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FIXED POINT THEOREMS FOR $\alpha - \beta_M$–GERAGHTY TYPE CONTRACTIONS WITH APPLICATIONS IN MATRIX EQUATIONS

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Abstract. In this manuscript, we present a novel notion of $\alpha - \beta_M$-Geraghty type contractions, which are employed to establish the existence and uniqueness of a fixed point in complete b-metric spaces. To emphasize the significance of our results, illustrative examples are provided. Furthermore, we utilize the obtained results to demonstrate the existence of a solution for matrix equations. Thus, our results provide a suitable extension in this respect.

1. Introduction

In complete metric spaces, Geraghty [10] established a generalization of the Banach contraction principle by considering an auxiliary function, which has captured the attention of many researchers. In 2012, Samet *et al.* [19] presented fixed point theorems for α-ψ-contractive-type mappings and derived theorems regarding fixed points for such mappings. Subsequently, Karapinar and Samet [13] generalized the results presented in [19]. They achieved this by introducing a novel concept known as generalized α-ψ-contractive type mappings. In 2013, Karapinar *et al.* [12] introduced the notion of α-ψ-Meir-Keeler contractive mappings via a triangular α-admissible mapping in metric spaces. Afterwards, Cho *et al.* [7] introduced the concept of α-Geraghty contraction mappings in metric spaces and provided a proof of some fixed point results of such mappings. In 2014, Popescu [17] conducted research on fixed point theorems for mappings that belong to the generalized α-Geraghty type contraction in complete metric spaces. The notion of φ_E-Geraghty contractions was introduced by Fulga and Proca [9], and they established a fixed point result in complete metric spaces.

In 1989, Bakhtin [4] introduced the concept of a b-metric space, which was further utilized by Czerwick [8] to establish various fixed point results on this platform. The study of b-metric space holds a crucial place in fixed point theory from multiple perspectives. In 2019, Aydi *et al.* [3] extended the results of [9] by presenting the concept of α-β_E-Geraghty type contraction mappings and establishing the existence as well as the uniqueness of a fixed point for such mappings in the context of b-metric spaces.

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Matrix equations play a crucial role in numerous engineering and applied mathematics problems. Stability analysis [22], ladder networks [2], control theory [23, 27], and system theory [24, 25, 26] all involve various matrix equations. To test the existence of solutions to matrix equations, several advanced methods exist. One such method involves the application of tools from fixed point theory. By utilizing fixed point results, many researchers have successfully verified the existence and uniqueness of solutions to matrix equations [5, 14, 18, 20].

Inspired by [3], we introduce a new concept of α-β_M-Geraghty type contractions to demonstrate the existence and uniqueness of a fixed point in complete b-metric spaces. We provide examples to support our results. Moreover, we utilize the derived results to establish the existence of solutions to matrix equations.

2. Preliminaries

The following section is dedicated to recalling certain important notations and definitions that are imperative to comprehend the main results. Throughout this paper, let X represent a nonempty set.

DEFINITION 1. [4, 8] A mapping $d : X \times X \to [0,\infty)$ is referred to as a b-metric on X if it satisfies the following conditions for all $u,v,w \in X$ and some $s \geqslant 1$:

(i) $d(u,v) = 0$ if and only if $u = v$;

(ii) $d(u,v) = d(v,u)$;

(iii) $d(u,w) \leqslant s[d(u,v)+d(v,w)]$.

Furthermore, the triplet (X,d,s) is identified as a b-metric space.

It is noteworthy to mention that each metric space can be regarded as a b-metric space with a constant value of $s = 1$. In general, a b-metric is not continuous. Below, we present some examples of b-metric spaces.

EXAMPLE 1. 1. Let $X = \mathbb{R}$, and define the mapping $d : X \times X \to [0,\infty)$ as

$$d(u,v) = (u-v)^2 \quad \text{for all } u,v \in X.$$

This construction ensures that $(X,d,2)$ is a b-metric. It is obvious that, here, $s = 2$.

