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SOME GENERALIZATIONS OF NUMERICAL RADII AND SCHATTEN p–NORMS INEQUALITIES

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Abstract. In this paper, we present some generalizations and further refinements for the numerical radii of sectorial matrices and Schatten p-norms inequalities of accretive-dissipative matrices, which generalized some results of Kittaneh et al. Moreover, we also give some n-tuple power inequality for sectorial matrices by Yang [22].

1. Introduction

Let $\mathbb{H}$ be a complex Hilbert space with inner product $\langle .,. \rangle$ and $B(\mathbb{H})$ be the collection of all bounded linear operator on $\mathbb{H}$. For $A \in B(\mathbb{H})$, A^* denote the conjugate of A, it is called accretive if $\Re A > 0$, and A is an accretive-dissipative if $\Re A > 0$ and $\Im A > 0$. Here $\Re A = \frac{1}{2}(A+A^*)$ and $\Im A = \frac{1}{2i}(A-A^*)$ are the real part and imaginary parts of A, respectively. The numerical radius of $A \in B(\mathbb{H})$ is defined by

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, ||x|| = 1\},$$

and the operator norm of A is denoted by

$$||A|| = \sup\{|\langle Ax, y\rangle| : x \in \mathbb{C}^n, ||x|| = ||y|| = 1\}.$$

It is well known that

$$\frac{1}{2}||A|| \leqslant w(A) \leqslant ||A||. \tag{1.1}$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if A is normal.

Let $\mathbb{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \left\{\langle Ax, x\rangle : x \in \mathbb{C}^n,\ ||x|| = 1\right\}.$$

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If $W(A) \subset (0,\infty)$, we say that A is positive and we write $A > 0$. In addition, a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be sectorial if, for some $\alpha \in [0, \frac{\pi}{2})$, we have

$$W(A) \subset S_\alpha := \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leqslant (\Re z)\tan\alpha\}.$$

It is well known that if $W(A) \subset S_\alpha$, then

$$W(A^t) \subset S_\alpha \tag{1.2}$$

for $t \in (0,1)$. In fact, Drury [5] showed that

$$W(A^t) \subset S_{t\alpha} \tag{1.3}$$

under the same conditions as in (1.2). Moreover, Nasiri and Furuichi [18] proved that $W(A) \subseteq S_\alpha$ implies $W(A^{-1}) \subseteq S_\alpha$ when A is nonsingular.

Kittaneh [11, 12] improved (1.1) as follows

$$w(A) \leqslant \frac{1}{2}|||A| + |A^*||| \leqslant \frac{1}{2}\left(||A|| + ||A^2||^{\frac{1}{2}}\right) \tag{1.4}$$

and

$$\frac{1}{4}||A^*A + AA^*|| \leqslant w^2(A) \leqslant \frac{1}{2}||A^*A + AA^*||, \tag{1.5}$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A. El-Haddad and Kittaneh [6] showed the following generalizations of the first inequality in (1.4) and the second inequality in (1.5),

$$w^r(A) \leqslant \frac{1}{2}|||A|^{2\alpha r} + |A^*|^{2(1-\alpha)r}|| \tag{1.6}$$

and

$$w^{2r}(A) \leqslant ||\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}||, \tag{1.7}$$

where $0 < \alpha < 1$ and $r \geqslant 1$. Let $A \in B(\mathbb{H})$ with the cartesian decomposition $A = B + iC$ and $r \geqslant 2$. Then the authors [6] got the following inequality

$$w^r(A) \leqslant 2^{\frac{r}{2}-1}|||B|^r + |C|^r||. \tag{1.8}$$

In 2007, Yamazaki [21] proved $w(A) = \sup_{\theta\in\mathbb{R}} ||\Re(e^{i\theta}A)||$. As an alternative formula for the numerical radius, the identity has been used by many researchers. Very recently, Sheikhhosseini et al. [19] defined the weighted numerical radius as

$$w_v(A) = \sup_{\theta\in\mathbb{R}} ||\Re_v(e^{i\theta}A)||,$$

where $0 \leqslant v \leqslant 1$ and $\Re_v(A) = vA + (1-v)A^*$. Here, the function $w_v(\cdot) : B(\mathbb{H}) \to [0,\infty)$ is a norm. They [19] also defined $\Im_v(A) = -ivA + i(1-v)A^*$. New definition of the weighted numerical radius extended some existed results. For example [19],

$$||\Re_v(A)|| \leqslant w_v(A) \quad \text{and} \quad ||\Im_v(A)|| \leqslant w_v(A) \tag{1.9}$$

are coincides with the results $||\Re(A)|| \leqslant w(A)$ and $||\Im(A)|| \leqslant w(A)$ when $v = \frac{1}{2}$, obtained by Kittaneh et al. [13]. In addition, it is clear $w(A) = w_{\frac{1}{2}}(A)$.

