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MORE ACCURATE FORM OF HALF–DISCRETE HILBERT–TYPE INEQUALITY WITH A GENERAL KERNEL

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Abstract. In this work, by constructing a new kernel function in general form which includes both the homogeneous and the non-homogeneous cases, a half-discrete Hilbert-type inequality involving the newly constructed kernel function is established. Additionally, the equivalent Hardy-type inequalities are considered, and all the constant factors in the newly obtained inequalities are proved to be the best possible. Furthermore, by specializing the kernel function and introducing some special functions such as Beta function and Gamma function, some existing results and new examples are presented at the end of the paper.

1. Introduction

Let $p>1$, $a_n,\nu_n>0$ and $\boldsymbol{a}=\{a_n\}_{n=n_0}^{\infty}$. The sequence space $l_{p,\nu}$ is defined as follows:

$$l_{p,\nu}:=\left\{\boldsymbol{a}:\|\boldsymbol{a}\|_{p,\nu}:=\left(\sum_{n=n_0}^{\infty}a_n^p\nu_n\right)^{\frac{1}{p}}<\infty\right\}.$$

Specially, we abbreviate $\|\boldsymbol{a}\|_{p,\nu}$ to $\|\boldsymbol{a}\|_p$ and $l_{p,\nu}$ to l_p for $\nu_n=1$.

Let S be a measurable set and $p>1$. Suppose that $f(x)$ and $\mu(x)$ are two non-negative measurable functions defined on S. The function space $L_{p,\mu}$ is defined as follows:

$$L_{p,\mu}(S):=\left\{f:\|f\|_{p,\mu}:=\left[\int_S f^p(x)\mu(x)\mathrm{d}x\right]^{\frac{1}{p}}<\infty\right\}.$$

Specially, if $\mu(x)=1$, then $\|f\|_{p,\mu}$ and $L_{p,\mu}(S)$ are abbreviated as $\|f\|_p$ and $L_p(S)$ respectively.

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Consider a pair of conjugate parameters (p,q), $\frac{1}{p}+\frac{1}{q}=1$ $(p>1)$, and two real-valued functions $f,g \geqslant 0$, $f,g \in L_p(\mathbb{R}^+)$. Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\frac{\pi}{p}} \|f\|_p \|g\|_q. \tag{1.1}$$

The discrete form corresponding to (1.1) is as follows:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\frac{\pi}{p}} \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q, \tag{1.2}$$

where $\boldsymbol{a}=\{a_m\}_{m=1}^\infty \in l_p$, $\boldsymbol{b}=\{b_n\}_{n=1}^\infty \in l_q$, and the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ in both (1.1) and (1.2) is the best possible.

Generally, inequalities (1.1) and (1.2) are referred to as Hilbert-type inequalities [5]. Additionally, we have some other classical Hilbert-type inequalities, such as the following two related to logarithmic mean:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\log\frac{m}{n}}{m-n} < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2 \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q, \tag{1.3}$$

$$\int_0^\infty \int_0^\infty \frac{\log\frac{x}{y}}{x-y} f(x)g(y) \mathrm{d}x\mathrm{d}y < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2 \|f\|_p \|g\|_q. \tag{1.4}$$

Although these classical inequalities have been proposed by mathematicians for more than 100 years, their parameter extensions, more accurate forms and high dimensional generalizations have always attracted many researchers (see [2, 4, 9, 10, 16, 21, 22, 23, 24, 26]). Moreover, by the introduction of new kernel functions and parameters, and using the techniques of modern analysis, a large number of new discrete and integral Hilbert-type inequalities have been established in the past 30 years (see [3, 6, 9, 17, 18, 19, 24, 27, 28, 29]).

It should be pointed out that Hilbert-type inequalities also appear in half-discrete form, such as the following two [20, 25]:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{\log\frac{x}{n}}{x-n} a_n \mathrm{d}x < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2 \|f\|_p \|\boldsymbol{a}\|_q, \tag{1.5}$$

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{x+n} \mathrm{d}x < \frac{\pi}{\sin\frac{\pi}{p}} \|f\|_p \|\boldsymbol{a}\|_q. \tag{1.6}$$

Regarding other half-discrete inequalities, we refer to [1, 7, 8].

In addition to the above-mentioned Hilbert-type inequalities with specific kernel functions, some general forms of Hilbert-type inequalities have also been established by researchers in the past few years (see [11, 12, 13, 14]). However, these results with general form are usually obtained on the premise that the kernel function is homogeneous. In this paper, we will construct a general kernel function including both the homogeneous and non-homogeneous cases, and then establish a new more accurate half-discrete Hilbert-type inequality involving the newly constructed kernel function. Detailed definitions and lemmas will be presented in Section 2, and the main result and some examples will be presented in Section 3 and Section 4, respectively.

2. Definitions and lemmas

DEFINITION 2.1. Let $x,y>0$, and define Beta function [30] as follows:

$$B(x,y) := \int_0^{\infty} \frac{z^{x-1}}{(1+z)^{x+y}}\mathrm{d}z = B(y,x).$$

DEFINITION 2.2. For $x>0$, define Gamma function [30] as follows:

$$\Gamma(x) := \int_0^{\infty} z^{x-1}e^{-z}\mathrm{d}z.$$

In particular, if $x\in\mathbb{N}^+$, then $\Gamma(x)=(x-1)!$.

LEMMA 2.3. [15] *Let* $f^{(4)}(x)\in C(0,\infty)$, $(-1)^j f^{(j)}(x)>0$ $(j=0,1,2,3,4)$, *and* $f^{(j)}(\infty)=0$ $(j=0,1)$. *Then*

$$\sum_{m=0}^{\infty} f(m) < \int_0^{\infty} f(x)\mathrm{d}x + \frac{1}{2}f(0) - \frac{1}{12}f'(0).$$

LEMMA 2.4. *Let* $\alpha\neq 0$, $0<\beta\leqslant 1$, *and* $0<\beta\gamma\leqslant 1$. *Suppose that* $0\leqslant a\leqslant 1$, $S=(a,\infty)$ *for* $\alpha<0$, *and* $a\geqslant 1$, $S=(0,a)$ *for* $\alpha>0$. *Let*

$$b \geqslant \frac{1}{12}\max\left\{2\beta\sqrt{3\gamma(\gamma+1)}, 3\beta\gamma+\sqrt{3\beta\gamma(4-\beta\gamma)}\right\}.$$

Let $\kappa:\mathbb{R}^+\to\mathbb{R}^+$ *be a four-order differentiable function,* $(-1)^j\kappa^{(j)}(u)>0$, $j=1,2,3,4$, *and*

$$\lim_{\substack{u\to 0^+ \\ (u\to\infty)}} \kappa(u)u^{\gamma} = \lim_{\substack{u\to 0^+ \\ (u\to\infty)}} \kappa'(u)u^{\gamma+1} = 0.$$

Let

$$K(x,y) := \kappa\left(x^{\alpha}(y+b)^{\beta}\right),\ \ x>0,\ y\geqslant 0, \tag{2.1}$$

$$C(\kappa,\gamma) := \int_0^{\infty} \kappa(u)u^{\gamma-1}\mathrm{d}u. \tag{2.2}$$