2. Let $X = \{p,q,r\}$, and define the mapping $d : X \times X \to [0,\infty)$ as follows:

$$d(p,p) = d(q,q) = d(r,r) = 0, \ d(p,q) = d(q,p) = 2,$$

$$d(q,r) = d(r,q) = 3 \text{ and } d(p,r) = d(r,p) = 10.$$

Then, $(X,d,2)$ is a b-metric. It is obvious that, here, $s = 2$.

3. [6] Let $X = l_q(\mathbb{R})$ with $0<q<1$ where $l_q(\mathbb{R}) := \{\{u_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |u_n|^q < \infty\}$, and define the mapping $d : X \times X \to [0,\infty)$ by

$$d(u,v) = \left(\sum_{n=1}^{\infty} |u_n - v_n|^q\right)^{1/q},$$

where $u = \{u_n\}, v = \{v_n\}$. Then, d is a b-metric with a constant value of $s = 2^{1/q}$.

4. Let $X = \mathbb{R} \setminus \{0\}$, the mapping $d : X \times X \to [0,\infty)$ defined by

$$d(u,v) = (u-v)^2 + \left(\frac{1}{u} - \frac{1}{v}\right)^2 \quad \text{for all } u,v \in X.$$

It can be shown that d is a b-metric on X with a constant value of $s = 2$.

For additional intriguing examples of b-metric spaces, see e.g. [1, 11]. In conclusion, the above examples demonstrate the versatility and applicability of b-metric spaces in diverse settings.

DEFINITION 2. [8] A sequence $\{u_n\}$ in a b-metric space (X,d,s) is Cauchy if for any $\varepsilon > 0$, there exists a positive integer N such that $d(u_n,u_m) < \varepsilon$, for all $n,m \geqslant N$, and converges to $u \in X$ if for any $\varepsilon > 0$, there exists a positive integer N such that $d(u_n,u) < \varepsilon$, for any $n \geqslant N$. We use the notation as $\lim_{n,m\to\infty} d(u_n,u_m) = 0$ and $\lim_{n\to\infty} d(u_n,u) = 0$, respectively. It is worth noting that (X,d,s) is complete if every Cauchy sequence converges in X.

In 2014, Popescu [17] has extended the concept of a triangular α-admissible mapping [12] as follows:

DEFINITION 3. [17] For a function $\alpha : X \times X \to \mathbb{R}$, a mapping $f : X \to X$ is referred to as a triangular α-orbital admissible if the given conditions hold:

(T1) if $\alpha(u,fu) \geqslant 1$ then $\alpha(fu,f^2u) \geqslant 1$.

(T2) if $\alpha(u,v) \geqslant 1$ and $\alpha(v,fv) \geqslant 1$ then $\alpha(u,fv) \geqslant 1$, for all $u,v \in X$.

Furthermore, the collection of functions $\beta : [0,\infty) \to [0,\frac{1}{s})$, where $s \geqslant 1$, that satisfy the condition if $\lim_{n\to\infty} \beta(x_n) = \frac{1}{s}$ then $\lim_{n\to\infty} x_n = 0$. This collection is denoted as $\mathscr{F}_s$. If $s = 1$, then $\mathscr{F} = \mathscr{F}_1$. Geraghty type contraction mappings within the context of b-metric spaces were introduced by Aydi *et al.* [3] through the concept of $\alpha-\beta_E$-Geraghty type.

DEFINITION 4. [3] Let (X,d,s) denote a b-metric space, $\alpha : X \times X \to \mathbb{R}$, and $f : X \to X$. If there is $\beta \in \mathscr{F}_s$ such that

$$\alpha(u,v) \geqslant 1 \quad \text{implies that} \quad d(fu,fv) \leqslant \beta(E(u,v))E(u,v),$$

for any $u,v \in X$, where $E(u,v) = d(u,v) + |d(u,fu) - d(v,fv)|$, then f is referred to as an $\alpha - \beta_E$-Geraghty type contraction.

