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n-\widehat{\nu})^\beta)\frac{(U_m-\widehat{\mu})^{\alpha\lambda_1-1}U'(x)}{(V_n-\widehat{\nu})^{\beta(\lambda_1-\lambda)}}dx\\ &< \sum_{m=1}^{\infty}\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} k_\lambda((U(x)-\widehat{\mu})^\alpha,(V_n-\widehat{\nu})^\beta)\frac{(U(x)-\widehat{\mu})^{\alpha\lambda_1-1}U'(x)}{(V_n-\widehat{\nu})^{\beta(\lambda_1-\lambda)}}dx\\ &= \int_{\frac{1}{2}}^{\infty} k_\lambda((U(x)-\widehat{\mu})^\alpha,(V_n-\widehat{\nu})^\beta)\frac{(U(x)-\widehat{\mu})^{\alpha\lambda_1-1}}{(V_n-\widehat{\nu})^{\beta(\lambda_1-\lambda)}}d(U(x)-\widehat{\mu})\\ &= \frac{1}{\alpha}\int_{\frac{(U(\frac{1}{2})-\widehat{\mu})^\alpha}{(V_n-\widehat{\nu})^\beta}}^{\frac{(U\infty x)-\widehat{\mu})^\alpha}{(V_n-\widehat{\nu})^\beta}} k_\lambda(u,1)u^{\lambda_1-1}du\leqslant\frac{1}{\alpha}\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du=\frac{1}{\alpha}k_\lambda(\lambda_1).\end{aligned}$$

Hence, we have (12).

The lemma is proved. □

LEMMA 3. *The following inequality is valid:*

$$
\begin{aligned}
I &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^{\alpha},(V_n-\widehat{\nu})^{\beta})a_m b_n \\
&< \frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\sum_{n=1}^{\infty}(V_n-\widehat{\nu})^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right\}^{\frac{1}{q}}. \qquad (13)
\end{aligned}
$$

Proof. By Hölder's inequality with weight (cf. [13]), we obtain

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^{\alpha},(V_n-\widehat{\nu})^{\beta})\left[\frac{(V_n-\widehat{\nu})^{(\beta\lambda_2-1\}/p}\nu_{n+1}^{1/p}a_m}{(U_m-\widehat{\mu})^{(\alpha\lambda_1-1)/q}\mu_{m+1}^{1/q}}\right] \\
&\quad\times\left[\frac{(U_m-\widehat{\mu})^{(\alpha\lambda_1-1)/q}\mu_{m+1}^{1/q}b_n}{(V_n-\widehat{\nu})^{(\beta\lambda_2-1\}/p}\nu_{n+1}^{1/p}}\right] \\
&\leqslant \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^{\alpha},(V_n-\widehat{\nu})^{\beta})\frac{(V_n-\widehat{\nu})^{(\beta\lambda_2-1\}}\nu_{n+1}a_m^p}{(U_m-\widehat{\mu})^{(\alpha\lambda_1-1)(p-1)}\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^{\alpha},(V_n-\widehat{\nu})^{\beta})\frac{(U_m-\widehat{\mu})^{(\alpha\lambda_1-1)}\mu_{m+1}b_n^q}{(V_n-\widehat{\nu})^{(\beta\lambda_2-1\}(q-1)}\nu_{n+1}^{q-1}}\right]^{\frac{1}{q}} \\
&= \left\{\sum_{m=1}^{\infty}\omega(\lambda_2,m)(U_m-\widehat{\mu})^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\sum_{n=1}^{\infty}\varpi(\lambda_1,n)(V_n-\widehat{\nu})^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Then, using (11) and (12), we obtain (13).

The lemma is proved. □

REMARK 1. By (13) with the assumption, for $\lambda_1+\lambda_2=\lambda$, we find

$$
0<\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p(1-\alpha\lambda_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}<\infty,
$$
$$
0<\sum_{n=1}^{\infty}(V_n-\widehat{\nu})^{q(1-\beta\lambda_2)-1}\frac{b_n^q}{\nu_{n+1}^{q-1}}<\infty,
$$

and the following inequality:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n-\widehat{\nu})^\beta)a_m b_n \\
&< \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left[\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p(1-\alpha\lambda_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}} \\
&\times\left[\sum_{n=1}^{\infty}(V_n-\widehat{\nu})^{q(1-\beta\lambda_2)-1}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right]^{\frac{1}{q}}
\end{aligned} \tag{14}$$

In particular, for $\widehat{\mu}=\widehat{\nu}=0,$ we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m^\alpha,V_n^\beta)a_m b_n \\
&< \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left[\sum_{m=1}^{\infty}U_m^{p(1-\alpha\lambda_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}V_n^{q(1-\beta\lambda_2)-1}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right]^{\frac{1}{q}}
\end{aligned} \tag{15}$$

Hence, inequality (14) is a more accurate extension of (15).