Then

$$\omega(n) := \int_S K(x,n)x^{\alpha\gamma-1}\mathrm{d}x \leqslant \frac{C(\kappa,\gamma)}{|\alpha|(n+b)^{\beta\gamma}},\ \ n\in\mathbb{N}, \tag{2.3}$$

$$\varpi(x) := \sum_{n=0}^{\infty} K(x,n)(n+b)^{\beta\gamma-1} < \frac{C(\kappa,\gamma)}{\beta x^{\alpha\gamma}},\ \ x\in\mathbb{R}^+. \tag{2.4}$$

Proof. For $\alpha < 0$, setting $x^{\alpha}(n+b)^{\beta} = u$, and using (2.2), it follows that

$$\begin{aligned}\int_S K(x,n)x^{\alpha\gamma-1}\mathrm{d}x &= \int_a^{\infty} \kappa\left(x^{\alpha}(n+b)^{\beta}\right)x^{\alpha\gamma-1}\mathrm{d}x \\ &\leqslant \frac{1}{\mid \alpha \mid (n+b)^{\beta\gamma}}\int_0^{\infty}\kappa(u)u^{\gamma-1}\mathrm{d}u \\ &= \frac{C(\kappa,\gamma)}{\mid \alpha \mid (n+b)^{\beta\gamma}}. \end{aligned} \tag{2.5}$$

Therefore, (2.3) is obtained for $\alpha < 0$. Similarly, (2.3) can also be proved for $\alpha > 0$.

For arbitrary $x > 0,\ y \geqslant 0$, let

$$K(x,y) = \kappa(z),\ \ z = x^{\alpha}(y+b)^{\beta}.$$

In view of $0 < \beta \leqslant 1$, we get $\frac{\partial z}{\partial y} > 0$ and $(-1)^j\frac{\partial^j z}{\partial y^j} \leqslant 0,\ j = 2,3,4$. Observing that $(-1)^j\frac{\mathrm{d}^j\kappa}{\mathrm{d}z^j} > 0,\ j = 1,2,3,4,$ we have

$$\frac{\partial K}{\partial y} = \frac{\mathrm{d}\kappa}{\mathrm{d}z}\frac{\partial z}{\partial y} < 0, \tag{2.6}$$

$$\frac{\partial^2 K}{\partial y^2} = \frac{\mathrm{d}^2\kappa}{\mathrm{d}z^2}\left(\frac{\partial z}{\partial y}\right)^2 + \frac{\mathrm{d}\kappa}{\mathrm{d}z}\frac{\partial^2 z}{\partial y^2} > 0, \tag{2.7}$$

$$\frac{\partial^3 K}{\partial y^3} = \frac{\mathrm{d}^3\kappa}{\mathrm{d}z^3}\left(\frac{\partial z}{\partial y}\right)^3 + 3\frac{\mathrm{d}^2\kappa}{\mathrm{d}z^2}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial y^2} + \frac{\mathrm{d}\kappa}{\mathrm{d}z}\frac{\partial^3 z}{\partial y^3} < 0, \tag{2.8}$$

$$\begin{aligned}\frac{\partial^4 K}{\partial y^4} = \frac{\mathrm{d}^4\kappa}{\mathrm{d}z^4}\left(\frac{\partial z}{\partial y}\right)^4 &+ 6\frac{\mathrm{d}^3\kappa}{\mathrm{d}z^3}\left(\frac{\partial z}{\partial y}\right)^2\frac{\partial^2 z}{\partial y^2} + 3\frac{\mathrm{d}^2\kappa}{\mathrm{d}z^2}\left(\frac{\partial^2 z}{\partial y^2}\right)^2 \\ &+ 4\frac{\mathrm{d}^2\kappa}{\mathrm{d}z^2}\frac{\partial z}{\partial y}\frac{\partial^3 z}{\partial y^3} + \frac{\mathrm{d}\kappa}{\mathrm{d}z}\frac{\partial^4 z}{\partial y^4} > 0. \end{aligned} \tag{2.9}$$

That is,

$$(-1)^j\frac{\partial^j K}{\partial y^j} > 0,\ \ j = 1,2,3,4. \tag{2.10}$$

Let

$$F(x,y) := K(x,y)(y+b)^{\beta\gamma-1} := K(x,y)h(y).$$

Owing to $0 < \gamma\beta \leqslant 1$, it follows that

$$(-1)^j\frac{\mathrm{d}^j h}{\mathrm{d}y^j} \geqslant 0,\ \ j = 1,2,3,4. \tag{2.11}$$

By Leibniz's formula for higher derivative, and using (2.10) and (2.11), we get

$$(-1)^j \frac{\partial^j F}{\partial y^j} = (-1)^j \sum_{i=0}^{j} C_j^i \frac{\partial^i K}{\partial y^i} \frac{\mathrm{d}^{j-i} h}{\mathrm{d} y^{j-i}} > 0, \quad j = 1,2,3,4. \tag{2.12}$$

Additionally, in view of

$$\lim_{u\to\infty} \kappa(u) u^{\gamma} = \lim_{u\to\infty} \kappa'(u) u^{\gamma+1} = 0,$$

we have

$$\lim_{y\to\infty} F(x,y) = x^{-\alpha\gamma+\frac{\alpha}{\beta}} \lim_{u\to\infty} \kappa(u) u^{\gamma-\frac{1}{\beta}} = 0,$$

$$\lim_{y\to\infty} F_y'(x,y) = (\beta\gamma-1) x^{-\alpha\gamma+\frac{2\alpha}{\beta}} \lim_{u\to\infty} \kappa(u) u^{\gamma-\frac{2}{\beta}} + \beta x^{-\alpha\gamma+\frac{2\alpha}{\beta}} \lim_{u\to\infty} \kappa'(u) u^{\gamma+1-\frac{2}{\beta}} = 0.$$

By Lemma 2.3, we obtain

$$\begin{aligned} \varpi(x) = \sum_{n=0}^{\infty} F(x,n) &< \int_0^{\infty} F(x,y)\mathrm{d}y + \frac{1}{2} F(x,0) - \frac{1}{12} F_y'(x,0) \\ &= \int_0^{\infty} F(x,y)\mathrm{d}y + \frac{1}{12} K(x,0) b^{\beta\gamma-2}(6b-\beta\gamma+1) - \frac{1}{12} K_y'(x,0) b^{\beta\gamma-1} \\ &= \int_0^{\infty} F(x,y)\mathrm{d}y + \frac{1}{12} \kappa\left(x^{\alpha} b^{\beta}\right) b^{\beta\gamma-2}(6b-\beta\gamma+1) - \frac{\beta}{12} \kappa'\left(x^{\alpha} b^{\beta}\right) x^{\alpha} b^{\beta\gamma+\beta-2}. \end{aligned} \tag{2.13}$$

