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A NEW REFINEMENT OF JENSEN–TYPE INEQUALITY WITH RESPECT TO UNIFORMLY CONVEX FUNCTIONS WITH APPLICATIONS IN INFORMATION THEORY

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Abstract. In this paper, we establish a new refinement of Jensen-type inequality for uniformly convex functios. Furthermore, we apply those results in information theory and we obtain strong and more precise bounds for Shannon's entropy.

1. Introduction and preliminaries

Jensen's inequality probably plays a vital role in some aspect of mathematics (such as inequality, indeed arithmetic-geometric mean inequality, Hermite-Hadamard inequality, Hölder inequality, Minkowski inequality, and Ky Fan's inequality), statistics, and information theory (such as approximate bound of entropies) and etc.

The well-known Jensen's inequality (see [7], [17], [18], [19]) for convex function asserts that:

Let $f : I \longrightarrow \mathbb{R}$ be a convex function. Then the inequality

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i)$$

holds for each convex combination $\sum_{i=1}^{n} p_i x_i$ of points $x_i \in I$ $(interval\ \ I)$.

DEFINITION 1.1. ([8]) Assume that X is a random variable. Also, the range $R = \{x_1, \cdots, x_n\}$ is a the probability distribution of point p_i, $i = 1, \cdots, n$, $p_i > 0$, then the Shannon entropy $H(X)$ defined by

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i.$$

The following definitions can be found in [3, 4, 15, 20, 21].

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DEFINITION 1.2. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a function. Then f is uniformly convex if there exists $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ with $\phi(t)\geqslant 0$ and vanishes only at 0, and

$$f(\alpha x+(1-\alpha)y)+\alpha(1-\alpha)\phi(|x-y|)\leqslant \alpha f(x)+(1-\alpha)f(y)$$

for every $\alpha\in[0,1]$ and $x,y\in[a,b]$. The function ϕ is called the modulus of f. Also, if $\phi(x)=cx^2$ then f is called strongly convex with modulus c.

THEOREM 1.1. [12] *Let $f:I\longrightarrow\mathbb{R}$ be an uniformly convex function with modulus $\phi:\mathbb{R}\longrightarrow[0,+\infty]$ on I, $\{x_k\}_{k=1}^n\subseteq[a,b]$ be a sequence and let π be a permutation on $\{1,\dots,n\}$ such that $x_{\pi(1)}\leqslant x_{\pi(2)}\leqslant\dots\leqslant x_{\pi(n)}$. Then the inequality*

$$f\left(\sum_{k=1}^n p_kx_k\right)\leqslant\sum_{k=1}^n p_kf(x_k)-\sum_{k=1}^{n-1}p_{\pi(k)}p_{\pi(k+1)}\phi(x_{\pi(k+1)}-x_{\pi(k)}) \tag{1.1}$$

holds for every convex combination $\sum_{k=1}^n p_kx_k$ of points $x_k\in I$.

2. Main results

In the following theorem we obtain new bound for Jensen's inequality with respect to uniformly convex functions. The proof techniques in this section are similar to the proof techniques in [12].

THEOREM 2.1. *Let f be a uniformly convex function and $1\leqslant k\leqslant n$. If $x_1\leqslant x_2\leqslant\dots\leqslant x_n$ (or $x_n\leqslant\dots\leqslant x_2\leqslant x_1$). Then*

$$\begin{aligned}&f\left(\sum_{i=1}^n p_ix_i\right)+p_k(1-p_k)\phi\left(\left|\frac{\sum_{i=1,i\neq k}^n p_ix_i}{1-p_k}-x_k\right|\right)\\
&\leqslant(1-p_k)f\left(\frac{\sum_{i=1,i\neq k}^n p_ix_i}{1-p_k}\right)+p_kf(x_k)\\
&\leqslant\sum_{i=1}^n p_if(x_i)-\sum_{i=1,i\neq k,k-1}^{n-1}\frac{p_ip_{i+1}}{1-p_k}\phi(x_{i+1}-x_i)\\
&\quad-\frac{p_{k-1}p_{k+1}}{1-p_k}\phi(x_{k+1}-x_{k-1}),\end{aligned}$$

where $p_0p_2\phi(x_2-x_0)=p_{n-1}p_{n+1}\phi(x_{n+1}-x_{n-1})=0$, $p_0=p_{n+1}:=0$, $x_0:=x_2$, $x_{n+1}:=x_{n-1}$.