Bedrani et al. [1] extended the well known power inequality $w(A^k) \leqslant w^k(A)$ $(A \in \mathbb{M}_n(\mathbb{C}))$ for $k = 1, 2, \cdots$) to accretive matrices as follows

$$\cos(t\alpha)\cos^t(\alpha)w^t(A) \leqslant w(A^t) \leqslant \sec(t\alpha)\sec^{2t}(\alpha)w^t(A), \tag{1.10}$$

where $A \in \mathbb{M}_n(\mathbb{C})$, $W(A) \subset S_\alpha$ and $t \in (0,1)$.

On the other hand, Kittaneh and Sakkijha [14] presented the following Schatten p-norm inequalities for accretive-dissipative matrices $T, S \in \mathbb{M}_n(\mathbb{C})$,

$$2^{-\frac{p}{2}}\left(||T||_p^p + ||S||_p^p\right) \leqslant ||T+S||_p^p \leqslant 2^{\frac{3p}{2}-1}\left(||T||_p^p + ||S||_p^p\right) \tag{1.11}$$

for $p \geqslant 1$.

Recently, Yang [22] showed the following n-tuple power inequality for sectorial matrices

$$w^t\left(\sum_{j=1}^{k} x_j A_j\right) \leqslant \cos^{2t}(\alpha) w\left(\sum_{j=1}^{k} x_j A_j^t\right) \tag{1.12}$$

and

$$w\left(\left(\sum_{j=1}^{k} x_j A_j\right)^t\right) \leqslant \cos^{2t}(\alpha)\sec(t\alpha) w\left(\sum_{j=1}^{k} x_j A_j^t\right), \tag{1.13}$$

where $A_j \in \mathbb{M}_n(\mathbb{C})$ are such that $W(A_j) \subset S_\alpha$, x_j are positive real numbers with $\sum_{j=1}^{k} x_j = 1$ and $t \in [-1, 0]$.

Throughout this paper, we assume every function is continuous and all functions satisfy the following conditions : J is a subinterval of $(0,\infty)$ and $f : J \to (0,\infty)$.

In this paper, we intend to give some generalizations and further refinements of inequalities (1.5)–(1.11). Moreover, we also show the reverse of (1.12)–(1.13).

2. Main results

In order to get our results, we will list some necessary lemmas in front of each theorem. Firstly, we give a generalization and further refinements of the first inequality in (1.5).

THEOREM 1. *Let* $A \in B(\mathbb{H})$ *and* $0 \leqslant v \leqslant 1$. *Then*

$$v(1-v)||A^*A + AA^*|| \leqslant \frac{1}{4}\left(||\Re_v(A) + \Im_v(A)||^2 + ||\Re_v(A) - \Im_v(A)||^2\right) \leqslant w_v^2(A).$$

Proof. We have the following chain of inequalities

$$\begin{aligned} &v(1-v)||A^*A + AA^*|| \\ &= \frac{1}{4}||4v(1-v)(A^*A + AA^*)|| \\ &= \frac{1}{4}||(\Re_v(A) + \Im_v(A))^2 + (\Re_v(A) - \Im_v(A))^2|| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{4}\left(||(\Re_v(A)+\Im_v(A))^2||+||(\Re_v(A)-\Im_v(A))^2||\right)\\
&\leqslant \frac{1}{4}\left(||\Re_v(A)+\Im_v(A)||^2+||\Re_v(A)-\Im_v(A)||^2\right)\\
&= \frac{1}{4}\left(2\left(||\Re_v(A)||^2+||\Im_v(A)||^2\right)\right)\\
&\leqslant \frac{1}{4}\left(2\left(w_v^2(A)+w_v^2(A)\right)\right) \quad \text{(by (1.9))}\\
&= w_v^2(A). \quad \square
\end{aligned}
$$

Next, we give a generalization of the inequality (1.6). Before that, we need a lemma which is known as the generalized mixed Schwarz inequality.

LEMMA 1. ([15]) *Let $A \in B(\mathbb{H})$ and $0 \leqslant v \leqslant 1$. Then for all $x, y \in \mathbb{H}$, we have*

$$|\langle Ax, y\rangle|^2 \leqslant \langle |A|^{2v}x, x\rangle\langle |A^*|^{2(1-v)}y, y\rangle.$$

LEMMA 2. ([7] p. 118) (Operator Jensen inequality for convex function [17]). *Let $A \in B(\mathbb{H})$ be a self-adjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If $f(t)$ is a convex function on $[m, M]$, then*

$$f\left(\langle Ax, x\rangle\right) \leqslant \left\langle f(A)x, x\right\rangle$$

for every unit vector $x \in \mathbb{H}$.