Furthermore, setting $x^{\alpha}(y+b)^{\beta} = u$, we get

$$\begin{aligned} \int_0^{\infty} F(x,y)\mathrm{d}y &= \int_0^{\infty} \kappa\left(x^{\alpha}(y+b)^{\beta}\right)(y+b)^{\beta\gamma-1}\mathrm{d}y = \frac{1}{\beta x^{\alpha\gamma}} \int_{x^{\alpha} b^{\beta}}^{\infty} \kappa(u) u^{\gamma-1}\mathrm{d}u \\ &= \frac{1}{\beta x^{\alpha\gamma}} \left[\int_0^{\infty} \kappa(u) u^{\gamma-1}\mathrm{d}u - \int_0^{x^{\alpha} b^{\beta}} \kappa(u) u^{\gamma-1}\mathrm{d}u \right]. \end{aligned} \tag{2.14}$$

By the formula of integration by parts, and observing that $\kappa''(u) > 0$, and

$$\lim_{u\to 0^+} \kappa(u) u^{\gamma} = \lim_{u\to 0^+} \kappa'(u) u^{\gamma+1} = 0,$$

it follow that

$$\begin{aligned} \int_0^{x^{\alpha} b^{\beta}} \kappa(u) u^{\gamma-1}\mathrm{d}u &= \frac{\left(x^{\alpha} b^{\beta}\right)^{\gamma}}{\gamma} \kappa\left(x^{\alpha} b^{\beta}\right) - \frac{1}{\gamma} \int_0^{x^{\alpha} b^{\beta}} \kappa'(u) u^{\gamma}\mathrm{d}u \\ &= \frac{\left(x^{\alpha} b^{\beta}\right)^{\gamma}}{\gamma} \kappa\left(x^{\alpha} b^{\beta}\right) - \frac{\left(x^{\alpha} b^{\beta}\right)^{\gamma+1}}{\gamma(\gamma+1)} \kappa'\left(x^{\alpha} b^{\beta}\right) \\ &\quad + \frac{1}{\gamma(\gamma+1)} \int_0^{x^{\alpha} b^{\beta}} \kappa''(u) u^{\gamma+1}\mathrm{d}u \\ &> \frac{\left(x^{\alpha} b^{\beta}\right)^{\gamma}}{\gamma} \kappa\left(x^{\alpha} b^{\beta}\right) - \frac{\left(x^{\alpha} b^{\beta}\right)^{\gamma+1}}{\gamma(\gamma+1)} \kappa'\left(x^{\alpha} b^{\beta}\right). \end{aligned} \tag{2.15}$$

Applying (2.15) to (2.14), and using (2.2), we get

$$\int_0^\infty F(x,y)\mathrm{d}y < \frac{C(\kappa,\gamma)}{\beta x^{\alpha\gamma}} - \frac{b^{\beta\gamma}}{\beta\gamma}\kappa\left(x^\alpha b^\beta\right) + \frac{x^\alpha b^{\beta\gamma+\beta}}{\beta\gamma(\gamma+1)}\kappa'\left(x^\alpha b^\beta\right). \tag{2.16}$$

Combining (2.13) and (2.16), we get

$$\begin{aligned}\varpi(x) < {} & \frac{C(\kappa,\gamma)}{\beta x^{\alpha\gamma}} - \frac{b^{\beta\gamma-2}}{12\beta\gamma}\kappa\left(x^\alpha b^\beta\right)\left(12b^2-6b\beta\gamma+\beta^2\gamma^2-\beta\gamma\right) \\ & + \frac{x^\alpha b^{\beta\gamma+\beta-2}}{12\beta\gamma(\gamma+1)}\kappa'\left(x^\alpha b^\beta\right)\left[12b^2-\beta^2\gamma(\gamma+1)\right].\end{aligned} \tag{2.17}$$

By virtue of

$$b \geqslant \frac{1}{12}\max\left\{2\beta\sqrt{3\gamma(\gamma+1)}, 3\beta\gamma+\sqrt{3\beta\gamma(4-\beta\gamma)}\right\},$$

we have

$$\begin{aligned}12b^2-6b\beta\gamma+\beta^2\gamma^2-\beta\gamma = {} & 12\left[b-\frac{1}{12}\left(3\beta\gamma+\sqrt{3\beta\gamma(4-\beta\gamma)}\right)\right] \\ & \times\left[b-\frac{1}{12}\left(3\beta\gamma-\sqrt{3\beta\gamma(4-\beta\gamma)}\right)\right] \geqslant 0,\end{aligned} \tag{2.18}$$

$$12b^2-\beta^2\gamma(\gamma+1) = 12\left(b-\frac{\beta}{6}\sqrt{3\gamma(\gamma+1)}\right)\left(b+\frac{\beta}{6}\sqrt{3\gamma(\gamma+1)}\right) \geqslant 0. \tag{2.19}$$

Applying (2.18) and (2.19) to (2.17), and in view of $\kappa > 0$, $\kappa' < 0$, we get (2.4). Lemma 2.4 is proved. □

LEMMA 2.5. *Let $\alpha \neq 0$, $0 < \beta \leqslant 1$, $\gamma > 1$, and $0 < \gamma\beta \leqslant 1$. Suppose that $0 \leqslant a \leqslant 1$, $S = (a,\infty)$ for $\alpha < 0$, and $a \geqslant 1$, $S = (0,a)$ for $\alpha > 0$. Assume that $b > 0$, and $\kappa(u)$, $K(x,y)$ are defined via Lemma 2.4. For an arbitrary positive natural number l which is large enough, set*

$$\tilde{\boldsymbol{a}} := \{\tilde{a}_n\}_{n=0}^\infty := \left\{(n+b)^{\beta\gamma-1-\frac{\beta}{ql}}\right\}_{n=0}^\infty,$$

$$\tilde{f}(x) := \begin{cases} x^{\alpha\gamma-1+\frac{\alpha}{pl}} & x \in I \\ 0 & x \in S\setminus I \end{cases},$$

where $I = \{x : x > 0,\ x^{\operatorname{sgn}\alpha} < 1\}$. Then

$$\begin{aligned}\tilde{L} := {} & \sum_{n=0}^\infty \tilde{a}_n \int_I K(x,n)\tilde{f}(x)\mathrm{d}x = \int_I \tilde{f}(x)\sum_{n=0}^\infty \tilde{a}_n K(x,n)\mathrm{d}x \\ & > \frac{l}{\mid\alpha\beta\mid}\left(\int_1^\infty \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u + b^{\frac{-\beta}{l}}\int_0^1 \kappa(u)u^{\gamma+\frac{1}{pl}-1}\mathrm{d}u\right).\end{aligned} \tag{2.20}$$

Proof. Observing that $0 < \beta\gamma \leqslant 1$, and using (2.6), we have

$$\tilde{L} > \int_I x^{\alpha\gamma-1+\frac{\alpha}{pl}} \int_0^\infty K(x,y)(y+b)^{\beta\gamma-1-\frac{\beta}{ql}} \mathrm{d}y\mathrm{d}x. \tag{2.21}$$

Setting $x^\alpha(y+b)^\beta = u$, we obtain

$$\begin{aligned}\tilde{L} &> \frac{1}{\beta}\int_I x^{\frac{\alpha}{l}-1}\left[\int_{x^\alpha b^\beta}^\infty \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u\right]\mathrm{d}x \\ &= \frac{1}{\beta}\int_I x^{\frac{\alpha}{l}-1}\left[\int_1^\infty \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u\right]\mathrm{d}x + \frac{1}{\beta}\int_I x^{\frac{\alpha}{l}-1}\left[\int_{x^\alpha b^\beta}^1 \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u\right]\mathrm{d}x \\ &= \frac{l}{\mid\alpha\beta\mid}\int_1^\infty \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u + \frac{1}{\beta}\int_I x^{\frac{\alpha}{l}-1}\left[\int_{x^\alpha b^\beta}^1 \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u\right]\mathrm{d}x. \end{aligned} \tag{2.22}$$