For $\lambda=\alpha=\beta=1,\ k_1(x,y)=\frac{1}{x+y},\ \lambda_1=\frac{1}{q},\ \lambda_2=\frac{1}{p}$, inequality (14) reduces to the following more accurate extension of (4) (replacing $\mu_{m+1}\ (\nu_{n+1})$ by $\mu_m\ (\nu_n)$): for $\widehat{\kappa}\in[0,\frac{\mu_1+\nu_1}{}],$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n-\widehat{\kappa}} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right)^{\frac{1}{q}}. \tag{16}$$

In particular, for $\mu_m=\nu_n=\widehat{\kappa}=1$, inequality (16) reduces to (1).

LEMMA 4. *If $U(\infty)=V(\infty)=\infty$, then the constant factor $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ in (14) is the best possible.*

Proof. For any $\varepsilon>0$, we set

$$\widetilde{a}_m := U_m{}^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}\mu_{m+1}, \widetilde{b}_n := V_n{}^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}\nu_{n+1}\ \ (m,n\in\mathbf{N}).$$

If there exists a constant $M\ (\leqslant\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}})$, such that (14) is valid when we replace $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ by M. Then in particular, for $\widehat{\mu}=\widehat{\nu}=0,$ we have

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m^\alpha,V_n^\beta)\widetilde{a}_m\widetilde{b}_n \\
&< M\left[\sum_{m=1}^{\infty}U_m^{p(1-\alpha\lambda_1)-1}\frac{\widetilde{a}_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}V_n^{q(1-\beta\lambda_2)-1}\frac{\widetilde{b}_n^q}{\nu_{n+1}^{q-1}}\right]^{\frac{1}{q}}.
\end{aligned}$$

We obtain

$$
\begin{aligned}
\widetilde{I} &< M\left[\sum_{m=1}^{\infty} U_m^{p(1-\alpha\lambda_1)-1}\frac{U_m^{p\alpha(\lambda_1-\frac{\varepsilon}{p})-p}\mu_{m+1}^{p}}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} V_n^{q(1-\beta\lambda_2)-1}\frac{V_n^{q\beta(\lambda_2-\frac{\varepsilon}{q})-q}\nu_{n+1}^{q}}{\nu_{n+1}^{q-1}}\right]^{\frac{1}{q}}\\
&\leqslant M\left(\mu_1^{-\alpha\varepsilon-1}\mu_2+\sum_{m=2}^{\infty}U_m^{-\alpha\varepsilon-1}\mu_m\right)^{\frac{1}{p}}\left(\nu_1^{-\beta\varepsilon-1}\nu_2+\sum_{n=2}^{\infty}V_n^{-\beta\varepsilon-1}\nu_n\right)^{\frac{1}{q}}\\
&= M\left[\mu_1{}^{-\alpha\varepsilon-1}\mu_2+\sum_{m=2}^{\infty}\int_{m-1}^{m}U_m{}^{-\alpha\varepsilon-1}U'(x)dx\right]^{\frac{1}{p}}\\
&\quad\times\left[(\nu_1{}^{-\beta\varepsilon-1}\nu_2+\sum_{n=2}^{\infty}\int_{n-1}^{n}V_n{}^{-\beta\varepsilon-1}V'(y)dy\right]^{\frac{1}{q}}\\
&< M\left[\mu_1{}^{-\alpha\varepsilon-1}\mu_2+\sum_{m=2}^{\infty}\int_{m-1}^{m}U^{-\alpha\varepsilon-1}(x)dU(x)\right]^{\frac{1}{p}}\\
&\quad\times\left[\nu_1{}^{-\beta\varepsilon-1}\nu_2+\sum_{n=2}^{\infty}\int_{n-1}^{n}V^{-\beta\varepsilon-1}(y)dV(y)\right]^{\frac{1}{q}}\\
&= M\left(\mu_1{}^{-\alpha\varepsilon-1}\mu_2+\int_{1}^{\infty}U^{-\alpha\varepsilon-1}(x)dU(x)\right)^{\frac{1}{p}}\\
&\quad\times\left(\nu_1{}^{-\beta\varepsilon-1}\nu_2+\int_{1}^{\infty}V^{-\beta\varepsilon-1}(y)dV(y)\right)^{\frac{1}{q}}\\
&= \frac{1}{\varepsilon}M\left(\varepsilon\mu_1^{-\alpha\varepsilon-1}\mu_2+\frac{1}{\alpha}\mu_1^{-\alpha\varepsilon}\right)^{\frac{1}{p}}\left(\varepsilon\nu_1^{-\beta\varepsilon-1}\nu_2+\frac{1}{\beta}\nu_1^{-\beta\varepsilon}\right)^{\frac{1}{q}}.
\end{aligned}
$$