Proof. Assume that $1\leqslant k\leqslant n$ then

$$\begin{aligned}f\left(\sum_{i=1}^n p_ix_i\right)&=f\left((1-p_k)\frac{\sum_{i=1}^n p_ix_i-p_kx_k}{1-p_k}+p_kx_k\right)\\
&\leqslant(1-p_k)f\left(\frac{\sum_{i=1}^n p_ix_i-p_kx_k}{1-p_k}\right)+p_kf(x_k)\\
&\quad-p_k(1-p_k)\phi\left(\left|\frac{\sum_{i=1}^n p_ix_i-p_kx_k}{1-p_k}-x_k\right|\right)\end{aligned}$$

Let us first assume that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. By the use of Theorem 1.1, we get

$$\begin{aligned}
&(1-p_k)f\left(\frac{\sum_{i=1}^n p_ix_i - p_kx_k}{1-p_k}\right) + p_kf(x_k) \\
&= (1-p_k)f\left(\frac{\sum_{i=1,i\neq k}^n p_ix_i}{1-p_k}\right) + p_kf(x_k) \\
&\leqslant (1-p_k)\left[\frac{\sum_{i=1,i\neq k}^n p_if(x_i)}{1-p_k} - \sum_{i=1,i\neq k,k-1}^{n-1} \frac{p_ip_{i+1}}{(1-p_k)^2}\phi(x_{i+1}-x_i)\right. \\
&\quad \left. - \frac{p_{k-1}p_{k+1}}{(1-p_k)^2}\phi(x_{k+1}-x_{k-1})\right] + p_kf(x_k) \\
&\leqslant \sum_{i=1}^n p_if(x_i) - \sum_{i=1,i\neq k,k-1}^{n-1} \frac{p_ip_{i+1}}{1-p_k}\phi(x_{i+1}-x_i) \\
&\quad - \frac{p_{k-1}p_{k+1}}{1-p_k}\phi(x_{k+1}-x_{k-1}).
\end{aligned}$$

Similarly, the results hold if $x_n \leqslant \ldots \leqslant x_2 \leqslant x_1$. □

Another result with respect to Jensen's inequality as follows:

THEOREM 2.2. *Assume that f is a uniformly convex function with modulus ϕ also, $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ with $1 < k < n-1$ then*

$$\begin{aligned}
&f\left(\sum_{i=1}^n p_ix_i\right) + \sum_{i=1}^{k-1} p_ip_{i+1}\phi(x_{i+1}-x_i) + p_ks_k\phi(x_{k+1}-x_k) \\
&\leqslant s_kf\left(\frac{1}{s_k}\sum_{i=k+1}^n p_ix_i\right) + \sum_{i=1}^k p_if(x_i) \\
&\leqslant \sum_{i=1}^n p_if(x_i) - \frac{1}{s_k}\sum_{i=k+1}^n p_ip_{i+1}\phi(x_{i+1}-x_i),
\end{aligned}$$

where $s_k = \sum_{i=k+1}^n p_i$.

Proof. Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, $1 < k < n-1$. We have

$$\begin{aligned}
I &= f\left(\sum_{i=1}^n p_ix_i\right) \\
&= f\left[\left(\sum_{i=k+1}^n p_i\right)\left(\frac{\sum_{i=1}^n p_ix_i - \sum_{i=1}^k p_ix_i}{\sum_{i=k+1}^n p_i}\right) + \sum_{i=1}^k p_ix_i\right].
\end{aligned}$$

In view of Theorem 1.1 we obtain

$$\begin{aligned} I &\leqslant \left(\sum_{i=k+1}^{n} p_i\right) f\left(\frac{\sum_{i=k+1}^{n} p_i x_i}{\sum_{i=k+1}^{n} p_i}\right) + \sum_{i=1}^{k} p_i f(x_i) - \sum_{i=1}^{k-1} p_i p_{i+1} \phi(x_{i+1}-x_i) \\ &\quad - p_k \left(\sum_{i=k+1}^{n} p_i\right) \phi\left(\frac{\sum_{i=k+1}^{n} p_i x_i}{\sum_{i=k+1}^{n} p_i} - x_k\right) \\ &\leqslant \sum_{i=k+1}^{n} p_i f(x_i) - \frac{1}{\sum_{i=k+1}^{n} p_i} \sum_{i=k+1}^{n-1} p_i p_{i+1} \phi(x_{i+1}-x_i) + \sum_{i=1}^{k} p_i f(x_i) \\ &\quad - \sum_{i=1}^{k-1} p_i p_{i+1} \phi(x_{i+1}-x_i) - p_k \left(\sum_{i=k+1}^{n} p_i\right) \phi\left(\frac{\sum_{i=k+1}^{n} p_i x_i}{\sum_{i=k+1}^{n} p_i} - x_k\right). \end{aligned}$$