Denote $Fix(f)$ to be the collection of fixed points belonging to the mapping f. Aydi *et al.* [3] proved a fixed point result for $\alpha - \beta_E$-Geraghty type contraction mappings within the context of b-metric spaces as follows:

THEOREM 1. [3] *Let (X,d,s) be a b-metric space, $\alpha : X \times X \to \mathbb{R}$, and $f : X \to X$ be a mapping on X. Under the given conditions, which are as follows:*

(i) f is $\alpha - \beta_E$-Geraghty type contraction;

(ii) f is triangular α-orbital admissible;

(iii) $\alpha(u_0, fu_0) \geqslant 1$ for some $u_0 \in X$;

(iv) f is continuous,

it is concluded that $Fix(f)$ is nonempty and $\{f^n u_0\}$ converges to $w \in Fix(f)$.

3. Main results

In this particular segment, our objective is to broaden the scope of the outcomes expounded in the publication by [3] to encompass a more comprehensive range of mappings.

DEFINITION 5. Let (X,d,s) denote a b-metric space, with $\alpha : X \times X \to \mathbb{R}$, and $f : X \to X$ representing a mapping on X. If there exists $\beta \in \mathscr{F}_s$ such that

$$\alpha(u,v) \geqslant 1 \quad \text{implies that} \quad d(fu,fv) \leqslant \beta(M(u,v))M(u,v), \tag{1}$$

for all $u,v \in X$, where

$$M(u,v) = \max \left\{ \begin{array}{c} d(u,v) + |d(u,fu) - d(v,fv)|, \\ d(u,fu) + |d(u,v) - d(v,fv)|, \\ d(v,fv) + |d(u,v) - d(u,fu)| \end{array} \right\}.$$

Then, f is $\alpha - \beta_M$-Geraghty type contraction.

Our first main theorem offers a sufficient condition for the existence of a fixed point for the previously mentioned mappings in a b-metric space. A deduction of Theorem 1 is obtained, which broadens its scope.

THEOREM 2. *Let (X,d,s) be a b-metric space, with $\alpha: X\times X\to\mathbb{R}$, and $f: X\to X$ representing a mapping on X. We assume that f satisfies the following conditions:*

(i) f is $\alpha-\beta_M$-Geraghty type contraction;

(ii) f is triangular α-orbital admissible;

(iii) $\alpha(u_0,fu_0)\geqslant 1$ for some $u_0\in X$;

(iv) f is continuous.

It follows that the set $Fix(f)$ is nonempty and the sequence $\{f^n u_0\}$ converges to $w\in Fix(f)$.

Proof. By assuming the condition (iii), there exists an element $u_0\in X$ satisfying $\alpha(u_0,fu_0)\geqslant 1$. A sequence $\{u_n\}$ in a metric space (X,d,s) is established by defining $u_n=fu_{n-1}=f^n u_0$ for all $n\geqslant 1$. If there exists a nonnegative real number n such that $u_n=u_{n+1}=fu_n$, the proof is thereby concluded. Throughout the proof, it is assumed that $u_n\neq u_{n+1}$ for any nonnegative real number n.

It is known that $\alpha(u_0,u_1)=\alpha(u_0,fu_0)\geqslant 1$, and by condition (ii), it can be concluded that $\alpha(u_n,u_{n+1})=\alpha(f^n u_0,f^{n+1}u_0)\geqslant 1$ for all $n\geqslant 0$. This process can be repeated to obtain the inequality

$$\text{if}\quad \alpha(u_n,u_{n+1})\geqslant 1\quad\text{and}\quad \alpha(u_{n+1},fu_{n+1})\geqslant 1\quad\text{then}\quad \alpha(u_n,u_{n+2})\geqslant 1.$$

Using induction, it can be deduced that

$$\alpha(u_n,u_m)\geqslant 1,\quad \text{for any } m\geqslant n\geqslant 0.$$

For convenience, let $d_n=d(u_{n-1},u_n)$ for all $n\geqslant 1$. From Equation (1), it follows that

$$0<d_{n+1}=d(fu_{n-1},fu_n)\leqslant\beta(M(u_{n-1},u_n))M(u_{n-1},u_n),\quad\text{for all } n\geqslant 1. \tag{2}$$