By the decreasingness property of series and Fubini theorem (cf. [14]), in view of $U(\infty)=V(\infty)=\infty$, we find

$$
\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda(U_m^{\alpha},V_n^{\beta})\frac{U_m^{\alpha\lambda_1-1}\mu_{m+1}}{U_m^{\alpha\varepsilon/p}}\frac{V_n^{\beta\lambda_2-1}\nu_{n+1}}{V_n^{\beta\varepsilon/q}}\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{n}^{n+1}\int_{m}^{m+1}k_\lambda(U_m^{\alpha},V_n^{\beta})\frac{U_m^{\alpha\lambda_1-1}}{U_m^{\alpha\varepsilon/p}}\frac{V_n^{\beta\lambda_2-1}}{V_n^{\beta\varepsilon/q}}U'(x)V'(y)dxdy\\
&\geqslant \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{n}^{n+1}\left[\int_{m}^{m+1}k_\lambda(U^{\alpha}(x),V^{\beta}(y))\frac{U^{\alpha\lambda_1-1}(x)}{U^{\alpha\varepsilon/p}(x)}U'(x)dx\right]\frac{V^{\beta\lambda_2-1}(y)}{V^{\beta\varepsilon/q}(y)}V'(y)dy\\
&= \int_{1}^{\infty}\left[\int_{1}^{\infty}k_\lambda(U^{\alpha}(x),V^{\beta}(y))\frac{U^{\alpha\lambda_1-1}(x)}{U^{\alpha\varepsilon/p}(x)}U'(x)dx\right]\frac{V^{\beta\lambda_2-1}(y)}{V^{\beta\varepsilon/q}(y)}V'(y)dy.
\end{aligned}
$$

Setting $u=\frac{U^{\alpha}(x)}{V^{\beta}(y)}$, we have

$$\begin{aligned}
\widetilde{I} &= \frac{1}{\alpha}\int_{1}^{\infty}\left(\int_{\frac{U^{\alpha}(1)}{V^{\beta}(y)}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)\frac{V^{\beta\lambda_2-1}(y)}{V^{\beta\varepsilon/q}(y)}V'(y)dy \\
&= \frac{1}{\alpha\beta}\int_{\nu_1^{\beta}}^{\infty}\left(\int_{\frac{\mu_1^{\alpha}}{t}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)t^{-\varepsilon-1}dt \\
&= \frac{1}{\alpha\beta}\left[\int_{\nu_1^{\beta}}^{\infty}\left(\int_{\frac{\mu_1^{\alpha}}{t}}^{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)t^{-\varepsilon-1}dt\right. \\
&\quad\left.+\int_{\nu_1^{\beta}}^{\infty}\left(\int_{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)t^{-\varepsilon-1}dt\right] \\
&= \frac{1}{\alpha\beta}\left[\int_{0}^{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}\left(\int_{\frac{\mu_1^{\alpha}}{u}}^{\infty} t^{-\varepsilon-1}dt\right)k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du+\frac{1}{\varepsilon\nu_1^{\beta\varepsilon}}\int_{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right] \\
&= \frac{1}{\varepsilon\alpha\beta}\left[\frac{1}{\mu_1^{\alpha\varepsilon}}\int_{0}^{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}} k_{\lambda}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du+\frac{1}{\nu_1^{\beta\varepsilon}}\int_{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right].
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\frac{1}{\alpha\beta}\left[\frac{1}{\mu_1^{\alpha\varepsilon}}\int_{0}^{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}} k_{\lambda}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du+\frac{1}{\nu_1^{\beta\varepsilon}}\int_{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right] \\
&\leqslant \varepsilon\widetilde{I}\leqslant M\left(\varepsilon\mu_1^{-\alpha\varepsilon-1}\mu_2+\frac{1}{\alpha}\mu_1^{-\alpha\varepsilon}\right)^{\frac{1}{p}}\left(\varepsilon\nu_1^{-\beta\varepsilon-1}\nu_2+\frac{1}{\beta}\nu_1^{-\beta\varepsilon}\right)^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon\to 0^{+}$, by Fatou lemma (cf. [14]), we find