Finally, according to the following relations

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{k+1} \leqslant \frac{\sum_{i=k+1}^{n} p_i x_i}{\sum_{i=k+1}^{n} p_i}, \quad \frac{\sum_{i=k+1}^{n} p_i x_i}{\sum_{i=k+1}^{n} p_i} - x_k \geqslant x_{k+1} - x_k,$$

and this fact that ϕ is increasing we conclude the result. □

PROPOSITION 2.1. *Assume that f is a unifmly convex function with modulus ϕ, $1 \leqslant k \leqslant n$, $x_1 \leqslant \cdots \leqslant x_n$ then*

$$\begin{aligned} &f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \frac{n-1}{n^2} \phi\left(\left|\frac{\sum_{i=1, i\neq k}^{n} x_i}{n-1} - x_k\right|\right) \\ &\leqslant \frac{n-1}{n} f\left(\frac{\sum_{i=1, i\neq k}^{n} x_i}{n-1}\right) + \frac{1}{n} f(x_k) \\ &\leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{1}{n(n-1)} \sum_{i=1, i\neq k, k-1}^{n-1} \phi(x_{i+1}-x_i) - \frac{1}{n(n-1)} \phi(x_{k+1}-x_{k-1}), \end{aligned}$$

where $\phi(x_2 - x_0) = \phi(x_{n+1} - x_{n-1}) = 0$.

Proof. In Theorem 2.1 put $p_i = \frac{1}{n}$, $i = 1, \cdots, n$. □

LEMMA 2.1. *Let $\Phi : I \to \mathbb{R}$ be a twice-differentiable function and let*

$$m := \inf\left\{\Phi''(c) : c \in I\right\}.$$

Then $\Phi(x) - \frac{m}{2}x^2$ is a convex function on I.

In particular if $m > 0$ then Φ is strongly convex, that is, Φ is uniformly convex with modulus $\varphi(u) = \frac{m}{2}u^2$.

Proof. It is obvious that φ is increasing and vanishes only at 0. We consider two fixed points $c, d \in I$ and define

$$\varphi(\lambda) := \lambda\Phi(c) + (1-\lambda)\Phi(d) - \Phi(\lambda c + (1-\lambda)d) - \frac{m\lambda(1-\lambda)}{2}(c-d)^2$$

for all $\lambda \in [0,1]$. Now, we show that $\varphi(\lambda) \geqslant 0$, for all $\lambda \in [0,1]$. Since $\varphi(0) = \varphi(1) = 0$ and

$$\frac{d^2\varphi}{d\lambda^2} = m(c-d)^2 - (d-c)^2\Phi''(\lambda c + (1-\lambda)d) \leqslant 0,$$

$\Phi(\lambda) \geqslant 0$ for every $c,d \in [\mu, \nu]$ and $\lambda \in [0,1]$. Hence,

$$\lambda\Phi(c) + (1-\lambda)\Phi(d) \geqslant \Phi(\lambda c + (1-\lambda)d) + \frac{m\lambda(1-\lambda)}{2}(c-d)^2.$$

Therefore, the proof is complete. □

For example, it can be shown that the function $f(x) = \log x$, $x \in [1,d]$ satisfies

$$m = \inf\{f''(c) : c \in [1,d]\} = -1.$$

Therefore, $g(x) = \log x - \frac{m}{2}x^2 = \log x + \frac{1}{2}x^2$ is a convex function on $[1,d]$.

EXAMPLE 2.1. [12] If $a \geqslant 0$ and $f : [a,b] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\log(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then f is strongly convex with modulus $c := \frac{1}{2b}$.

THEOREM 2.3. *Assume that* $X = \{p_1, \cdots, p_n\}$, $1 \leqslant k \leqslant n$ *and* $p_1 \leqslant \cdots \leqslant p_n$ *then*

$$\begin{aligned} &H(X) + \frac{1}{2(n-1)}\left[\sum_{i=1}^{n-1}(p_{i+1}-p_i)^2 + (p_{k+1}-p_{k-1})^2\right] \\ &\leqslant (1-p_k)\log\frac{n-1}{1-p_k} - p_k\log p_k \\ &\leqslant \log n - \frac{(1-np_k)^2}{2n(n-1)}, \end{aligned}$$

where $p_0 := p_2$, $p_{n+1} := p_{n-1}$.

