Journal of Mathematical Inequalities
Volume 17, Number 4 (2023), 1427–1431
doi:10.7153/jmi-2023-17-93

SKEW–CIRCULANT MATRIX AND CRITICAL POINTS OF POLYNOMIALS

YUNLAN WEI, YANPENG ZHENG* AND ZHAOLIN JIANG

(*Communicated by L. Mihoković*)

Abstract. In this paper, we first prove a relation between the critical points of the skew-circulant matrix and the eigenvalues of its principal matrix. Furthermore, we reprove the inequality about the zeros of a polynomial and its critical points by using the properties of skew-circulant matrix, which is to show that we can not only find the skew-circulant matrix, but also give more structure matrices to prove this inequality.

1. Introduction

There are some interesting relations between critical points and zeros of a polynomial. Since Schoenberg [8] conjectured that the following quadratic inequality

$$\sum_{k=1}^{n-1} |\omega_k|^2 \leqslant \frac{1}{n^2}\left|\sum_{j=1}^{n} z_j\right|^2 + \frac{n-2}{n}\sum_{j=1}^{n} |z_j|^2,$$

where z_j $(j=1,2,\cdots,n)$ are the zeros and ω_k $(k=1,2,\cdots,n-1)$ are the critical points of the polynomial $p(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$, respectively. The studies centring Schoenberg's conjecture have a history. Further, de Bruin and Sharma [1] proposed a higher order conjecture, the quartic inequality. In 2016, Kushel and Tyaglov [4] reconfirmed these conjectures by circulant matrices. Further, Lin [5] solved some problems proposed by Kushel and Tyaglov, and gave a more general case about the above inequalities.

In this paper, we consider the $n\times n$ skew-circulant matrix

$$S=\begin{pmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ -s_{n-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_1 \\ -s_1 & \cdots & -s_{n-1} & s_0 \end{pmatrix}.$$

Mathematics subject classification (2020): 14A25, 15A42, 15B05.
Keywords and phrases: Critical point, eigenvalue, inequality, skew-circulant matrix.
* Corresponding author.

Paper JMI-17-93

Firstly, we prove the relations between the critical points of the skew-circulant matrix and the eigenvalues of its principal matrix. As with circulant matrices, the relations given can be used to prove the inequality between the zeros and critical values of two polynomials in [5]. Besides, based on the properties of skew-circulant matrix, we reprove the inequality about the zeros of a polynomial and its critical points.

2. Results

THEOREM 1. *Let S be an $n\times n$ skew circulant matrix and let $p(z)=\det(zI-S)$. Then the critical points of $p(z)$ coincide with the eigenvalues of S_n, where S_n is the $(n-1)$th principal submatrix of S.*

Proof. Suppose that the function $f_{ij}(x)$ $(i,j=1,2,\cdots,n)$ is derivable on the interval J. Then the determinant function

$$f(x)=\begin{vmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{vmatrix}$$

can be derivable on the interval J, and

$$f'(x)=\sum_{i=1}^{n}\begin{vmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ f'_{i1}(x) & f'_{i2}(x) & \cdots & f'_{in}(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{vmatrix} \tag{1}$$

Denote S_i as the principal submatrix of skew-circulant matrix S by removing the i-th row and column, where $i=1,2,\cdots,n$. According to Eq. (1), we calculate the derivate of the characteristic polynomial

$$p(z)=\det(zI-S).$$

Then we have

$$p'(z)=\sum_{i=1}^{n}\det(zI-S_i). \tag{2}$$

Perform some similarity transformation on the characteristic polynomial $p(z)$, we have

$$\det(zI-S_i)=\det(zI-S_n), \quad \text{for } i=1,2,\cdots,n-1. \tag{3}$$

Therefore, we can derive from Eqs. (2) and (3) that

$$p'(z) = n\det(zI - S_n).$$

The proof is complete. $\square$

REMARK 1. In [4], Kushel and Tyaglov proved the relations between the critical points of a circulant matrix and the eigenvalues of its principal submatrix using Cheung and Ng's [2] technique. Here, we just use the derivation technique and the properties of skew-circulant matrix to prove the above relations satisfied by skew-circulant matrix.

LEMMA 1. [3] *Let $T = (t_{kj})_{k,j=1}^{n}$ is an $n \times n$ complex matrix with eigenvalues $u_1, u_2, \cdots u_n$. Then*

$$\sum_{j=1}^{n} |u_j|^q \leqslant \sum_{j=1}^{n} \left(\sum_{k=1}^{n} |t_{kj}|^{\frac{q}{q-1}} \right)^{q-1} \quad \textit{for } q \geqslant 2.$$

Next, with the help of Lemma 1 and the properties of shew-circulant matrix, we reprove the Theorem 3.2 in [5].