$$\begin{aligned}
\frac{1}{\alpha\beta}k_{\lambda}(\lambda_1) &= \frac{1}{\alpha\beta}\left[\lim_{\varepsilon\to 0^{+}}\frac{1}{\mu_1^{\alpha\varepsilon}}\int_{0}^{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}\lim_{\varepsilon\to 0^{+}} k_{\lambda}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du\right. \\
&\qquad\left.+\lim_{\varepsilon\to 0^{+}}\frac{1}{\nu_1^{\beta\varepsilon}}\int_{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}^{\infty}\lim_{\varepsilon\to 0^{+}} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right] \\
&\leqslant \frac{1}{\alpha\beta}\lim_{\varepsilon\to 0^{+}}\left[\frac{1}{\mu_1^{\alpha\varepsilon}}\int_{0}^{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}} k_{\lambda}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du\right. \\
&\qquad\left.+\frac{1}{\nu_1^{\beta\varepsilon}}\int_{\frac{\mu_1^{\alpha}}{\nu_1^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right]\leqslant M\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\beta}\right)^{\frac{1}{q}},
\end{aligned}$$

namely, $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1) \leqslant M$. Hence, $M = \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1)$ is the best possible constant factor of (14).

The lemma is proved. □

REMARK 2. Setting $\widehat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\widehat{\lambda}_1 + \widehat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda,$$

$$\widehat{\lambda}_1 \leqslant \frac{1}{p\alpha} + \frac{1}{q\alpha} = \frac{1}{\alpha}, \quad \widehat{\lambda}_2 \leqslant \frac{1}{q\beta} + \frac{1}{p\beta} = \frac{1}{\beta},$$

and by Hölder's inequality (cf. [13]), we obtain

$$\begin{aligned} 0 &< k_\lambda(\lambda - \widehat{\lambda}_2) = k_\lambda(\widehat{\lambda}_1) = k_\lambda\left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}\right) \\ &= \int_0^\infty k_\lambda(u,1) u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} - 1} du = \int_0^\infty k_\lambda(u,1)(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}}) du \\ &\leqslant \left(\int_0^\infty k_\lambda(u,1) u^{\lambda-\lambda_2-1} du\right)^{\frac{1}{p}} \left(\int_0^\infty k_\lambda(u,1) u^{\lambda_1-1} du\right)^{\frac{1}{q}} \\ &= (k_\lambda(\lambda-\lambda_2))^{\frac{1}{p}} (k_\lambda(\lambda_1))^{\frac{1}{q}} < \infty. \end{aligned} \tag{17}$$

We can reduce (13) as follows:

$$\begin{aligned} I &< \frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^\infty (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{n=1}^\infty (V_n - \widehat{\nu})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{\nu_{n+1}^{q-1}}\right]^{\frac{1}{q}}, \end{aligned} \tag{18}$$

LEMMA 5. *If the constant factor* $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in* (13) *is the best possible, then.* $\lambda_1 + \lambda_2 = \lambda$.

Proof. If the constant factor $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (13) is the best possible, then by (18) and (14) (for $\lambda_1 = \widehat{\lambda}_1, \lambda_2 = \widehat{\lambda}_2$), we have the following inequality:

$$\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1) \leqslant \frac{k_\lambda(\widehat{\lambda}_1)}{\beta^{1/p}\alpha^{1/q}},$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \leqslant k_\lambda(\widehat{\lambda}_1)$. We observe that inequality (17) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [13]) $Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}$ *a.e.* in $\mathbf{R}_+$. Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_1-\lambda_2} = B/A$ *a.e.* in $\mathbf{R}_+$, and then $\lambda - \lambda_1 - \lambda_2 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

The lemma is proved. □

3. Main results and some particular cases

THEOREM 1. *Inequality* (18) *(or* (13)*) is equivalent to the following inequality:*

$$J := \left\{ \sum_{n=1}^{\infty} (V_n - \widehat{\nu})^{p\beta\widehat{\lambda}_2 - 1} \nu_{n+1} \left[\sum_{m=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{\nu})^\beta) a_m \right]^p \right\}^{\frac{1}{p}}$$
$$< \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right]^{\frac{1}{p}}. \tag{19}$$

If the constant factor in (18) *is the best possible, then, so is the constant factor in* (19).