THEOREM 2. *Let $A \in B(\mathbb{H})$ and f be an increasing convex function. If $0 \leqslant v \leqslant 1$, then*

$$f(w(A)) \leqslant \frac{1}{2}||f(|A|^{2v}) + f(|A^*|^{2(1-v)})||.$$

Proof. For every unit vector $x \in \mathbb{H}$, we have

$$
\begin{aligned}
f(|\langle Ax, x\rangle|) &\leqslant f\left(\langle |A|^{2v}x, x\rangle^{\frac{1}{2}}\langle |A^*|^{2(1-v)}x, x\rangle^{\frac{1}{2}}\right) \quad \text{(by Lemma 1)}\\
&\leqslant f\left(\frac{\langle |A|^{2v}x, x\rangle + \langle |A^*|^{2(1-v)}x, x\rangle}{2}\right) \quad \text{(by AM} - \text{GM inequality)}\\
&\leqslant \frac{1}{2}\left(f(\langle |A|^{2v}x, x\rangle) + f(\langle |A^*|^{2(1-v)}x, x\rangle)\right)\\
&\leqslant \frac{1}{2}\left(\langle f(|A|^{2v})x, x\rangle + \langle f(|A^*|^{2(1-v)})x, x\rangle\right) \quad \text{(by Lemma 2)}\\
&= \frac{1}{2}\left\langle \left(f(|A|^{2v}) + f(|A^*|^{2(1-v)})\right)x, x\right\rangle.
\end{aligned}
$$

Hence

$$\begin{aligned} f(w(A)) &= f\Big(\sup_{\|x\|=1} |\langle Ax,x\rangle|\Big) \\ &= \sup_{\|x\|=1} f\big(|\langle Ax,x\rangle|\big) \\ &\leqslant \sup_{\|x\|=1} \frac{1}{2}\big\langle \big(f(|A|^{2v}) + f(|A^*|^{2(1-v)})\big)x,x\big\rangle \\ &= \frac{1}{2}\|f(|A|^{2v}) + f(|A^*|^{2(1-v)})\|. \quad \square \end{aligned}$$

REMARK 1. It is clear that the inequality (1.6) is a special case of Theorem 2 for $f(t)=t^r$ when $r \geqslant 1$.

We now give a generalization of the inequality (1.7).

LEMMA 3. ([15]) *Let $A \in B(\mathbb{H})$ be positive, and let $x \in \mathbb{H}$ be any unit vector. Then*

$$\begin{aligned} \langle A^v x,x\rangle &\leqslant \langle Ax,x\rangle^v \text{ for } 0 < v \leqslant 1; \\ \langle Ax,x\rangle^v &\leqslant \langle A^v x,x\rangle \text{ for } v \geqslant 1. \end{aligned}$$

THEOREM 3. *Let $A \in B(\mathbb{H})$ and f be an increasing convex function. If $0 \leqslant v \leqslant 1$, then*

$$f(w^2(A)) \leqslant \|vf(|A|^2) + (1-v)f(|A^*|^2)\|.$$

Proof. For every unit vector $x \in \mathbb{H}$, we have

$$\begin{aligned} f(|\langle Ax,x\rangle|^2) &\leqslant f\big(\langle |A|^{2v}x,x\rangle\langle |A^*|^{2(1-v)}x,x\rangle\big) \text{ (by Lemma 1)} \\ &\leqslant f\big(\langle |A|^2x,x\rangle^v\langle |A^*|^2x,x\rangle^{1-v}\big) \text{ (by Lemma 3)} \\ &\leqslant f\big(v\langle |A|^2x,x\rangle + (1-v)\langle |A^*|^2x,x\rangle\big) \\ &\leqslant vf\big(\langle |A|^2x,x\rangle\big) + (1-v)f\big(\langle |A^*|^2x,x\rangle\big) \\ &\leqslant v\big\langle f(|A|^2)x,x\big\rangle + (1-v)\big\langle f(|A^*|^2)x,x\big\rangle \\ &= \big\langle \big(vf(|A|^2) + (1-v)f(|A^*|^2)\big)x,x\big\rangle. \end{aligned}$$

Taking supremum over $x \in \mathbb{H}$ with $\|x\| = 1$, we can get Theorem 3. $\square$

REMARK 2. In a recent paper, the authors [9] presented the following numerical radius inequalities

$$f(w^2(A)) \leqslant \frac{1}{2}f\big(w(|A||A^*|)\big) + \frac{1}{4}\|f(|A|^2) + f(|A^*|^2)\| \tag{2.1}$$

under the same conditions as in Theorem 3. We now prove (2.1) improves Theorem 3 when $v=\frac{1}{2}$. In fact, we only need to prove

$$f\big(w(|A||A^*|)\big)\leqslant\frac{1}{2}||f(|A|^2)+f(|A^*|^2)||. \tag{2.2}$$

Estimate

$$\begin{aligned}
f\big(|\langle|A||A^*|x,x\rangle|\big)&=f\big(|\langle|A^*|x,|A|x\rangle|\big)\\
&\leqslant f\big(|||A^*|x||\cdot|||A|x||\big)\\
&=f\big(\langle|A^*|^2x,x\rangle^{\frac{1}{2}}\langle|A|^2x,x\rangle^{\frac{1}{2}}\big)\\
&\leqslant f\left(\frac{\langle|A^*|^2x,x\rangle+\langle|A|^2x,x\rangle}{2}\right)\\
&\leqslant\frac{1}{2}\Big(f\big(\langle|A^*|^2x,x\rangle\big)+f\big(\langle|A|^2x,x\rangle\big)\Big)\\
&\leqslant\frac{1}{2}\Big(\langle f(|A^*|^2)x,x\rangle+\big\langle f(|A|^2)x,x\big\rangle\Big)\\
&=\frac{1}{2}\big\langle\big(f(|A|^2)+f(|A^*|^2)\big)x,x\big\rangle.
\end{aligned}$$

Taking the supremum over unit vectors $x\in\mathbb{H}$ with $||x||=1$ implies the desired inequality (2.2).