If $\alpha > 0$, by Fubini's theorem, then we have

$$\begin{aligned}\int_I x^{\frac{\alpha}{l}-1}\left[\int_{x^\alpha b^\beta}^1 \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u\right]\mathrm{d}x &= \int_0^1 \kappa(u)u^{\gamma-\frac{1}{ql}-1}\left(\int_0^{u^{\frac{1}{\alpha}}b^{\frac{-\beta}{\alpha}}} x^{\frac{\alpha}{l}-1}\mathrm{d}x\right)\mathrm{d}u \\ &= \frac{lb^{\frac{-\beta}{l}}}{\alpha}\int_0^1 \kappa(u)u^{\gamma+\frac{1}{pl}-1}\mathrm{d}u. \end{aligned} \tag{2.23}$$

If $\alpha < 0$, by Fubini's theorem again, then we obtain

$$\begin{aligned}\int_I x^{\frac{\alpha}{l}-1}\left[\int_{x^\alpha b^\beta}^1 \kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u\right]\mathrm{d}x &= \int_0^1 \kappa(u)u^{\gamma-\frac{1}{ql}-1}\left(\int_{u^{\frac{1}{\alpha}}b^{\frac{-\beta}{\alpha}}}^\infty x^{\frac{\alpha}{l}-1}\mathrm{d}x\right)\mathrm{d}u \\ &= \frac{lb^{\frac{-\beta}{l}}}{\mid\alpha\mid}\int_0^1 \kappa(u)u^{\gamma+\frac{1}{pl}-1}\mathrm{d}u. \end{aligned} \tag{2.24}$$

Combining (2.22), (2.23) and (2.24), we arrive at (2.20). Lemma 2.5 is proved. □

3. Main results

THEOREM 3.1. *Let $\alpha \neq 0$, $0 < \beta \leqslant 1$ and $0 < \beta\gamma \leqslant 1$. Suppose that $0 \leqslant a \leqslant 1$, $S = (a,\infty)$ for $\alpha < 0$, and $a \geqslant 1$, $S = (0,a)$ for $\alpha > 0$. Let*

$$b \geqslant \frac{1}{12}\max\left\{2\beta\sqrt{3\gamma(\gamma+1)}, 3\beta\gamma + \sqrt{3\beta\gamma(4-\beta\gamma)}\right\}.$$

Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\mu(x) = x^{p(1-\alpha\gamma)-1}$ and $\nu_n = (n+b)^{q(1-\beta\gamma)-1}$. Suppose that $f(x)$, $a_n \geqslant 0$ with $f(x) \in L_{p,\mu}(S)$ and $\boldsymbol{a} = \{a_n\}_{n=0}^\infty \in l_{q,\nu}$. Let $\kappa(u)$, $K(x,y)$ and $C(\kappa,\gamma)$ be defined via Lemma 2.4. Then the following inequalities hold and are equiv-

alent:

$$L_1 := \sum_{n=0}^{\infty}(n+b)^{p\beta\gamma-1}\left(\int_S K(x,n)f(x)\mathrm{d}x\right)^p < \left[|\,\alpha\,|^{-\frac{1}{q}}\,\beta^{-\frac{1}{p}}C(\kappa,\gamma)\right]^p \|f\|_{p,\mu}^p, \quad (3.1)$$

$$L_2 := \int_S x^{q\alpha\gamma-1}\left(\sum_{n=0}^{\infty}K(x,n)a_n\right)^q \mathrm{d}x < \left[|\,\alpha\,|^{-\frac{1}{q}}\,\beta^{-\frac{1}{p}}C(\kappa,\gamma)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \quad (3.2)$$

$$\begin{aligned} L &:= \sum_{n=0}^{\infty}a_n\int_S K(x,n)f(x)\mathrm{d}x = \int_S f(x)\sum_{n=0}^{\infty}K(x,n)a_n\mathrm{d}x \\ &<|\,\alpha\,|^{-\frac{1}{q}}\,\beta^{-\frac{1}{p}}C(\kappa,\gamma)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \end{aligned} \quad (3.3)$$

where the constant $|\,\alpha\,|^{-\frac{1}{q}}\,\beta^{-\frac{1}{p}}C(\kappa,\gamma)$ *in* (3.1), (3.2) *and* (3.3) *is the best possible.*

Proof. By Hölder's inequality and inequality (2.3), it follows that

$$\begin{aligned} &\left(\int_S K(x,n)f(x)\mathrm{d}x\right)^p \\ &= \left[\int_S \left((K(x,n))^{\frac{1}{p}}x^{\frac{1-\alpha\gamma}{q}}f(x)\right)\left((K(x,n))^{\frac{1}{q}}x^{\frac{\alpha\gamma-1}{q}}\right)\mathrm{d}x\right]^p \\ &\leqslant \int_S K(x,n)x^{\frac{p(1-\alpha\gamma)}{q}}f^p(x)\mathrm{d}x\left(\int_S K(x,n)x^{\alpha\gamma-1}\mathrm{d}x\right)^{p-1} \\ &= (\omega(n))^{p-1}\int_S K(x,n)x^{\frac{p(1-\alpha\gamma)}{q}}f^p(x)\mathrm{d}x \\ &\leqslant \left[\frac{C(\kappa,\gamma)}{|\,\alpha\,|\,(n+b)^{\beta\gamma}}\right]^{p-1}\int_S K(x,n)x^{\frac{p(1-\alpha\gamma)}{q}}f^p(x)\mathrm{d}x. \end{aligned} \quad (3.4)$$

Plugging (3.4) back into the left hand side of inequality (3.1), and using Lebesgue term-by-term integration theorem, we get

$$L_1 \leqslant \left[\frac{C(\kappa,\gamma)}{|\,\alpha\,|}\right]^{p-1}\int_S f^p(x)x^{\frac{p(1-\alpha\gamma)}{q}}\sum_{n=0}^{\infty}K(x,n)(n+b)^{\beta\gamma-1}\mathrm{d}x. \quad (3.5)$$

Applying (2.4) to (3.5), we arrive at (3.1). In a similar way, it can also be proved inequality (3.2) holds true. In fact, by the use of Hölder's inequality again and (2.4), we obtain

$$\begin{aligned} \left(\sum_{n=0}^{\infty}K(x,n)a_n\right)^q &= \left[\sum_{n=0}^{\infty}K(x,n)(n+b)^{\frac{\beta\gamma-1}{p}}\left((n+b)^{\frac{1-\beta\gamma}{p}}a_n\right)\right]^q \\ &\leqslant [\varpi(x)]^{q-1}\sum_{n=0}^{\infty}K(x,n)(n+b)^{\frac{q(1-\beta\gamma)}{p}}a_n^q \\ &< \left[\frac{C(\kappa,\gamma)}{\beta x^{\alpha\gamma}}\right]^{q-1}\sum_{n=0}^{\infty}K(x,n)(n+b)^{\frac{q(1-\beta\gamma)}{p}}a_n^q. \end{aligned} \quad (3.6)$$