Proof. Assuming that (19) is valid, by Hölder's inequality (cf. [13]), we find

$$I = \sum_{n=1}^{\infty} \left[\frac{(V_n - \widehat{\nu})^{\frac{-1}{p} + \beta\widehat{\lambda}_2}}{\nu_{n+1}^{-1/p}} \sum_{m=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{\nu})^\beta) a_m \right]$$
$$\times \left[(V_n - \widehat{\nu})^{\frac{1}{p} - \beta\widehat{\lambda}_2} \frac{b_n}{\nu_{n+1}^{1/p}} \right]$$
$$\leqslant J \left[\sum_{n=1}^{\infty} (V_n - \widehat{\nu})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{\nu_{n+1}^{q-1}} \right]^{\frac{1}{q}}. \tag{20}$$

Then by (19), we obtain (18).

On the other hand, assuming that (18) is valid, we set

$$b_n := (V_n - \widehat{\nu})^{p\beta\widehat{\lambda}_2 - 1} \nu_{n+1} \left[\sum_{m=1}^{\infty} k_\lambda((U_m - \widehat{\mu})^\alpha, (V_n - \widehat{\nu})^\beta) a_m \right]^{p-1}, \quad n \in \mathbf{N}.$$

If $J = 0$, then (19) is naturally valid; if $J = \infty$, then it is impossible that to make (19) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (18), it follows that

$$\sum_{n=1}^{\infty} (V_n - \widehat{\nu})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{\nu_{n+1}^{q-1}}$$
$$= J^p = I < \frac{k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^{\infty} (U_m - \widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1} \frac{a_m^p}{\mu_{m+1}^{p-1}} \right]^{\frac{1}{p}}$$
$$\times \left[\sum_{n=1}^{\infty} (V_n - \widehat{\nu})^{q(1-\beta\widehat{\lambda}_2)-1} \frac{b_n^q}{\nu_{n+1}^{q-1}} \right]^{\frac{1}{q}},$$

$$J=\left[\sum_{n=1}^{\infty}(V_n-\widehat{\nu})^{q(1-\beta\widehat{\lambda}_2)-1}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right]^{\frac{1}{p}}$$
$$<\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}},$$

namely, (19) follows, which is equivalent to (18).

If the constant factor in (18) is the best possible, then so is constant factor in (19). Otherwise, by (20) (for $\lambda_1+\lambda_2=\lambda$), we would reach a contradiction that the constant factor in (14) is not the best possible.

The theorem is proved. □

THEOREM 2. *If $U(\infty)=V(\infty)=\infty$, then the following statements (i), (ii), (iii), (iv) and (v) are equivalent:*

(i) Both $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ and $k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)$ are independent of p,q;

(ii) We have the following inequality:

$$k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\leqslant k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right);$$

(iii) $\lambda_1+\lambda_2=\lambda$;

(iv) the constant factor $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (18) (resp. (19)) is the best possible.

If the statement (iii) follows, namely, $\lambda_1+\lambda_2=\lambda$, then we have the following inequality equivalent to (14) with the best possible constant factor $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$:

$$\left\{\sum_{n=1}^{\infty}(V_n-\widehat{\nu})^{p\beta\lambda_2-1}\nu_{n+1}\left[\sum_{m=1}^{\infty}k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n-\widehat{\nu})^\beta)a_m\right]^p\right\}^{\frac{1}{p}}$$
$$<\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left[\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p(1-\alpha\lambda_1)-1}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}. \tag{21}$$

Proof. $(i)\Rightarrow(ii)$. Since both $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ and $k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are independent of p,q, we find

$$k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=\lim_{p\to\infty}\lim_{q\to1^+}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=k_\lambda(\lambda_1),$$

and by Fatou lemma (cf. [14]), we have the following inequality:

$$k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=\lim_{p\to\infty}\lim_{q\to1^+}k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)$$
$$\geqslant k_\lambda(\lambda_1)=k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1).$$

$(ii) \Rightarrow (iii)$. If $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) \leqslant k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (17) keeps the form of equality. Based on the proof of Lemma 5, it follows that $\lambda_1+\lambda_2=\lambda$.

$(iii) \Rightarrow (i)$. If $\lambda_1+\lambda_2=\lambda$, then we have

$$k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=k_\lambda(\lambda_1).$$

Both $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ and $k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are independent of p,q.

Hence, it follows that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

$(iii) \Rightarrow (iv)$. By Lemm 4 and Theorem 1, we obtain the conclusions.

$(iv) \Rightarrow (iii)$. By Theorem 1 and Lemma 5, we obtain $\lambda_1+\lambda_2=\lambda$.