THEOREM 2. *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the roots of a polynomial p of degree $n \geqslant 2$, and let $\omega_1, \omega_2, \cdots, \omega_{n-1}$ be the roots of the derivative p'. Then*

$$\sum_{k=1}^{n-1} |\omega_k|^q \leqslant \frac{(n-1)^{q-2}}{n^q} \left| \sum_{j=1}^{n} \lambda_j \right|^q + \frac{(n-1)^{q-2}(n-2)}{n^{q/2}} \left(\sum_{j=1}^{n} |\lambda_j|^2 \right)^{q/2} \quad \textit{for } q \geqslant 2.$$

Proof. Let $C = (c_{ij})$ be a complex matrix of order $n-1$. According to the monotonicity of the generalized mean function, we have

$$\left(\frac{\sum_{i=1}^{n-1} |c_{ij}|^{\frac{q}{q-1}}}{n-1} \right)^{q-1} \leqslant \frac{\sum_{i=1}^{n-1} |c_{ij}|^q}{n-1} \quad \text{for } q \geqslant 1, j = 1, \ldots, n-1.$$

Arranging the above inequality, we get

$$\left(\sum_{i=1}^{n-1} |c_{ij}|^{\frac{q}{q-1}} \right)^{q-1} \leqslant (n-1)^{q-2} \sum_{i=1}^{n-1} |c_{ij}|^q \quad \text{for } q \geqslant 2, j = 1, \ldots, n-1.$$

Summing the above inequality with respect to the variable j

$$\sum_{j=1}^{n-1} \left(\sum_{i=1}^{n-1} |c_{ij}|^{\frac{q}{q-1}} \right)^{q-1} \leqslant (n-1)^{q-2} \sum_{i,j=1}^{n-1} |c_{ij}|^q \quad \text{for } q \geqslant 2. \tag{4}$$

According to Lemma 1, and considering S_{n-1} as the matrix C in Eq. (4), we have

$$\begin{aligned}\sum_{k=1}^{n-1}|\omega_k|^q &\leqslant (n-1)^{q-2}\left(|s_0|^q+(n-2)\sum_{k=0}^{n-1}|s_k|^q\right)\\ &\leqslant (n-1)^{q-2}\left(|s_0|^q+(n-2)\left(\sum_{k=0}^{n-1}|s_k|^2\right)^{q/2}\right)\end{aligned}\tag{5}$$

for $q \geqslant 2$. According to the properties of skew-circulant matrix S, we have

$$s_0=\frac{1}{n}\sum_{j=1}^{n}\lambda_j,\ \sum_{k=0}^{n-1}|s_k|^2=\frac{1}{n}\sum_{j=1}^{n}|\lambda_j|^2.\tag{6}$$

Further, taking Eq. (6) into Eq. (5), we get

$$\begin{aligned}\sum_{k=1}^{n-1}|\omega_k|^q &\leqslant (n-1)^{q-2}\left(\frac{1}{n^q}\left|\sum_{j=1}^{n}\lambda_j\right|^q+(n-2)\left(\frac{1}{n}\sum_{j=1}^{n}|\lambda_j|^2\right)^{q/2}\right)\\ &=\frac{(n-1)^{q-2}}{n^q}\left|\sum_{j=1}^{n}\lambda_j\right|^q+\frac{(n-1)^{q-2}(n-2)}{n^{q/2}}\left(\sum_{j=1}^{n}|\lambda_j|^2\right)^{q/2}\end{aligned}$$

for $q \geqslant 2$. The proof is complete. □

REMARK 2. We no longer use the properties of the circulant matrix [5] but the skew-circulant matrix as a bridge to prove the above inequality in Theorem 2. This is to show that we can not only find the skew-circulant matrix, but also give more structure matrices to prove this inequality, like the Toeplitz matrix.

Funding. The research was supported by Natural Science Foundation of Shandong Province (Grant Nos. ZR2022QA057, ZR2022MA092 and ZR2020QA035) and National Natural Science Foundation of China (Grant No. 12001257).

Competing Interests. All authors disclosed no relevant relationships.

REFERENCES

[1] M. BRUIN AND A. SHARMA, *On a Schoenberg-type conjecture*, J. Comput. Appl. Math., **105**, 1 (1999), 221–228.

[2] W. CHEUNG AND T. NG, *Relationships between the zeros of two polynomials*, Linear Algebra Appl., **432**, 1 (2011), 107–115.

[3] W. JOHNSON, H. KÖNIG, B. MAUREY AND J. RETHERFORD, *Eigenvalues of p-summing and p-type operators in Banach spaces*, J. Funct. Anal., **32**, 1 (1979), 353–380.

[4] O. KUSHEL AND M. TYAGLOV, *Circulants and critical points of polynomials*, J. Math. Anal. Appl., **439**, 1 (2016), 434–450.

[5] M. LIN, M. XIE AND J. ZHANG, *Remarks on circulant matrices and critical points of polynomials*, J. Math. Anal. Appl., **502**, 1 (2021), 125233.

[6] S. MALAMUD, *Inverse spectral problem for normal matrices and the Gauss-Lucas theorem*, Transl. Am. Math. Soc., **357**, 1 (2004), 4043–4064.

[7] R. PEREIRA, *Differentiators and the geometry of polynomials*, J. Math. Anal. Appl., **285**, 1 (2003), 336–348.

[8] I. SCHOENBERG, *A conjectured analogue of Rolles theorem for polynomials with real or complex coefficients*, Am. Math. Mon., **93**, 1 (1986), 8–13.

(Received June 29, 2023)

Yunlan Wei
School of Mathematics and Statistics
Linyi University
Linyi 276000, China
e-mail: wyl19910110@163.com

Yanpeng Zheng
School of Automation and Electrical Engineering
Linyi University
Linyi 276000, China
e-mail: zhengyanpeng0702@sina.com

Zhaolin Jiang
School of Mathematics and Statistics
Linyi University
Linyi 276000, China
e-mail: jzh1208@sina.com