Note that

$$M(u_{n-1},u_n)=\max\{d_n+|d_n-d_{n+1}|,d_{n+1}\}.$$

Suppose there exists an integer $n>0$ such that $d_n\leqslant d_{n+1}$. Utilizing equation (2), we can derive the following inequality,

$$d_{n+1}\leqslant\beta(d_{n+1})d_{n+1}<s^{-1}d_{n+1},$$

this leads to a contradiction. So, we conclude that $d_{n+1}<d_n$, for any $n>0$. Consequently, it follows that

$$M(u_{n-1},u_n)=2d_n-d_{n+1},\quad\text{for any } n\geqslant 1.$$

Since the sequence $\{d_n\}$ is decreasing and bounded below by 0, there exists a value $t\geqslant 0$ such that $\lim_{n\to\infty}d_n=t$. It is assumed that $t>0$. By taking $n\to\infty$ in (2), we derive

$$s^{-1}t=s^{-1}\lim_{n\to\infty}d_{n+1}\leqslant\lim_{n\to\infty}d_{n+1}\leqslant\lim_{n\to\infty}\beta(M(u_{n-1},u_n))M(u_{n-1},u_n)\leqslant s^{-1}t.$$

This yields
$$\lim_{n\to\infty}\beta(M(u_{n-1},u_n))M(u_{n-1},u_n)=s^{-1}t,$$
and therefore, we can conclude that
$$\lim_{n\to\infty}\beta(M(u_{n-1},u_n))=s^{-1}.$$
Since $\beta\in\mathscr{F}_s$,
$$t=\lim_{n\to\infty}M(u_{n-1},u_n)=0.$$
This leads to a contradiction, and hence we can deduce that
$$\lim_{n\to\infty}d(u_{n-1},u_n)=0. \tag{3}$$

We shall prove the Cauchy property of the sequence $\{u_n\}$ by contradiction. To be precise, for every i, we can identify a positive value $\varepsilon>0$ such that we can locate subsequences $\{u_{m(i)}\}$ and $\{u_{n(i)}\}$ of $\{u_n\}$, where $m(i)>n(i)>i$, and
$$d(u_{m(i)},u_{n(i)})\geqslant\varepsilon. \tag{4}$$
Moreover, for every $n(i)$, we can select $m(i)$ as the smallest integer greater than $n(i)$ that satisfies (4). Consequently,
$$d(u_{m(i)-1},u_{n(i)})<\varepsilon. \tag{5}$$
By virtue of the inequality $\alpha(u_{n(i)},u_{m(i)})\geqslant 1$, it can be inferred from equations (1) and (4) that the following holds true:
$$\begin{aligned} s\varepsilon\leqslant sd(u_{n(i)},u_{m(i)}) &\leqslant s\beta(M(u_{n(i)-1},u_{m(i)-1}))M(u_{n(i)-1},u_{m(i)-1})\\ &< M(u_{n(i)-1},u_{m(i)-1}), \end{aligned} \tag{6}$$
where
$$M(u_{n(i)-1},u_{m(i)-1})=\max\left\{\begin{array}{l} d(u_{n(i)-1},u_{m(i)-1})+|d(u_{n(i)-1},u_{n(i)})-d(u_{m(i)-1},u_{m(i)})|,\\ d(u_{n(i)-1},u_{n(i)})+|d(u_{n(i)-1},u_{m(i)-1})-d(u_{m(i)-1},u_{m(i)})|,\\ d(u_{m(i)-1},u_{m(i)})+|d(u_{n(i)-1},u_{m(i)-1})-d(u_{n(i)-1},u_{n(i)})| \end{array}\right\}.$$