Therefore, the statements (i), (ii), (iii), (iv) and (v) are equivalent.

The theorem is proved. □

REMARK 3. (i) For $\lambda=\alpha=\beta=1, \lambda_1=\frac{1}{q}, \lambda_1=\frac{1}{p}$ in (14) and (21), we have the following equivalent inequalities with the best possible constant factor:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_1(U_m-\widehat{\mu},V_n-\widehat{\nu})a_mb_n<k_1\left(\frac{1}{q}\right)\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right)^{\frac{1}{q}}, \tag{22}$$

$$\left[\sum_{n=1}^{\infty}\nu_{n+1}\left(\sum_{m=1}^{\infty}k_1(U_m-\widehat{\mu},V_n-\widehat{\nu})a_m\right)^p\right]^{\frac{1}{p}}<k_1(\frac{1}{q})\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right)^{\frac{1}{p}}. \tag{23}$$

(ii) For $\lambda=\alpha=\beta=1$, $\lambda_1=\frac{1}{p}$, $\lambda_1=\frac{1}{q}$ in (14) and (21), we have the following equivalent dual forms of (22) and (23) with the best possible constant factor:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_1(U_m-\widehat{\mu},V_n-\widehat{\nu})a_mb_n$$
$$<k_1\left(\frac{1}{p}\right)\left[\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p-2}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(V_n-\widehat{\nu})^{q-2}\frac{b_n^q}{\nu_{n+1}^{q-1}}\right]^{\frac{1}{q}}, \tag{24}$$

$$\left[\sum_{n=1}^{\infty}(V_n-\widehat{\nu})^{p-2}\nu_{n+1}\left(\sum_{m=1}^{\infty}k_1(U_m-\widehat{\mu},V_n-\widehat{\nu})a_m\right)^p\right]^{\frac{1}{p}}$$
$$<k_1\left(\frac{1}{p}\right)\left[\sum_{m=1}^{\infty}(U_m-\widehat{\mu})^{p-2}\frac{a_m^p}{\mu_{m+1}^{p-1}}\right]^{\frac{1}{p}}. \tag{25}$$

(iii) For $p=q=2$, both (22) and (24) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_1(U_m-\widehat{\mu},V_n-\widehat{\nu})a_mb_n<k_1\left(\frac{1}{2}\right)\left(\sum_{m=1}^{\infty}\frac{a_m^2}{\mu_{m+1}}\sum_{n=1}^{\infty}\frac{b_n^2}{\nu_{n+1}}\right)^{\frac{1}{2}}, \tag{26}$$

and both (23) and (25) reduce to the equivalent form of (26) as follows:

$$\left[\sum_{n=1}^{\infty} v_{n+1}\left(\sum_{m=1}^{\infty} k_1(U_m-\widehat{\mu},V_n-\widehat{\nu})a_m\right)^2\right]^{\frac{1}{2}} < k_1\left(\frac{1}{2}\right)\left(\sum_{m=1}^{\infty}\frac{a_m^2}{\mu_{m+1}}\right)^{\frac{1}{2}}. \quad (27)$$

4. Operator expressions and some particular inequalities

We set functions

$$\varphi(m) := (U_m-\widehat{\mu})^{p(1-\alpha\widehat{\lambda}_1)-1}\frac{1}{\mu_{m+1}^{p-1}}, \qquad \psi(n) := (V_n-\widehat{\nu})^{q(1-\beta\widehat{\lambda}_2)-1}\frac{1}{\nu_{n+1}^{q-1}},$$

wherefrom,

$$\psi^{1-p}(n) = (V_n-\widehat{\nu})^{p\beta\widehat{\lambda}_2-1}\nu_{n+1} \ \ (m,n\in\mathbf{N}).$$

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{a=\{a_m\}_{m=1}^{\infty}; ||a||_{p,\varphi} := \left(\sum_{m=1}^{\infty}\varphi(m)|a_m|^p\right)^{\frac{1}{p}} < \infty\right\},$$

$$l_{q,\psi} := \left\{b=\{b_n\}_{n=1}^{\infty}; ||b||_{q,\psi} := \left(\sum_{n=1}^{\infty}\psi(n)|b_n|^q\right)^{\frac{1}{q}} < \infty\right\},$$

$$l_{p,\psi^{1-p}} := \left\{c=\{c_n\}_{n=1}^{\infty}; ||c||_{p,\psi^{1-p}} := \left(\sum_{n=1}^{\infty}\psi^{1-p}(n)|c_n|^p\right)^{\frac{1}{p}} < \infty\right\}.$$

Assuming that $a\in l_{p,\varphi}$, setting

$$c=\{c_n\}_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^{\alpha},(V_n-\widehat{\nu})^{\beta})a_m, \ \ n\in\mathbf{N},$$

we can rewrite (19) as follows:

$$||c||_{p,\psi^{1-p}} < \frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi} < \infty,$$

namely, $c\in l_{p,\psi^{1-p}}$.