Proof. Using Lemma 2.1 the function $f(x) = x\log x$ on $[a,b]$ is uniformly convex with modulus $\phi(r) = \frac{r^2}{2b}$. In Proposition 2.1, put $f(x) = x\log x$, $x_i = p_i$, $a = 0$, $b = 1$ then by some calculus we have

$$\begin{aligned} &\frac{1}{n}\log\frac{1}{n} + \frac{n-1}{n^2}\frac{\left(\frac{1-p_k}{n-1} - p_k\right)^2}{2} \\ &\leqslant \frac{n-1}{n} \times \frac{1-p_k}{n-1}\log\frac{1-p_k}{n-1} + \frac{1}{n}p_k\log p_k \\ &\leqslant -\frac{1}{n}H(X) - \frac{1}{n(n-1)} \times \frac{\sum_{i=1, i\neq k,k-1}^{n-1}(p_{i+1}-p_i)^2}{2} \\ &\quad - \frac{1}{n(n-1)} \times \frac{(p_{k+1}-p_{k-1})^2}{2}. \end{aligned}$$ □

In the following propositions we give new bounds for the Shannon entropy $H(X)$.

PROPOSITION 2.2. *Let X be a random variable as in Theorem 2.3 with $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_n$. Then*

$$H(X) \leqslant \log n - \frac{(1-np_k)^2}{2n(n-1)}$$

for each $1 \leqslant k \leqslant n$.

Proof. This is an easy consequence of Theorem 2.3. □

PROPOSITION 2.3. *Assume that $X = \{p_1, \ldots, p_n\}$ is a random variable.*

1. *If $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$, $1 \leqslant k \leqslant n$ then*

$$H(X) + \frac{p_1^2}{2(1-p_k)} \sum_{i=1, i\neq k, k-1}^{n-1} \frac{(p_{i+1}-p_i)^2}{p_i p_{i+1}} + \frac{p_1^2}{2(1-p_k)} \times \frac{(p_{k+1}-p_{k-1})^2}{p_{k+1}p_{k-1}}$$
$$\leqslant (1-p_k)\log\frac{n-1}{1-p_k} - p_k \log p_k$$
$$\leqslant \log n - \frac{p_1^2}{2p_k(1-p_k)}(np_k-1)^2,$$

where $p_2 = p_0$, $p_{n+1} = p_{n-1}$.

2. *Let $\mu = \min_{1\leqslant i\leqslant n}\{p_i\}$. Then*

$$H(X) \leqslant \log n - \frac{\mu^2}{2p_k(1-p_k)}(np_k-1)^2$$

for each $1 \leqslant k \leqslant n$.

Proof.

1. Assume that $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$. Let $x_i = \frac{1}{p_i}$. By the use of Lemma 2.1 in [12] the function $f(x) = -\log x$ is uniformly convex with modulus $\phi(r) = \frac{p_1^2 r^2}{2}$ on $[1, \frac{1}{p_1}]$. Now, put $f(x) = -\log x$ in Theorem 2.1 then by some calculus we have

$$-\log n + p_k(1-p_k) \times \frac{p_1^2}{2} \times \left(\frac{n-1}{1-p_k} - \frac{1}{p_k}\right)^2$$
$$\leqslant (1-p_k)\log\frac{1-p_k}{n-1} + p_k \log p_k$$
$$\leqslant -H(X) - \sum_{i=1, i\neq k, k-1}^{n-1} \frac{p_i p_{i+1}}{1-p_k} \times \frac{p_1^2}{2} \times \left(\frac{1}{p_i} - \frac{1}{p_{i+1}}\right)^2$$
$$- \frac{p_{k-1}p_{k+1}}{1-p_k} \times \frac{p_1^2}{2} \times \left(\frac{1}{p_{k-1}} - \frac{1}{p_{k+1}}\right)^2,$$

which compelets the proof.