Before give the generalization of the inequality (1.8), we show the definition of geometrical convexity: a function f is said geometrically convex if $f(a^vb^{1-v})\leqslant(f(a))^v(f(b))^{1-v}$ for $0\leqslant v\leqslant 1$.

LEMMA 4. ([8] p. 26) *For $a,b\geqslant 0$, $0<v<1$, and $r\neq 0$, let $M_r(a,b,v)=(va^r+(1-v)b^r)^{\frac{1}{r}}$ and let $M_0(a,b,v)=a^vb^{1-v}$. Then*

$$M_r(a,b,v)\leqslant M_s(a,b,v)\ \text{ for }\ r\leqslant s.$$

THEOREM 4. *Let $A\in B(\mathbb{H})$ with the cartesian decomposition $A=B+iC$ and f be an increasing geometrically convex function. If f is convex and $f(1)=1$, then*

$$f^r\left(\frac{w(A)}{\sqrt{2}}\right)\leqslant\left|\left|\frac{f(|B|^r)+f(|C|^r)}{2}\right|\right|,$$

where $r\geqslant 2$.

Proof. For every unit vector $x\in\mathbb{H}$, we have

$$\begin{aligned}
f\left(\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right)&=f\left(\left(\frac{\langle Bx,x\rangle^2+\langle Cx,x\rangle^2}{2}\right)^{\frac{1}{2}}\right)\\
&\leqslant f\left(\left(\frac{\langle|B|x,x\rangle^2+\langle|C|x,x\rangle^2}{2}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant f\left(\left(\frac{\langle |B|x,x\rangle^r+\langle |C|x,x\rangle^r}{2}\right)^{\frac{1}{r}}\right) \quad \text{(by Lemma 4)}\\
&\leqslant \left(f\left(\frac{\langle |B|x,x\rangle^r+\langle |C|x,x\rangle^r}{2}\right)\right)^{\frac{1}{r}}\left(f(1)\right)^{1-\frac{1}{r}}\\
&\leqslant \left(f\left(\frac{\langle |B|^r x,x\rangle+\langle |C|^r x,x\rangle}{2}\right)\right)^{\frac{1}{r}} \quad \text{(by Lemma 3)}\\
&\leqslant \left(\frac{f(\langle |B|^r x,x\rangle)+f(\langle |C|^r x,x\rangle)}{2}\right)^{\frac{1}{r}}\\
&\leqslant \left(\frac{\langle f(|B|^r)x,x\rangle+\langle f(|C|^r)x,x\rangle}{2}\right)^{\frac{1}{r}}\\
&= \left(\frac{\langle \big(f(|B|^r)+f(|C|^r)\big)x,x\rangle}{2}\right)^{\frac{1}{r}}.
\end{aligned}
$$

Since f is continuous and increasing, we have

$$
\begin{aligned}
f^r\left(\frac{w(A)}{\sqrt{2}}\right) &= f^r\left(\sup_{||x||=1}\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right)\\
&= \sup_{||x||=1} f^r\left(\frac{|\langle Ax,x\rangle|}{\sqrt{2}}\right)\\
&\leqslant \sup_{||x||=1}\frac{\langle \big(f(|B|^r)+f(|C|^r)\big)x,x\rangle}{2}\\
&= \left|\left|\frac{f(|B|^r)+f(|C|^r)}{2}\right|\right|,
\end{aligned}
$$

as desired. □

REMARK 3. The inequality (1.8) comes from Theorem 4 when $f(t)=t$.

Next, we give a generalization of the inequality (1.9) as promised.

THEOREM 5. *Let* $T=\begin{pmatrix} 0 & A\\ B & 0\end{pmatrix}$. *Then*

$$
\max\left\{\frac{1}{2}||\Re_v(e^{i\theta}A)+\Re_v(e^{i\theta}B)||,\frac{1}{2}||\Im_v(e^{i\theta}A)-\Im_v(e^{i\theta}B)||\right\}\leqslant w_v(T),
$$

where $A,B\in B(\mathbb{H})$, $\theta\in\mathbb{R}$ *and* $0\leqslant v\leqslant 1$.