By Lebesgue term-by-term integration theorem and (2.3), it follows that

$$L_2 < \left[\frac{C(\kappa,\gamma)}{\beta}\right]^{q-1} \sum_{n=0}^{\infty}(n+b)^{\frac{q(1-\beta\gamma)}{p}} a_n^q \int_S K(x,n)x^{\alpha\gamma-1}\mathrm{d}x$$
$$< \left[|\alpha|^{-\frac{1}{q}} \beta^{-\frac{1}{p}} C(\kappa,\gamma)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q.$$

Inequalities (3.1) and (3.2) are proved. Moreover, we will prove (3.3) via (3.1). In fact, we can first get two expressions of L by Lebesgue term-by-term integration theorem, and then it follows from Hölder's inequality that

$$L = \sum_{n=0}^{\infty}\left[(n+b)^{\beta\gamma-\frac{1}{p}} \int_S K(x,n)f(x)\mathrm{d}x\right]\left[a_n(n+b)^{-\beta\gamma+\frac{1}{p}}\right]$$
$$\leqslant L_1^{\frac{1}{p}}\left[\sum_{n=0}^{\infty} a_n^q (n+b)^{q(1-\beta\gamma)-1}\right]^{\frac{1}{q}} = L_1^{\frac{1}{p}}\|\boldsymbol{a}\|_{q,\nu}. \tag{3.7}$$

Plugging (3.1) back into (3.7), we get (3.3). Conversely, assume (3.3) is valid, and set $\boldsymbol{b} = \{b_n\}_{n=0}^{\infty}$, where

$$b_n := (n+b)^{p\beta\gamma-1}\left(\int_S K(x,n)f(x)\mathrm{d}x\right)^{p-1}.$$

By virtue of (3.3), we obtain

$$L_1 = \sum_{n=0}^{\infty} b_n \int_S K(x,n)f(x)\mathrm{d}x < |\alpha|^{-\frac{1}{q}} \beta^{-\frac{1}{p}} C(\kappa,\gamma)\|f\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}$$
$$= |\alpha|^{-\frac{1}{q}} \beta^{-\frac{1}{p}} C(\kappa,\gamma)\|f\|_{p,\mu} L_1^{\frac{1}{q}}. \tag{3.8}$$

It follows from (3.8) that (3.1) holds true. Therefore, inequality (3.1) is equivalent to (3.3). For the sake of proving the equivalence of inequalities (3.1), (3.2) and (3.3), it is suffices to prove (3.2) is equivalent to (3.3). In fact, if (3.2) is supposed to be true, then

$$L = \int_S \left[x^{-\alpha\gamma+\frac{1}{q}} f(x)\right]\left[x^{\alpha\beta-\frac{1}{q}} \sum_{n=0}^{\infty} K(x,n)a_n\right]\mathrm{d}x \leqslant \|f\|_{p,\mu} L_2^{\frac{1}{q}}. \tag{3.9}$$

Apply (3.2) to (3.9), then it follows (3.3). On the contrary, Suppose that (3.3) holds true, and set

$$g(x) := x^{q\alpha\gamma-1}\left(\sum_{n=0}^{\infty} K(x,n)a_n\right)^{q-1}.$$

Then

$$L_2 = \int_S g(x)\sum_{n=0}^{\infty} K(x,n)a_n \mathrm{d}x < |\alpha|^{-\frac{1}{q}} \beta^{-\frac{1}{p}} C(\kappa,\gamma)\|g\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}$$
$$= |\alpha|^{-\frac{1}{q}} \beta^{-\frac{1}{p}} C(\kappa,\gamma)\|\boldsymbol{a}\|_{q,\nu} L_2^{\frac{1}{p}}. \tag{3.10}$$

Inequality (3.1) follows naturally from (3.10), and the proof of the equivalence of inequalities (3.1), (3.2) and (3.3) is completed.

In order to complete the proof of Theorem 3.1, we also need to prove that the constant $|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\kappa,\gamma)$ in (3.1), (3.2) and (3.3) is the best possible. Assume that there exists a real number c satisfying

$$0<c\leqslant|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\kappa,\gamma) \tag{3.11}$$

so that (3.3) holds, that is,

$$\sum_{n=0}^{\infty}a_n\int_S K(x,n)f(x)\mathrm{d}x=\int_S f(x)\sum_{n=0}^{\infty}K(x,n)a_n\mathrm{d}x<c\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{3.12}$$

Replace f and a_n in (3.12) with $\tilde{f}$ and $\tilde{a}_n$ defined in Lemma 2.5, respectively. By (2.20) and (3.12), it follows that

$$\begin{aligned}
&\frac{l}{|\alpha\beta|}\left(\int_1^{\infty}\kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u+b^{\frac{-\beta}{l}}\int_0^1\kappa(u)u^{\gamma+\frac{1}{pl}-1}\mathrm{d}u\right)\\
&<c\|\tilde{f}\|_{p,\mu}\|\tilde{\boldsymbol{a}}\|_{q,\nu}=c\left(\int_I x^{\frac{\alpha}{l}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left[\sum_{n=0}^{\infty}(n+b)^{-\frac{\beta}{l}-1}\right]^{\frac{1}{q}}\\
&=c\left(\frac{l}{|\alpha|}\right)^{\frac{1}{p}}\left[b^{-\frac{\beta}{l}-1}+\sum_{n=1}^{\infty}(n+b)^{-\frac{\beta}{l}-1}\right]^{\frac{1}{q}}\\
&<c\left(\frac{l}{|\alpha|}\right)^{\frac{1}{p}}\left[b^{-\frac{\beta}{l}-1}+\int_0^{\infty}(y+b)^{-\frac{\beta}{l}-1}\mathrm{d}y\right]^{\frac{1}{q}}\\
&=c\left(\frac{l}{|\alpha|}\right)^{\frac{1}{p}}\left(b^{-\frac{\beta}{l}-1}+\frac{lb^{-\frac{\beta}{l}}}{\beta}\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.13}$$

That is,

$$\int_1^{\infty}\kappa(u)u^{\gamma-\frac{1}{ql}-1}\mathrm{d}u+b^{\frac{-\beta}{l}}\int_0^1\kappa(u)u^{\gamma+\frac{1}{pl}-1}\mathrm{d}u<c\beta|\alpha|^{\frac{1}{q}}\left(\frac{b^{-\frac{\beta}{l}-1}}{l}+\frac{b^{-\frac{\beta}{l}}}{\beta}\right)^{\frac{1}{q}}.$$

Letting $l\to+\infty$, and using (2.2), we have

$$c\geqslant|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\kappa,\gamma). \tag{3.14}$$

Combining (3.11) and (3.14), we get $c=|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\kappa,\gamma)$, and it follows therefore that $|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\kappa,\gamma)$ in (3.3) is the best possible. Additionally, from the equivalence of (3.1), (3.2) and (3.3), the constant $|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\kappa,\gamma)$ in (3.1) and (3.2) can also be proved to be the best possible. Theorem 3.1 is proved. □

4. Some examples

In this section, it is assumed that α,β,γ,a,b satisfy the conditions of Theorem 3.1, except where specially noted. Let $\kappa(u)$ take some specific functions, we can get several examples as follows.