2. Straightforward from 1. □

Proof. This follows from Proposition 2.3. □

PROPOSITION 2.4. *If f is a uniformly convex with modulus ϕ and $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ with $1 < k < n-1$ then*

$$\begin{aligned} & f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \frac{1}{n^2}\sum_{i=1}^{k-1}\phi(x_{i+1}-x_i) + \frac{n-k}{n^2}\phi(x_{k+1}-x_k) \\ & \leqslant \frac{n-k}{n} f\left(\frac{1}{n-k}\sum_{i=k+1}^{n} x_i\right) + \frac{1}{n}\sum_{i=1}^{k} f(x_i) \\ & \leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \frac{1}{n(n-k)}\sum_{i=k+1}^{n-1}\phi(x_{i+1}-x_i). \end{aligned}$$

Proof. For proof of the theorem, put $p_i = \frac{1}{n}$, $1 \leqslant i \leqslant n$ in Theorem 2.2. □

THEOREM 2.4. *Assume that $p_1 \leqslant \cdots \leqslant p_n$, $1 < k < n-1$ then*

$$\begin{aligned} & H(X) + \frac{1}{2(n-k)}\sum_{i=k+1}^{n-1}(p_{i+1}-p_i)^2 \\ & \leqslant s_k \log\frac{n-k}{s_k} - \sum_{i=1}^{k} p_i \log p_i \\ & \leqslant \log n - \frac{1}{2n}\sum_{i=1}^{k-1}(p_{i+1}-p_i)^2 - \frac{n-k}{2n}(p_{k+1}-p_k)^2. \end{aligned}$$

Proof. In view of Lemma 2.1 the function $f(x) = x\log x$ on $[0,1]$ is strongly convex with modulus $c = \frac{1}{2}$. In Proposition 2.4 put $f(x) = x\log x$ and $x_i = p_i$ for each $i = 1,\cdots,n$ we have

$$\begin{aligned} & \frac{1}{n}\log\frac{1}{n} + \frac{1}{n^2}\frac{1}{2}\sum_{i=1}^{k-1}(p_{i+1}-p_i)^2 + \frac{n-k}{2n^2}(p_{k+1}-p_k)^2 \\ & \leqslant \frac{n-k}{n}\frac{\sum_{i=k+1}^{n} p_i}{n-k}\log\left(\frac{\sum_{i=k+1}^{n} p_i}{n-k}\right) \\ & + \frac{1}{n}\sum_{i=1}^{k} p_i \log p_i \leqslant -\frac{1}{n}H(X) - \frac{1}{n(n-k)}\frac{1}{2}\sum_{i=k+1}^{n-1}(p_{i+1}-p_i)^2, \end{aligned}$$

which completes the proof. □

COROLLARY 2.1. *Assume that $p_1 \leqslant \cdots \leqslant p_n$, $1 < k < n-1$ then*

$$H(X) \leqslant \log n - \frac{n-k}{2n}(p_{k+1}-p_k)^2.$$

COROLLARY 2.2. *Let f be a uniformly convex function on I, $x_1,\dots,x_n \in I$ and $x_n \leqslant \cdots \leqslant x_2 \leqslant x_1$, $1 \leqslant k \leqslant n-1$. Then*

$$\begin{aligned} & f\left(\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{k-1} p_i p_{i+1} \phi(x_{i+1}-x_i) + p_k s_k \phi(x_k - x_{k+1}) \\ \leqslant & \, s_k f\left(\frac{1}{s_k}\sum_{i=k+1}^{n} p_i x_i\right) + \sum_{i=1}^{k} p_i f(x_i) \\ \leqslant & \sum_{i=1}^{n} p_i f(x_i) - \frac{1}{s_k}\sum_{i=k+1}^{n-1} p_i p_{i+1} \phi(x_i - x_{i+1}), \end{aligned}$$

where $s_k = \sum_{i=k+1}^{n} p_i$.

Proof. It follows from Theoren 2.2. □

THEOREM 2.5. *Assume that $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$ then*

$$\begin{aligned} & H(X) + \frac{p_1^2}{2}\sum_{i=k+1}^{n-1} \frac{(p_{i+1}-p_i)^2}{p_i p_{i+1}} \\ \leqslant & \, s_k \log \frac{n-k}{s_k} - \sum_{i=1}^{k} p_i \log p_i \\ \leqslant & \log n - \frac{p_1^2}{2}\sum_{i=1}^{k-1} \frac{(p_{i+1}-p_i)^2}{p_i p_{i+1}} - \frac{p_1^2}{2} p_k s_k \left(\frac{1}{p_k} - \frac{1}{p_{k+1}}\right)^2. \end{aligned}$$