Proof. Let $M_\theta=\Re_v(e^{i\theta}T)$ and $U=\begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}$. Then we have

$$
M_\theta+U^*M_\theta U=\begin{pmatrix} 0 & \Re_v(e^{i\theta}A)+\Re_v(e^{i\theta}B)\\ \Re_v(e^{i\theta}A)+\Re_v(e^{i\theta}B) & 0\end{pmatrix}.
$$

With the fact $\left|\left|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right|\right| = \left|\left|\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right|\right| = \max\left\{||A||, ||B||\right\}$, we get

$$\begin{aligned} ||\Re_v(e^{i\theta}A) + \Re_v(e^{i\theta}B)|| &= ||M_\theta + U^*M_\theta U|| \\ &\leqslant ||M_\theta|| + ||U^*M_\theta U|| \\ &\leqslant 2||M_\theta|| \\ &\leqslant 2w_v(T), \end{aligned}$$

that is

$$\frac{1}{2}||\Re_v(e^{i\theta}A) + \Re_v(e^{i\theta}B)|| \leqslant w_v(T). \tag{2.3}$$

Similarly,

$$M_\theta - U^*M_\theta U = \begin{pmatrix} 0 & i\big(\Im_v(e^{i\theta}A) - \Im_v(e^{i\theta}B)\big) \\ i\big(\Im_v(e^{i\theta}A) - \Im_v(e^{i\theta}B)\big) & 0 \end{pmatrix}.$$

We obtain

$$\begin{aligned} ||\Im_v(e^{i\theta}A) - \Im_v(e^{i\theta}B)|| &= ||i\big(\Im_v(e^{i\theta}A) - \Im_v(e^{i\theta}B)\big)|| \\ &= ||M_\theta - U^*M_\theta U|| \\ &\leqslant ||M_\theta|| + ||U^*M_\theta U|| \\ &\leqslant 2w_v(T), \end{aligned}$$

that is

$$\frac{1}{2}||\Im_v(e^{i\theta}A) - \Im_v(e^{i\theta}B)|| \leqslant w_v(T). \quad \square \tag{2.4}$$

REMARK 4. We can get the inequalities (1.9) by (2.3) and (2.4) when $A = B$ and $A = -B$, respectively.

Next, we give some $n-$tuple numerical radii inequalities for sectorial matrices which generalized (1.10).

LEMMA 5. ([4]) *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$ and $t \in [0,1]$. Then*

$$\cos^{2t}(\alpha)\Re(A^t) \leqslant \Re^t(A) \leqslant \Re(A^t).$$

LEMMA 6. ([1]) *Let $A \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A) \subset S_\alpha$. Then*

$$\cos(\alpha)w(A) \leqslant w(\Re A) \leqslant w(A).$$

LEMMA 7. ([3]) *Let $A_1, A_2, \cdots A_n \geqslant 0$. Then for every non-negative concave function f on $[0,\infty)$ and for every unitarily invariant norm $|||\cdot|||$,*

$$\left|\left|\left|f\Big(\sum_{j=1}^{n} A_j\Big)\right|\right|\right| \leqslant \left|\left|\left|\sum_{j=1}^{n} f(A_j)\right|\right|\right|.$$

THEOREM 6. *Let $A_i \in \mathbb{M}_n(\mathbb{C})$ with $W(A_i) \subset S_\alpha$ and $t \in [0,1]$. Then*

$$\cos^t(\alpha) w^t\left(\sum_{i=1}^k A_i\right) \leqslant w\left(\sum_{i=1}^k A_i^t\right).$$

Proof. Under the conditions, we have the following chain of inequalities

$$\begin{aligned}
\cos^t(\alpha) w^t\left(\sum_{i=1}^k A_i\right) &\leqslant w^t\left(\Re(\sum_{i=1}^k A_i)\right) \quad \text{(by Lemma 6)}\\
&= \left\|\Re(\sum_{i=1}^k A_i)\right\|^t\\
&= \left\|\left(\Re(\sum_{i=1}^k A_i)\right)^t\right\|\\
&\leqslant \left\|\sum_{i=1}^k \Re^t(A_i)\right\| \quad \text{(by Lemma 7)}\\
&\leqslant \left\|\sum_{i=1}^k \Re(A_i^t)\right\| \quad \text{(by Lemma 5)}\\
&= \left\|\Re(\sum_{i=1}^k A_i^t)\right\|\\
&= w\left(\Re(\sum_{i=1}^k A_i^t)\right)\\
&\leqslant w\left(\sum_{i=1}^k A_i^t\right) \quad \text{(by Lemma 6)}. \qquad \square
\end{aligned}$$

COROLLARY 1. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$ and $t \in [0,1]$. Then*

$$\cos^t(\alpha) w^t(A) \leqslant w(A^t).$$

Proof. Let $k = 1$ in Theorem 6. $\square$

REMARK 5. Corollary 1 is a refinement of the left-hand side in (1.10).

Next, we present some relations between $w\left(\sum_{i=1}^k A_i^t\right)$ and $w\left(\left(\sum_{i=1}^k A_i\right)^t\right)$ when A_i are sectorial matrices, which can be regarded as a complement of Theorem 6.