DEFINITION 2. Define a more accurate Hilbert-type operator $T: l_{p,\varphi}\to l_{p,\psi^{1-p}}$ as follows: For any $a\in l_{p,\varphi}$, there exists a unique representation $Ta=c\in l_{p,\psi^{1-p}}$, satisfying for any $n\in\mathbf{N}, Ta(n)=c_n$. Define the formal inner product of Ta and $b\in l_{q,\psi}$, and the norm of T as follows:

$$(Ta,b) := \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} k_\lambda((U_m-\widehat{\mu})^{\alpha},(V_n-\widehat{\nu})^{\beta})a_m\right)b_n = I,$$

$$||T|| := \sup_{a(\neq 0)\in l_{p,\varphi}} \frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 1 and Theorem 2, we have

THEOREM 3. *If $a \in l_{p,\varphi}$, $b \in l_{q,\psi}$, $||a||_{p,\varphi}$, $||b||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta,b) < \frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}||b||_{q,\psi}, \tag{28}$$

$$||Ta||_{p,\psi^{1-p}} < \frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}. \tag{29}$$

Moreover, if $U(\infty) = V(\infty) = \infty$, then, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor $\frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (28) (resp. (29)) is the best possible, namely,

$$||T|| = \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}. \tag{30}$$

EXAMPLE 1. We set $k_\lambda(x,y) = \frac{1}{(cx+y)^\lambda}$ $(c,\lambda > 0;\ x,y > 0)$. Then we find

$$k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n,-\widehat{\nu})^\beta) = \frac{1}{[c(U_m-\widehat{\mu})^\alpha + (V_n-\widehat{\nu})^\beta]^\lambda}.$$

For $0 < \alpha,\beta \leqslant 1$, $0 < \lambda_1$, $\lambda - \lambda_2 \leqslant \frac{1}{\alpha}$, $0 < \lambda_2$, $\lambda - \lambda_1 \leqslant \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is a strictly decreasing and strictly convex function with respect to $x,y > 0$, and for $\gamma = \lambda_1$, $\lambda - \lambda_2$

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}}{(cu+1)^\lambda} du = \frac{1}{c^\gamma} B(\gamma,\lambda-\gamma) \in \mathbf{R}_+.$$

In view of Theorem 3, it follows that if $U(\infty) = V(\infty) = \infty$, then, $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} = \frac{1}{\beta^{1/p}\alpha^{1/q}} \frac{1}{c^\gamma} B(\lambda_1,\lambda_2).$$

EXAMPLE 2. We set $k_\lambda(x,y) = \frac{\ln(cx/y)}{(cx)^\lambda - y^\lambda}$ $(c > 0,\ 0 < \lambda \leqslant 1;\ x,y > 0)$. Then we find

$$k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n,-\widehat{\nu})^\beta) = \frac{\ln[c(U_m-\widehat{\mu})^\alpha/(V_n-\widehat{\nu})^\beta]}{[c(U_m-\widehat{\mu})^\alpha]^\lambda - [(V_n-\widehat{\nu})^\beta]^\lambda}.$$

For $0 < \alpha,\beta \leqslant 1$, $0 < \lambda_1$, $\lambda - \lambda_2 \leqslant \frac{1}{\alpha}$, $0 < \lambda_2$, $\lambda - \lambda_1 \leqslant \frac{1}{\beta}$, $k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is a strictly decreasing and strictly convex function with respect to $x,y>0$ (cf. [25], Example 2.2.1), and for $\gamma=\lambda_1,\ \lambda-\lambda_2$

$$k_\lambda(\gamma)=\int_0^\infty \frac{\ln(cu)}{(cu)^\lambda-1}u^{\gamma-1}du=\frac{1}{c^\gamma}\left[\frac{\pi}{\sin(\pi\gamma/\lambda)}\right]^2\in\mathbf{R}_+.$$

In view of Theorem 3, it follows that if $U(\infty)=V(\infty)=\infty$, then, $\lambda_1+\lambda_2=\lambda$ if and only if

$$||T||=\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}=\frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{1}{c^\gamma}\left[\frac{\pi}{\sin(\pi\gamma/\lambda)}\right]^2.$$