Proof. If $p_1 \leqslant \cdots \leqslant p_n$ then in view of [[12], Lemma 2.1] the function $f(x) = -\log x$ on $[1, \frac{1}{p_1}]$ is a uniformly convex with modulus $\phi(r) = \frac{p_1^2 r^2}{2}$. In Corollary 2.2 put $f(x) = -\log x$, $x_i = \frac{1}{p_i}$ for each $1 \leqslant i \leqslant n$ we have

$$\begin{aligned} & -\log n + \sum_{i=1}^{k-1} p_i p_{i+1} \frac{p_1^2}{2}\left(\frac{1}{p_{i+1}} - \frac{1}{p_i}\right)^2 + p_k s_k \left(\frac{1}{p_k} - \frac{1}{p_{k+1}}\right)^2 \\ \leqslant & -s_k \log\left(\frac{n-k}{s_k}\right) + \sum_{i=1}^{k} p_i \log p_i \\ \leqslant & \sum_{i=1}^{n} p_i \log p_i - \frac{1}{s_k}\sum_{i=k+1}^{n-1} p_i p_{i+1} \frac{p_1^2}{2}\left(\frac{1}{p_i} - \frac{1}{p_{i+1}}\right)^2 \end{aligned}$$

which $s_k = \sum_{i=k+1}^{n} p_i$. □

Some applications to special means will be presented below.

REMARK 2.1. Under conditions of Proposition 2.4(and Theorem 2.2), we have

$$
\begin{aligned}
& f(\mathscr{A}(x_1,x_2,\ldots,x_n)) + \frac{k-1}{n^2}\cdot\mathscr{A}(\phi(x_2-x_1),\phi(x_3-x_2),\ldots,\phi(x_k-x_{k-1})) \\
&\quad + \frac{n-k}{n^2}\phi(x_{k+1}-x_k) \\
&\leqslant \frac{n-k}{n} f(\mathscr{A}(x_{k+1},x_{k+2},\ldots,x_{k+n})) + \frac{k}{n}\mathscr{A}(f(x_1),f(x_2),\ldots,f(x_k)) \\
&\leqslant \mathscr{A}(f(x_1),f(x_2),\ldots,f(x_n)) \\
&\quad - \frac{n-k-1}{n(n-k)}\mathscr{A}(\phi(x_{k+2}-x_{k+1}),\phi(x_{k+3}-x_{k+2}),\ldots,\phi(x_n-x_{n-1})),
\end{aligned}
$$

where $\mathscr{A}(x_1,x_2,\ldots,x_n) = \frac{x_1+x_2+\ldots+x_n}{n}$ is the arithmetic mean of the numbers $x_1,x_2,\ldots,x_n$.

EXAMPLE 2.2. If $c\in[0,1]$ then the function $f(x)=(1-x)^2$, $x\in[a,b]$ is strongly convex with modulus c.

Proof. For $x,y\in[a,b]$ we define $F:[0,1]\longrightarrow\mathbb{R}$ by

$$F(t) := t(1-x)^2 + (1-t)(1-y)^2 - (1-tx-(1-t)y)^2 - ct(1-t)(x-y)^2.$$

It is obvious that F is twice-differentiable on $[0,1]$ and

$$F''(t) = 2(y-x)^2(c-1) \leqslant 0,$$

for every $t\in[0,1]$. On the other hand, $F(0)=F(1)=0$, thus $F(t)\geqslant 0$ for every $t\in[0,1]$ or f is strongly convex with modulus c. □

EXAMPLE 2.3. The function $f(x)=x^3$, $x\in[1,2]$ is uniformly convex with modulus $\phi(t)=t^3$.

Proof. For $x,y\in[1,2]$ we define $F:[0,1]\longrightarrow\mathbb{R}$ by

$$F(t) := (tx+(1-t)y)^3 + t(1-t)|x-y|^3 - tx^3 - (1-t)y^3.$$

It is obvious that F is twice-differentiable on $[0,1]$ and

$$F''(t) = 2(x-y)^2(3(tx+(1-t)y) - |x-y|) \geqslant 0,$$

for every $t\in[0,1]$. On the other hand, $F(0)=F(1)=0$, thus $F(t)\leqslant 0$ for every $t\in[0,1]$ or f uniformly convex with modulus $\phi(t)=t^3$. □