LEMMA 8. ([4]) *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $W(A) \subset S_\alpha$ and $t \in [-1,0]$. Then*

$$\Re(A^t) \leqslant \Re^t(A) \leqslant \cos^{2t}(\alpha)\Re(A^t).$$

LEMMA 9. ([16]) *Let $A_1, A_2, \cdots A_n \geqslant 0$. Then for every non-negative convex function f on $[0,\infty)$ with $f(0)=0$ and for every unitarily invariant norm $|||\cdot|||$,*

$$\left|\left|\left|\sum_{j=1}^{n} f(A_j)\right|\right|\right| \leqslant \left|\left|\left|f\Big(\sum_{j=1}^{n} A_j\Big)\right|\right|\right|.$$

THEOREM 7. *Let $A_i \in \mathbb{M}_n(\mathbb{C})$ with $W(A_i) \subset S_\alpha$ and $t \in [-1,0]$. Then*

$$\sec^{2t}(\alpha)\cos(\alpha) w\left(\sum_{i=1}^{k} A_i^t\right) \leqslant w\left((\sum_{i=1}^{k} A_i)^t\right),$$

where $i = 1, 2, \cdots, k$.

Proof. Compute

$$\begin{aligned}
w\left((\sum_{i=1}^{k} A_i)^t\right) &\geqslant w\left(\Re\big((\sum_{i=1}^{k} A_i)^t\big)\right) \quad \text{(by Lemma 6)}\\
&= \left\|\Re\big((\sum_{i=1}^{k} A_i)^t\big)\right\|\\
&\geqslant \sec^{2t}(\alpha)\left\|\big(\Re(\sum_{i=1}^{k} A_i)\big)^t\right\| \quad \text{(by Lemma 8)}\\
&= \sec^{2t}(\alpha)\left\|\big(\sum_{i=1}^{k} \Re(A_i)\big)^t\right\|\\
&\geqslant \sec^{2t}(\alpha)\left\|\sum_{i=1}^{k} \Re^t(A_i)\right\| \quad \text{(by Lemma 9)}\\
&\geqslant \sec^{2t}(\alpha)\left\|\sum_{i=1}^{k} \Re(A_i^t)\right\| \quad \text{(by Lemma 8)}\\
&= \sec^{2t}(\alpha)\left\|\Re\big(\sum_{i=1}^{k} A_i^t\big)\right\|\\
&= \sec^{2t}(\alpha) w\left(\Re\big(\sum_{i=1}^{k} A_i^t\big)\right)\\
&\geqslant \sec^{2t}(\alpha)\cos(\alpha) w\left(\sum_{i=1}^{k} A_i^t\right) \quad \text{(by Lemma 6)}. \qquad \square
\end{aligned}$$

We now give a reverse of Theorem 7.

THEOREM 8. *Let $A_i \in \mathbb{M}_n(\mathbb{C})$ with $W(A_i) \subset S_\alpha$ and $t \in [0,1]$. Then*

$$w\left((\sum_{i=1}^{k} A_i)^t\right) \leqslant \sec^{2t}(\alpha)\sec(t\alpha) w\left(\sum_{i=1}^{k} A_i^t\right),$$

where $i = 1, 2, \cdots, k$.

Proof. We have the following chain of inequalities

$$
\begin{aligned}
w\Big((\sum_{i=1}^{k}A_i)^t\Big) &\leqslant \sec(t\alpha)w\Big(\Re\big((\sum_{i=1}^{k}A_i)^t\big)\Big) \quad \text{(by (1.3) and Lemma 6)}\\
&= \sec(t\alpha)\Big|\Big|\Re\big((\sum_{i=1}^{k}A_i)^t\big)\Big|\Big|\\
&\leqslant \sec^{2t}(\alpha)\sec(t\alpha)\Big|\Big|\big(\Re(\sum_{i=1}^{k}A_i)\big)^t\Big|\Big| \quad \text{(by Lemma 5)}\\
&= \sec^{2t}(\alpha)\sec(t\alpha)\Big|\Big|\big(\sum_{i=1}^{k}\Re(A_i)\big)^t\Big|\Big|\\
&\leqslant \sec^{2t}(\alpha)\sec(t\alpha)\Big|\Big|\sum_{i=1}^{k}\Re^t(A_i)\Big|\Big| \quad \text{(by Lemma 7)}\\
&\leqslant \sec^{2t}(\alpha)\sec(t\alpha)\Big|\Big|\sum_{i=1}^{k}\Re(A_i^t)\Big|\Big| \quad \text{(by Lemma 5)}\\
&= \sec^{2t}(\alpha)\sec(t\alpha)\Big|\Big|\Re(\sum_{i=1}^{k}A_i^t)\Big|\Big|\\
&= \sec^{2t}(\alpha)\sec(t\alpha)w\Big(\Re(\sum_{i=1}^{k}A_i^t)\Big)\\
&\leqslant \sec^{2t}(\alpha)\sec(t\alpha)w\Big(\sum_{i=1}^{k}A_i^t\Big) \quad \text{(by Lemma 6)}. \qquad \square
\end{aligned}
$$

Next, we give some generalizations and further refinements of Schatten p-norms inequalities (1.11) for accretive-dissipative matrices.