EXAMPLE 4.1. Let $\kappa(u)=\frac{1}{(1+u)^{\lambda}}$, $\lambda>\gamma>0$. Then $(-1)^{j}\frac{\mathrm{d}^{j}\kappa}{\mathrm{d}u^{j}}>0,\ j=1,2,3,4,$

$$\lim_{\substack{u\to 0^{+}\\(u\to\infty)}}\kappa(u)u^{\gamma}=\lim_{\substack{u\to 0^{+}\\(u\to\infty)}}\kappa^{'}(u)u^{\gamma+1}=0,$$

and

$$C(\kappa,\gamma)=\int_{0}^{\infty}\frac{u^{\gamma-1}}{(1+u)^{\lambda}}\mathrm{d}u=B(\gamma,\lambda-\gamma):=B(\gamma,\tau).$$

Therefore, inequalities (3.1), (3.2), and (3.3) reduce to

$$\sum_{n=0}^{\infty}(n+b)^{p\beta\gamma-1}\left[\int_{S}\left(1+x^{\alpha}(n+b)^{\beta}\right)^{-\lambda}f(x)\mathrm{d}x\right]^{p}<\left[|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}B(\gamma,\tau)\right]^{p}\|f\|_{p,\mu}^{p},\tag{4.1}$$

$$\int_{S}x^{q\alpha\gamma-1}\left[\sum_{n=0}^{\infty}\left(1+x^{\alpha}(n+b)^{\beta}\right)^{-\lambda}a_{n}\right]^{q}\mathrm{d}x<\left[|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}B(\gamma,\tau)\right]^{q}\|\boldsymbol{a}\|_{q,\nu}^{q},\tag{4.2}$$

$$\int_{S}f(x)\sum_{n=0}^{\infty}\left(1+x^{\alpha}(n+b)^{\beta}\right)^{-\lambda}a_{n}\mathrm{d}x<|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}B(\gamma,\tau)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},\tag{4.3}$$

where $\mu(x)=x^{p(1-\alpha\gamma)-1}$, $\nu_{n}=(n+b)^{q(1-\beta\gamma)-1}$, and $\gamma+\tau=\lambda$.

If $\alpha<0$, replacing α with $-\alpha$ $(\alpha>0)$, and $x^{\alpha\lambda}f(x)$ with $f(x)$, then inequalities (4.1), (4.2) and (4.3) are transformed into the following three inequalities:

$$\sum_{n=0}^{\infty}(n+b)^{p\beta\gamma-1}\left[\int_{a}^{\infty}\left(x^{\alpha}+(n+b)^{\beta}\right)^{-\lambda}f(x)\mathrm{d}x\right]^{p}<\left[\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}B(\gamma,\tau)\right]^{p}\|f\|_{p,\mu}^{p},\tag{4.4}$$

$$\int_{a}^{\infty}x^{q\alpha\tau-1}\left[\sum_{n=0}^{\infty}\left(x^{\alpha}+(n+b)^{\beta}\right)^{-\lambda}a_{n}\right]^{q}\mathrm{d}x<\left[\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}B(\gamma,\tau)\right]^{q}\|\boldsymbol{a}\|_{q,\nu}^{q},\tag{4.5}$$

$$\int_{a}^{\infty}f(x)\sum_{n=0}^{\infty}\left(x^{\alpha}+(n+b)^{\beta}\right)^{-\lambda}a_{n}\mathrm{d}x<\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}B(\gamma,\tau)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},\tag{4.6}$$

where $\mu(x)=x^{p(1-\alpha\tau)-1}$ and $\nu_{n}=(n+b)^{q(1-\beta\gamma)-1}$.

Setting $\alpha=\beta$, $b=s$ $(s\in\mathrm{N}^{+})$ in (4.6), we get a Hilbert-type inequality with a homogeneous kernel of degree $-\beta\lambda$ as follows:

$$\int_{a}^{\infty}f(x)\sum_{n=s}^{\infty}\left(x^{\beta}+n^{\beta}\right)^{-\lambda}a_{n}\mathrm{d}x<\frac{1}{\beta}B(\gamma,\tau)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\tag{4.7}$$

Setting $a=0$ and $s=\beta=1$ in (4.7), then (4.7) is transformed into the result of Yang [25]. If $\alpha>0$, setting $\alpha=\beta$ in (4.3), we get

$$\int_0^a f(x)\sum_{n=0}^{\infty}\left[1+x^\beta(n+b)^\beta\right]^{-\lambda}a_n\mathrm{d}x<\frac{1}{\beta}B(\gamma,\tau)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{4.8}$$

EXAMPLE 4.2. Let $\kappa(u)=e^{-\lambda u}$ $(\lambda>0)$ in Theorem 3.1. Obviously, $\kappa(u)$ satisfies the conditions of Theorem 3.1 (Lemma 2.4). Additionally, we have

$$C(\kappa,\gamma)=\int_0^\infty e^{-\lambda u}u^{\gamma-1}\mathrm{d}u=\lambda^{-\gamma}\Gamma(\gamma).$$

Then inequalities (3.1), (3.2), and (3.3) reduce to

$$\sum_{n=0}^{\infty}(n+b)^{p\beta\gamma-1}\left[\int_S e^{-\lambda x^\alpha(n+b)^\beta}f(x)\mathrm{d}x\right]^p<\left[|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}\lambda^{-\gamma}\Gamma(\gamma)\right]^p\|f\|_{p,\mu}^p, \tag{4.9}$$

$$\int_S x^{q\alpha\gamma-1}\left[\sum_{n=0}^{\infty}e^{-\lambda x^\alpha(n+b)^\beta}a_n\right]^q\mathrm{d}x<\left[|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}\lambda^{-\gamma}\Gamma(\gamma)\right]^q\|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.10}$$

$$\int_S f(x)\sum_{n=0}^{\infty}e^{-\lambda x^\alpha(n+b)^\beta}a_n\mathrm{d}x<|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}\lambda^{-\gamma}\Gamma(\gamma)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.11}$$

where $\mu(x)=x^{p(1-\alpha\gamma)-1}$, $\nu_n=(n+b)^{q(1-\beta\gamma)-1}$.

If $\alpha>0$, setting $\alpha=\beta$ and $b=s$ $(s\in\mathrm{N}^+)$ in (4.11), we get

$$\int_a^\infty f(x)\sum_{n=s}^{\infty}e^{-\lambda x^\beta n^\beta}a_n\mathrm{d}x<\frac{1}{\beta\lambda^\gamma}\Gamma(\gamma)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.12}$$

where $\mu(x)=x^{p(1-\beta\gamma)-1}$ and $\nu_n=n^{q(1-\beta\gamma)-1}$.

If $\alpha<0$, setting $\alpha=-\beta$ and $b=s$ $(s\in\mathrm{N}^+)$ in (4.11), we get the homogeneous case:

$$\int_0^a f(x)\sum_{n=s}^{\infty}e^{-\lambda\left(\frac{n}{x}\right)^\beta}a_n\mathrm{d}x<\frac{1}{\beta\lambda^\gamma}\Gamma(\gamma)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.13}$$

where $\mu(x)=x^{p(1+\beta\gamma)-1}$ and $\nu_n=n^{q(1-\beta\gamma)-1}$.