By utilizing the triangle inequality and equation (5), it is possible to deduce that
$$\begin{aligned} M(u_{n(i)-1},u_{m(i)-1}) &\leqslant d(u_{n(i)-1},u_{m(i)-1})+d(u_{n(i)-1},u_{n(i)})+d(u_{m(i)-1},u_{m(i)})\\ &\leqslant s[d(u_{n(i)-1},u_{n(i)})+d(u_{n(i)},u_{m(i)-1})]\\ &\quad +d(u_{n(i)-1},u_{n(i)})+d(u_{m(i)-1},u_{m(i)})\\ &\leqslant sd(u_{n(i)-1},u_{n(i)})+s\varepsilon+d(u_{n(i)-1},u_{n(i)})+d(u_{m(i)-1},u_{m(i)}). \end{aligned} \tag{7}$$
From (3), (6) and (7), we have
$$\lim_{i\to\infty}s\beta(M(u_{n(i)-1},u_{m(i)-1}))M(u_{n(i)-1},u_{m(i)-1})=\lim_{i\to\infty}M(u_{n(i)-1},u_{m(i)-1})=s\varepsilon. \tag{8}$$

This implies that

$$\lim_{i\to\infty}\beta(M(u_{n(i)-1},u_{m(i)-1}))=s^{-1}.$$

Since $\beta\in\mathscr{F}_s$,

$$\lim_{i\to\infty}M(u_{n(i)-1},u_{m(i)-1})=0,$$

this result is in contradiction with equation (8). Therefore, it can be inferred that the sequence $\{u_n\}$ is a Cauchy sequence. By completeness of b-metric space, there exists an element $\omega\in X$ such that

$$\lim_{n\to\infty}d(u_n,\omega)=0.$$

By continuity of f, we have $\omega=\lim_{n\to\infty}u_{n+1}=\lim_{n\to\infty}fu_n=f(\lim_{n\to\infty}u_n)=f\omega$, that is, $\omega\in Fix(f)$. Since $u_n=f^nu_0$, we can conclude $\{f^nu_0\}$ converges to ω. $\square$

Next, we proceed to present our second main theorem. We substitute the continuity condition of the mapping f from Theorem 2 with an alternative criterion.

THEOREM 3. *Let (X,d,s) be a b-metric space, with $\alpha:X\times X\to\mathbb{R}$, and $f:X\to X$ representing a mapping on X. We assume that f satisfies the following conditions:*

(i) f is $\alpha-\beta_M$-Geraghty type contraction;

(ii) f is triangular α-orbital admissible;

(iii) $\alpha(u_0,fu_0)\geqslant 1$ for some $u_0\in X$;

(iv) if a sequence $\{u_n\}$ convergences to $u\in X$ and complies with the condition $\alpha(u_n,u_{n+1})\geqslant 1$ for every n, then the existence of a subsequence $\{u_{n(i)}\}$ from $\{u_n\}$ is guaranteed, which satisfies $\alpha(u_{n(i)},u)\geqslant 1$ for every i.

It follows that the set $Fix(f)$ is nonempty and the sequence $\{f^nu_0\}$ converges to $w\in Fix(f)$.

Proof. According to the statements presented in Theorem 2, it can be deduced that the sequence, which is defined as $u_n=f^nu_0$, converges to a limit point $\omega\in X$. Through the application of condition (iv), there exists a subsequence denoted as $\{u_{n(i)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(i)},\omega)\geqslant 1$ for all i. Additionally, based on the condition (i), it can be established that

$$d(u_{n(i)+1},f\omega)\leqslant\beta(M(u_{n(i)},\omega))M(u_{n(i)},\omega), \tag{9}$$

where

$$M(u_{n(i)},\omega)=\max\left\{\begin{array}{l}d(u_{n(i)},\omega)+|d(u_{n(i)},u_{n(i)+1})-d(\omega,f\omega)|,\\ d(u_{n(i)},u_{n(i)+1})+|d(u_{n(i)},\omega)-d(\omega,f\omega)|,\\ d(\omega,f\omega)+|d(u_{n(i)},\omega)-d(u_{n(i)},u_{n(i)+1})|\end{array}\right\}.$$