LEMMA 10. ([10]) *Let A,B be positive and f be an increasing convex function on $[0,\infty)$. Then for every unitarily invariant norm $|||\cdot|||$,*

$$|||f(|A+iB|)||| \leqslant |||f(A+B)||| \leqslant |||f(\sqrt{2}\,|A+iB|)|||.$$

LEMMA 11. ([2]) *Let $A_1,A_2,\cdots A_n$ be positive and $p \geqslant 1$. Then*

$$\sum_{j=1}^{n}||A_j||_p^p \leqslant \Big|\Big|\sum_{j=1}^{n}A_j\Big|\Big|_p^p \leqslant n^{p-1}\sum_{j=1}^{n}||A_j||_p^p.$$

THEOREM 9. *Let $T_1,T_2,\cdots T_n \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative. Then for every increasing convex function f on $[0,\infty)$ with $f(0)=0$ and $p \geqslant 1$, we have*

$$\Big|\Big|f\big(\sqrt{2}\,|\sum_{j=1}^{n}T_j|\big)\Big|\Big|_p^p \geqslant \sum_{j=1}^{n}||f(|T_j|)||_p^p.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1,2,\cdots,n$, be the Cartesian decompositions of T_j. Then we have

$$\begin{aligned}
\left\|f\left(\sqrt{2}\,\left|\sum_{j=1}^{n} T_j\right|\right)\right\|_p^p &= \left\|f\left(\sqrt{2}\,\left|\sum_{j=1}^{n} A_j + i\sum_{j=1}^{n} B_j\right|\right)\right\|_p^p \\
&\geqslant \left\|f\left(\sum_{j=1}^{n} A_j + \sum_{j=1}^{n} B_j\right)\right\|_p^p \quad \text{(by Lemma 10)} \\
&= \left\|f\left(\sum_{j=1}^{n} (A_j + B_j)\right)\right\|_p^p \\
&\geqslant \left\|\sum_{j=1}^{n} f(A_j + B_j)\right\|_p^p \quad \text{(by Lemma 9)} \\
&\geqslant \sum_{j=1}^{n} \left\|f(A_j + B_j)\right\|_p^p \quad \text{(by Lemma 11)} \\
&\geqslant \sum_{j=1}^{n} \left\|f(|A_j + iB_j|)\right\|_p^p \quad \text{(by Lemma 10)} \\
&= \sum_{j=1}^{n} \left\|f(|T_j|)\right\|_p^p. \quad \square
\end{aligned}$$

REMARK 6. The left-hand side in (1.11) follows as a special case of Theorem 9 with $f(t) = t$ and $n = 2$.

LEMMA 12. ([10]) *Let A,B be positive and f be a non-negative increasing concave function on $[0,\infty)$. Then for every unitarily invariant norm $|||\cdot|||$,*

$$\frac{1}{2}|||f(2|A+iB|)||| \leqslant |||f(A+B)||| \leqslant |||f(\sqrt{2}\,|A+iB|)|||.$$

THEOREM 10. *Let $T_1, T_2, \cdots T_n \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative. Then for every non-negative increasing concave function f on $[0,\infty)$ and $p \geqslant 1$, we have*

$$\left\|f\left(2\,\left|\sum_{j=1}^{n} T_j\right|\right)\right\|_p^p \leqslant 2\cdot n^{p-1}\sum_{j=1}^{n}\left\|f\left(\sqrt{2}\,|T_j|\right)\right\|_p^p.$$

Proof. Let $T_j = A_j + iB_j$, $j = 1,2,\cdots,n$, be the cartesian decompositions of T_j. Then we have

$$\begin{aligned}
\frac{1}{2}\left\|f\left(2\,\left|\sum_{j=1}^{n} T_j\right|\right)\right\|_p^p &= \frac{1}{2}\left\|f\left(2\,\left|\sum_{j=1}^{n} A_j + i\sum_{j=1}^{n} B_j\right|\right)\right\|_p^p \\
&\leqslant \left\|f\left(\sum_{j=1}^{n} A_j + \sum_{j=1}^{n} B_j\right)\right\|_p^p \quad \text{(by Lemma 12)}
\end{aligned}$$

$$\begin{aligned}
&= \left|\left| f\left(\sum_{j=1}^{n}(A_j+B_j)\right)\right|\right|_p^p \\
&\leqslant \left|\left| \sum_{j=1}^{n} f(A_j+B_j)\right|\right|_p^p \quad \text{(by Lemma 7)} \\
&\leqslant n^{p-1}\sum_{j=1}^{n}\left|\left| f(A_j+B_j)\right|\right|_p^p \quad \text{(by Lemma 11)} \\
&\leqslant n^{p-1}\sum_{j=1}^{n}\left|\left| f\left(\sqrt{2}\,|A_j+iB_j|\right)\right|\right|_p^p \quad \text{(by Lemma 12)} \\
&= n^{p-1}\sum_{j=1}^{n}\left|\left| f\left(\sqrt{2}\,|T_j|\right)\right|\right|_p^p. \quad \square
\end{aligned}$$

COROLLARY 2. *Let $T,S \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative and $p \geqslant 1$. Then we have*

$$||T+S||_p^p \leqslant 2^{\frac{p}{2}}\left(||T||_p^p + ||S||_p^p\right).$$

Proof. Let $f(t)=t$ and $n=2$ in Theorem 10. $\square$

REMARK 7. Corollary 2 is a refinement of the right-hand side in (1.11).