EXAMPLE 4.3. Let $\kappa(u)=\operatorname{csch}(\lambda u)=\frac{2}{e^{\lambda u}-e^{-\lambda u}}$ $(\lambda>0)$ in Theorem 3.1. It can be verified that

$$\frac{\mathrm{d}\kappa}{\mathrm{d}u}=-2\lambda\frac{e^{\lambda u}+e^{-\lambda u}}{\left(e^{\lambda u}-e^{-\lambda u}\right)^2}<0, \tag{4.14}$$

$$\frac{\mathrm{d}^2\kappa}{\mathrm{d}u^2}=2\lambda^2\frac{e^{2\lambda u}+e^{-2\lambda u}+6}{\left(e^{\lambda u}-e^{-\lambda u}\right)^3}>0, \tag{4.15}$$

$$\frac{\mathrm{d}^3\kappa}{\mathrm{d}u^3} = -2\lambda^3 \frac{e^{3\lambda u} + 23e^{\lambda u} + 23e^{-\lambda u} + e^{-3\lambda u}}{\left(e^{\lambda u} - e^{-\lambda u}\right)^4} < 0, \tag{4.16}$$

$$\frac{\mathrm{d}^4\kappa}{\mathrm{d}u^4} = 2\lambda^4 \frac{e^{4\lambda u} + 76e^{2\lambda u} + 76e^{-2\lambda u} + e^{-4\lambda u}}{\left(e^{\lambda u} - e^{-\lambda u}\right)^5} > 0. \tag{4.17}$$

In addition to the conditions of Theorem 1, we also specify that $\gamma > 1$, then we get

$$\lim_{\substack{u\to 0^+ \\ (u\to\infty)}} \kappa(u)u^{\gamma} = \lim_{\substack{u\to 0^+ \\ (u\to\infty)}} \kappa'(u)u^{\gamma+1} = 0,$$

and

$$\begin{aligned} C(\kappa,\gamma) &= \int_0^\infty \operatorname{csch}(\lambda u)u^{\gamma-1}\mathrm{d}u = 2\int_0^\infty \sum_{k=0}^{\infty} e^{-(2k+1)\lambda u}u^{\gamma-1}\mathrm{d}u \\ &= 2\sum_{k=0}^{\infty}\int_0^\infty e^{-(2k+1)\lambda u}u^{\gamma-1}\mathrm{d}u = 2\sum_{k=0}^{\infty}\frac{\int_0^\infty e^{-t}t^{\gamma-1}\mathrm{d}t}{(2k\lambda+\lambda)^{\gamma}} \\ &= \sum_{k=0}^{\infty}\frac{2\Gamma(\gamma)}{(2k\lambda+\lambda)^{\gamma}} := C(\gamma,\lambda). \end{aligned} \tag{4.18}$$

Therefore, inequalities (3.1), (3.2), and (3.3) are transformed into

$$\sum_{n=0}^{\infty}(n+b)^{p\beta\gamma-1}\left[\int_S \operatorname{csch}\left(\lambda x^{\alpha}(n+b)^{\beta}\right)f(x)\mathrm{d}x\right]^p < \left[|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\gamma,\lambda)\right]^p \|f\|_{p,\mu}^p, \tag{4.19}$$

$$\int_S x^{q\alpha\gamma-1}\left[\sum_{n=0}^{\infty}\operatorname{csch}\left(\lambda x^{\alpha}(n+b)^{\beta}\right)a_n\right]^q \mathrm{d}x < \left[|\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\gamma,\lambda)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.20}$$

$$\int_S f(x)\sum_{n=0}^{\infty}\operatorname{csch}\left(\lambda x^{\alpha}(n+b)^{\beta}\right)a_n\mathrm{d}x < |\,\alpha\,|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}C(\gamma,\lambda)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.21}$$

where $\mu(x) = x^{p(1-\alpha\gamma)-1}$ and $\nu_n = (n+b)^{q(1-\beta\gamma)-1}$.

If $\alpha > 0$, then we set $\alpha = \beta$, $b = s$ and $\gamma = 2m$ $(s,m \in \mathrm{N}^+)$ in (4.21). Observing that [30]:

$$\sum_{k=0}^{\infty}\frac{2}{(2k+1)^{2m}} = \frac{\pi^{2m}}{(2m)!}(2^{2m}-1)B_m,$$

where B_m is the Bernoulli number, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $\cdots$, we have

$$\int_0^a f(x)\sum_{n=s}^{\infty}\operatorname{csch}\left(\lambda x^{\beta}n^{\beta}\right)a_n\mathrm{d}x < \left(2^{2m}-1\right)\frac{B_m}{2m\beta}\left(\frac{\pi}{\lambda}\right)^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.22}$$

where $\mu(x) = x^{p(1-2m\beta)-1}$ and $\nu_n = n^{q(1-2m\beta)-1}$.

If $\alpha < 0$, setting $\alpha = -\beta$ and $b = s$ $(s \in \mathrm{N}^{+})$ in (4.21), we get the homogeneous case of (4.22):

$$\int_{a}^{\infty} f(x) \sum_{n=s}^{\infty} \operatorname{csch}\left(\frac{\lambda n^{\beta}}{x^{\beta}}\right) a_n \mathrm{d}x < \left(2^{2m}-1\right) \frac{B_m}{2m\beta}\left(\frac{\pi}{\lambda}\right)^{2m} \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.23}$$

where $\mu(x) = x^{p(1+2m\beta)-1}$ and $\nu_n = n^{q(1-2m\beta)-1}$.

EXAMPLE 4.4. Let $\kappa(u) = \coth(\lambda u) - 1 = \frac{2e^{-\lambda u}}{e^{\lambda u}-e^{-\lambda u}}$ $(\lambda > 1)$ in Theorem 3.1. By Leibniz formula for higher derivative, and using (4.14), (4.15), (4.16) and (4.17), we get

$$(-1)^j \frac{\mathrm{d}^j \kappa}{\mathrm{d}u^j} = (-1)^j \sum_{i=0}^{j} C_j^i \left(e^{-\lambda u}\right)^{(i)} (\operatorname{csch}(\lambda u))^{(j-i)} > 0, \quad j = 1,2,3,4.$$

Similar to Example 4.3, we specify $\gamma > 1$, and then we get

$$\lim_{\substack{u\to 0^+\\(u\to\infty)}} \kappa(u)u^{\gamma} = \lim_{\substack{u\to 0^+\\(u\to\infty)}} \kappa'(u)u^{\gamma+1} = 0.$$

Furthermore, we have

$$\begin{aligned} C(\kappa,\gamma) &= \int_0^{\infty} (\coth(\lambda u) - 1)\, u^{\gamma-1}\mathrm{d}u = 2\sum_{k=0}^{\infty}\int_0^{\infty} e^{-2(k+1)\lambda u}u^{\gamma-1}\mathrm{d}u \\ &= \sum_{k=0}^{\infty} \frac{2\Gamma(\gamma)}{(2k\lambda+2\lambda)^{\gamma}} := C^*(\gamma,\lambda). \end{aligned} \tag{4.24}$$