EXAMPLE 3. For $s\in\mathbf{N}$, we set $k_\lambda(x,y)=\frac{1}{\prod_{k=1}^{s}(x^{\lambda/s}+c_ky^{\lambda/s})}$ $(0<c_1\leqslant\cdots\leqslant c_s,$ $0<\lambda\leqslant 1;\ x,y>0)$. Then we find

$$k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n,-\widehat{\nu})^\beta)=\frac{1}{\prod_{k=1}^{s}[(U_m-\widehat{\mu})^{\alpha\lambda/s}+c_k(V_n-\widehat{\nu})^{\beta\lambda/s}]}.$$

For $0<\alpha,\beta\leqslant 1,\ 0<\lambda_1,\ \lambda-\lambda_2\leqslant\frac{1}{\alpha},0<\lambda_2,\lambda-\lambda_1\leqslant\frac{1}{\beta},\ k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$ such that $k_\lambda(x,y)$ is a strictly decreasing and strictly convex function with respect to $x,y>0$, and for $\gamma=\lambda_1,\ \lambda-\lambda_2$, by Example 1 of [29], it follows that

$$\begin{aligned}k_\lambda^{(s)}(\gamma)&=\int_0^\infty\frac{u^{\gamma-1}}{\prod_{k=1}^{s}(u^{\lambda/s}+c_k)}du\\&=\frac{\pi s}{\lambda\sin(\frac{\pi s\gamma}{\lambda})}\sum_{k=1}^{s}c_k^{\frac{s\gamma}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j-c_k}\in\mathbf{R}_+.\end{aligned}$$

In view of Theorem 3, it follows that if $U(\infty)=V(\infty)=\infty$, then, $\lambda_1+\lambda_2=\lambda$ if and only if

$$||T||=\frac{k_\lambda^{(s)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}=\frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{\pi s}{\lambda\sin(\frac{\pi s\lambda_1}{\lambda})}\sum_{k=1}^{s}c_k^{\frac{s\lambda_1}{\lambda}-1}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j-c_k}.$$

In particular, for $c_1=\cdots=c_s=c$, we have $k_\lambda(x,y)=\frac{1}{(x^{\lambda/s}+cy^{\lambda/s})^s}$ and

$$k_\lambda^{(s)}(\gamma)=\int_0^\infty\frac{t^{\gamma-1}}{(t^{\lambda/s}+c)^s}dt=\frac{s}{\lambda c^{\lambda_2 s/\lambda}}B\left(\frac{s\lambda_1}{\lambda},\frac{s\lambda_2}{\lambda}\right).$$

If $s=1$, then we have $k_\lambda(x,y)=\frac{1}{x^\lambda+cy^\lambda}$,

$$k_\lambda((U_m-\widehat{\mu})^\alpha,(V_n,-\widehat{\nu})^\beta)=\frac{1}{(U_m-\widehat{\mu})^{\alpha\lambda}+c(V_n-\widehat{\nu})^{\beta\lambda}}.$$

and

$$||T|| = \frac{k_\lambda^{(s)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} = \frac{1}{\beta^{1/p}\alpha^{1/q}} \frac{\pi}{\lambda c^{\lambda_2/\lambda} \sin(\frac{\pi\lambda_1}{\lambda})}.$$

5. Conclusions

In this paper, by means of the weight coefficients, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate Hardy-Hilbert-type inequality with the general homogeneous kernel and the discrete intermediate variables is obtained, which is a more accurate extension of inequality (4). The equivalent forms are given in Theorem 1. The equivalent statements of the best possible constant factor related to some parameters are considered in Theorem 2. Some particular cases are given in Remark 3. As applications, the operator expressions and some examples are given in Theorem 3 and Example 1–3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Yong Hong
Department of Applied Mathematics
Guangzhou Huashang College
Guangdong, Guangzhou 511300, P. R. China
and
College of Mathematics and Statistics
Guangdong University and of Finance and Economics
Guangdong, Guangzhou 510320, P. R. China
e-mail: mathhongyong@163.com

Yanru Zhong
School of Computer Science and Information Security
Guilin and University of Electronic Technology
Guilin, Guangxi 541004, P. R. China
e-mail: bcyang@gdei.edu.cn

Bicheng Yang
School of Mathematics
Guangdong University of Education
Guangzhou, Guangdong 510303, P. R. China
e-mail: 18577399236@163.com