Next, we give a reverse of (1.12).

LEMMA 13. ([20]) *Let $A_1, A_2, \cdots A_k \geqslant 0$ and $x_1, x_2, \cdots, x_k$ be positive real numbers with $\sum_{j=1}^{k} x_j = 1$. Then for every unitarily invariant norm $|||\cdot|||$ on $M_n(\mathbb{C})$,*

$$\left|\left|\left|\sum_{j=1}^{n} x_j f(A_j)\right|\right|\right| \leqslant \left|\left|\left| f\left(\sum_{j=1}^{n} x_j A_j\right)\right|\right|\right|.$$

for every non-negative concave function f on $[0,\infty)$.

THEOREM 11. *Let $A_j \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A_j) \subset S_\alpha$ and x_j be positive real numbers with $\sum_{j=1}^{k} x_j = 1$. Then*

$$w^t\left(\sum_{j=1}^{k} x_j A_j\right) \geqslant \cos^{2t}(\alpha)\cos(\alpha) w\left(\sum_{j=1}^{k} x_j A_j^t\right),$$

where $j=1,2,\cdots,k$ and $t \in [0,1]$.

Proof. Compute

$$\begin{aligned}
w^t\left(\sum_{j=1}^{k} x_j A_j\right) &\geqslant w^t\left(\Re\left(\sum_{j=1}^{k} x_j A_j\right)\right) \quad \text{(by Lemma 6)} \\
&= \left|\left|\Re\left(\sum_{j=1}^{k} x_j A_j\right)\right|\right|^t
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \left(\sum_{j=1}^{k} x_j \Re(A_j) \right)^t \right\| \\
&\geqslant \left\| \sum_{j=1}^{k} x_j \Re^t(A_j) \right\| \quad \text{(by Lemma 13)} \\
&\geqslant \left\| \sum_{j=1}^{k} x_j \cos^{2t}(\alpha) \Re(A_j^t) \right\| \quad \text{(by Lemma 5)} \\
&= \cos^{2t}(\alpha) \left\| \Re\left(\sum_{j=1}^{k} x_j A_j^t \right) \right\| \\
&= \cos^{2t}(\alpha) w\left(\Re\left(\sum_{j=1}^{k} x_j A_j^t \right) \right) \\
&\geqslant \cos^{2t}(\alpha) \cos(\alpha) w\left(\sum_{j=1}^{k} x_j A_j^t \right) \quad \text{(by Lemma 6).} \quad \square
\end{aligned}
$$

Next, we give a reverse of (1.13).

THEOREM 12. *Let $A_j \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A_j) \subset S_\alpha$ and x_j be positive real numbers with $\sum_{j=1}^{k} x_j = 1$. Then*

$$
w\left(\left(\sum_{j=1}^{k} x_j A_j \right)^t \right) \geqslant \cos^{2t}(\alpha) \cos(\alpha) w\left(\sum_{j=1}^{k} x_j A_j^t \right),
$$

where $j = 1, 2, \cdots, k$ and $t \in [0, 1]$.

Proof. We have

$$
\begin{aligned}
w\left(\left(\sum_{j=1}^{k} x_j A_j \right)^t \right) &\geqslant w\left(\Re\left(\sum_{j=1}^{k} x_j A_j \right)^t \right) \quad \text{(by Lemma 6)} \\
&= \left\| \Re\left(\sum_{j=1}^{k} x_j A_j \right)^t \right\| \\
&\geqslant \left\| \Re^t\left(\sum_{j=1}^{k} x_j A_j \right) \right\| \quad \text{(by Lemma 5)} \\
&= \left\| \left(\sum_{j=1}^{k} x_j \Re(A_j) \right)^t \right\| \\
&\geqslant \cos^{2t}(\alpha) \cos(\alpha) w\left(\sum_{j=1}^{k} x_j A_j^t \right) \quad \text{(by Theorem 11).} \quad \square
\end{aligned}
$$

REMARK 8. As we can see that inequalities (1.3) is stronger than (1.2). However, it should be noticed that when $t \in [-1,0]$, inequality (1.3) implies $\alpha = 0$ instead of $\alpha \in [0, \frac{\pi}{2})$ with the definition of S_α. Now, under the same conditions as in (1.13), we rewrite it as follows:

$$w\left(\left(\sum_{j=1}^{k} x_j A_j\right)^t\right) \leqslant \cos^{2t}(\alpha)\sec(\alpha) w\left(\sum_{j=1}^{k} x_j A_j^t\right). \tag{2.5}$$

The proof of (2.5) is consistent with the rest of (1.13).

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