Therefore, inequalities (3.1), (3.2) and (3.3) reduce to

$$\sum_{n=0}^{\infty}(n+b)^{p\beta\gamma-1}\left[\int_S \left(\coth\left(\lambda x^{\alpha}(n+b)^{\beta}\right)-1\right) f(x)\mathrm{d}x\right]^p$$
$$< \left[|\,\alpha\,|^{-\frac{1}{q}}\,\beta^{-\frac{1}{p}} C^*(\gamma,\lambda)\right]^p \|f\|_{p,\mu}^p, \tag{4.25}$$

$$\int_S x^{q\alpha\gamma-1}\left[\sum_{n=0}^{\infty}\left(\coth\left(\lambda x^{\alpha}(n+b)^{\beta}\right)-1\right) a_n\right]^q \mathrm{d}x$$
$$< \left[|\,\alpha\,|^{-\frac{1}{q}}\,\beta^{-\frac{1}{p}} C^*(\gamma,\lambda)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.26}$$

$$\int_S f(x)\sum_{n=0}^{\infty}\left(\coth\left(\lambda x^{\alpha}(n+b)^{\beta}\right)-1\right) a_n \mathrm{d}x$$
$$<|\,\alpha\,|^{-\frac{1}{q}}\,\beta^{-\frac{1}{p}} C^*(\gamma,\lambda)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.27}$$

where $\mu(x) = x^{p(1-\alpha\gamma)-1}$, $\nu_n = (n+b)^{q(1-\beta\gamma)-1}$.

Setting $\gamma = 2m$ $(m \in \mathrm{N}^+)$ in (4.27), in view of [30]

$$\sum_{k=0}^{\infty} \frac{2}{(k+1)^{2m}} = \frac{B_m}{(2m)!}(2\pi)^{2m},$$

we have

$$\int_S f(x) \sum_{n=0}^{\infty} \left(\coth\left(\lambda x^{\alpha} (n+b)^{\beta} \right) - 1 \right) a_n \mathrm{d}x < \mid \alpha \mid^{-\frac{1}{q}} \beta^{-\frac{1}{p}} \frac{B_m}{2m} \left(\frac{\pi}{\lambda} \right)^{2m} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.28}$$

where $\mu(x) = x^{p(1-2m\alpha)-1}$ and $\nu_n = (n+b)^{q(1-2m\beta)-1}$.

REMARK 4.5. If we take $\kappa(u) = \mathrm{sech}(\lambda u)$ or $\kappa(u) = 1 - \tanh(\lambda u)$ in Theorem 3.1, it can be shown that $\kappa(u)$ does not satisfy the condition $(-1)^j \frac{\mathrm{d}^j \kappa}{\mathrm{d}u^j} > 0$, $j = 1,2,3,4$. Therefore, it can not be obtained any corresponding Hilbert-type inequalities with the kernel functions involving hyperbolic secant or hyperbolic tangent functions.

EXAMPLE 4.6. Let $\kappa(u) = \log\left(1 + \frac{1}{u}\right)$ in Theorem 3.1. It follows that

$$\frac{\mathrm{d}\kappa}{\mathrm{d}u} = \frac{1}{u+1} - \frac{1}{u} < 0, \quad \frac{\mathrm{d}^2\kappa}{\mathrm{d}u^2} = \frac{1}{u^2} - \frac{1}{(u+1)^2} > 0,$$

$$\frac{\mathrm{d}^3\kappa}{\mathrm{d}u^3} = \frac{2}{(u+1)^3} - \frac{2}{u^3} < 0, \quad \frac{\mathrm{d}^4\kappa}{\mathrm{d}u^4} = \frac{6}{u^4} - \frac{6}{(u+1)^4} > 0.$$

In addition to the conditions of Theorem 3.1, we also assume that $0 < \gamma < 1$. Then

$$\lim_{\substack{u \to 0^+ \\ (u \to \infty)}} \kappa(u) u^{\gamma} = \lim_{\substack{u \to 0^+ \\ (u \to \infty)}} \kappa'(u) u^{\gamma+1} = 0.$$

By the formula of integration by parts, we get

$$\begin{aligned} C(\kappa,\gamma) &= \int_0^{\infty} \log\left(1 + \frac{1}{u}\right) u^{\gamma-1} \mathrm{d}u = \frac{1}{\gamma} \int_0^{\infty} \frac{u^{\gamma-1}}{1+u} \mathrm{d}u \\ &= \frac{1}{\gamma} B(\gamma, 1-\gamma) = \frac{\pi}{\gamma \sin(\gamma\pi)}. \end{aligned}$$

Hence, inequalities (3.1), (3.2) and (3.3) are transformed into

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+b)^{p\beta\gamma-1} \left[\int_S \log\left(1 + x^{-\alpha}(n+b)^{-\beta}\right) f(x) \mathrm{d}x \right]^p \\ &\quad < \left[\mid \alpha \mid^{-\frac{1}{q}} \beta^{-\frac{1}{p}} \frac{\pi}{\gamma \sin(\gamma\pi)} \right]^p \|f\|_{p,\mu}^p, \end{aligned} \tag{4.29}$$

$$\int_S x^{q\alpha\gamma-1}\left[\sum_{n=0}^{\infty}\log\left(1+x^{-\alpha}(n+b)^{-\beta}\right)a_n\right]^q \mathrm{d}x$$
$$< \left[|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}\frac{\pi}{\gamma\sin(\gamma\pi)}\right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \tag{4.30}$$
$$\int_S f(x)\sum_{n=0}^{\infty}\log\left(1+x^{-\alpha}(n+b)^{-\beta}\right)a_n\mathrm{d}x$$
$$<|\alpha|^{-\frac{1}{q}}\beta^{-\frac{1}{p}}\frac{\pi}{\gamma\sin(\gamma\pi)}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.31}$$

where $\mu(x)=x^{p(1-\alpha\gamma)-1}$ and $\nu_n=(n+b)^{q(1-\beta\gamma)-1}$.

If $\alpha>0$, setting $\alpha=\beta$ and $b=s$ $(s\in\mathrm{N}^+)$ in (4.31), we get

$$\int_0^a f(x)\sum_{n=s}^{\infty}\log\left(1+\frac{1}{x^\beta n^\beta}\right)a_n\mathrm{d}x<\frac{\pi}{\beta\gamma\sin(\gamma\pi)}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.32}$$

where $\mu(x)=x^{p(1-\beta\gamma)-1}$ and $\nu_n=n^{q(1-\beta\gamma)-1}$.

If $\alpha<0$, setting $\alpha=-\beta$ and $b=s$ $(s\in\mathrm{N}^+)$ in (4.31), we get

$$\int_a^{\infty} f(x)\sum_{n=s}^{\infty}\log\left(1+\frac{x^\beta}{n^\beta}\right)a_n\mathrm{d}x<\frac{\pi}{\beta\gamma\sin(\gamma\pi)}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{4.33}$$

where $\mu(x)=x^{p(1+\beta\gamma)-1}$ and $\nu_n=n^{q(1-\beta\gamma)-1}$.

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