REMARK 2.2. If $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_n$, then by using Proposition 2.1 for the function $f(x) = (1-x)^2$ with $x_i = p_i$, $a = 0$, $b = 1$, we have

$$\begin{aligned}
&\frac{(\sum_{i=1}^{n}(1-p_i))^2}{n^2} + c\frac{1}{n^2}\frac{(\sum_{i=1,i\neq k}^{n}(p_i-p_k))^2}{n-1} \\
&\leqslant \frac{1}{n}\frac{(\sum_{i=1,i\neq k}^{n}(1-p_i))^2}{(n-1)} + \frac{1}{n}(1-p_k)^2 \\
&\leqslant \frac{1}{n}\sum_{i=1}^{n}(1-p_i)^2 - \frac{c}{n(n-1)}\sum_{i=1,i\neq k,k-1}^{n-1}(p_{i+1}-p_i)^2 - \frac{c}{n(n-1)}(p_{k+1}-p_{k-1})^2,
\end{aligned}$$

where $p_0 = p_2$, $p_{n+1} = p_{n-1}$.

REMARK 2.3. Under conditions of Proposition 2.1 for the strongly convex function $f(x) = x^2$ with modulus $c = 1$, we have

$$\begin{aligned}
&\frac{(\sum_{i=1}^{n}(x_i))^2}{n^2} + \frac{1}{n^2}\frac{\left(\sum_{i=1,i\neq k}^{n}(x_i-x_k)\right)^2}{n-1} \\
&\leqslant \frac{1}{n}\frac{\left(\sum_{i=1,i\neq k}^{n}(x_i)\right)^2}{(n-1)} + \frac{x_k^2}{n} \\
&\leqslant \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \frac{1}{n(n-1)}\sum_{i=1,i\neq k,k-1}^{n-1}(x_{i+1}-x_i)^2 - \frac{1}{n(n-1)}(x_{k+1}-p_k)^2.
\end{aligned}$$

REMARK 2.4. Under conditions of Proposition 2.4 for the strongly convex function $f(x) = -\log x$ with modulus $\phi(r) = \frac{p_1^2 r^2}{2}$ on $[1, \frac{1}{p_1}]$, we have

$$\begin{aligned}
&\log\frac{\mathscr{A}(x_1,x_2,\ldots,x_n)}{\mathscr{A}^{\frac{n-k}{n}}(x_{k+1},x_{k+2},\ldots,x_n)} \\
&\geqslant \frac{k}{n}\log G(x_1,x_2,\ldots x_k) + \frac{p_1^2}{2n^2}\left[\sum_{i=1}^{k-1}(x_{i+1}-x_i)^2 + (n-k)(x_{k+1}-x_k)^2\right],
\end{aligned}$$

and

$$\frac{n-k}{n}\log\mathscr{A}(x_{k+1},x_{k+2},\ldots,x_n) + \log\frac{G^{\frac{k}{n}}(x_1,x_2,\ldots x_k)}{G(x_1,x_2,\ldots x_n)} \geqslant \frac{p_1^2}{2n(n-k)}\sum_{i=k+1}^{n-1}(x_{i+1}-x_i)^2,$$

when $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, $x_i \geqslant 0$ and $1 < k < n-1$.

REMARK 2.5. Under conditions of Proposition 2.4 for the strongly convex function $f(x)=\frac{1}{x}$, $x\in[\lambda_1,\varepsilon_1]$ with modulus $c\in(0,\frac{1}{\varepsilon_1^3})$, $\phi(r)=cx^2$, we have

$$\begin{aligned}
&\frac{1}{\mathscr{A}(x_1,x_2,\dots,x_n)}+\frac{c}{n^2}\sum_{i=1}^{k-1}(x_{i+1}-x_i)^2+c\frac{n-k}{n^2}(x_{k+1}-x_k)^2\\
&\leqslant\frac{(n-k)^2}{n}\frac{1}{\mathscr{A}(x_{k+1},x_{k+2},\dots,x_n)}+\frac{k}{n}\frac{1}{H(x_1,\dots,x_k)}\\
&\leqslant\frac{1}{H(x_1,\dots,x_n)}-\frac{c}{n(n-k)}\sum_{i=k+1}^{n-1}(x_{i+1}-x_i)^2,
\end{aligned}$$

where $H(x_1,\dots,x_n)=\frac{n}{\frac{1}{x_1}+\dots+\frac{1}{x_n}}$ when $x_1\leqslant x_2\leqslant\dots\leqslant x_n$, $x_i>0$ and $1<k<n-1$.

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