Suppose that $d(\omega, f\omega) > 0$. Applying the triangle inequality and (9), we obtain for all i

$$\begin{aligned} s^{-1}d(\omega, f\omega) - d(\omega, u_{n(i)+1}) \leqslant d(u_{n(i)+1}, f\omega) &\leqslant \beta(M(u_{n(i)}, \omega))M(u_{n(i)}, \omega) \\ &\leqslant s^{-1}M(u_{n(i)}, \omega). \end{aligned}$$

Taking limit $i \to \infty$, we obtain

$$\lim_{i\to\infty} \beta(M(u_{n(i)}, \omega))M(u_{n(i)}, \omega) \leqslant \lim_{i\to\infty} s^{-1}M(u_{n(i)}, \omega) = s^{-1}d(\omega, f\omega). \tag{10}$$

We deduce that

$$\lim_{i\to\infty} \beta(M(u_{n(i)}, \omega)) = s^{-1}.$$

Since $\beta \in \mathscr{F}_S$, we get

$$\lim_{i\to\infty} M(u_{n(i)}, \omega) = 0,$$

this contradicts to (10). Henceforth, it can be inferred that $d(\omega, f\omega) = 0$, which implies that ω represents a fixed point of f. Furthermore, it can be observed that the sequence $\{f^n u_0\}$ exhibits convergence towards ω. $\square$

We shall now proceed to establish the uniqueness of such a fixed point.

THEOREM 4. *Suppose, in addition to the hypotheses of Theorem 2 (resp. Theorem 3), that*

$$(U)\colon\ \alpha(u,v) \geqslant 1, \text{ for all } u,v \in Fix(f).$$

Then, $Fix(f) = \{\omega\}$.

Proof. We present a proof by means of contradiction. Specifically, there exist $\omega, \upsilon \in X$ such that $\omega = f\omega$ and $\upsilon = f\upsilon$ with $\omega \neq \upsilon$. From assumption (U), we get $\alpha(\omega, \upsilon) \geqslant 1$. Thus, by (1), we have

$$\begin{aligned} s^{-1}d(\omega,\upsilon) = d(\omega,\upsilon) = d(f\omega, f\upsilon) &\leqslant \beta(M(\omega,\upsilon))M(\omega,\upsilon) \leqslant s^{-1}M(\omega,\upsilon) \\ &\leqslant s^{-1}\max\left\{\begin{array}{c} d(\omega,\upsilon) + |d(\omega,f\omega) - d(\upsilon,f\upsilon)|, \\ d(\omega,f\omega) + |d(\omega,\upsilon) - d(\upsilon,f\upsilon)|, \\ d(\upsilon,f\upsilon) + |d(\omega,\upsilon) - d(\omega,f\omega)| \end{array}\right\} = s^{-1}d(\omega,\upsilon), \end{aligned}$$

that is a contradiction. Therefore $\omega = \upsilon$. $\square$

We state the following corollary by setting $\alpha(u,v) = 1$ in Theorem 3.

COROLLARY 1. *Let (X,d,s) be a complete b-metric space and $f : X \to X$ representing a mapping on X. Suppose there exists $\beta \in \mathscr{F}_s$ such that*

$$d(fu, fv) \leqslant \beta(M(u,v))M(u,v) \tag{11}$$

for all $u,v \in X$, *where*

$$M(u,v)=\max\left\{\begin{array}{c} d(u,v)+|d(u,fu)-d(v,fv)|, \\ d(u,fu)+|d(u,v)-d(v,fv)|, \\ d(v,fv)+|d(u,v)-d(u,fu)| \end{array}\right\}.$$

Then, $Fix(f)=\{\omega\}$ *and* $\{f^n u_0\}$ *converges to* ω *for all* $u_0 \in X$.

We can also obtain the following two results.

COROLLARY 2. *Let* (X,d,s) *be a complete* b*-metric space and a mapping* $f: X \to X$ *satisfying the condition*

$$d(fu,fv) \leqslant \frac{M(u,v)}{s+M(u,v)}, \tag{12}$$

for all $u,v \in X$, *where*

$$M(u,v)=\max\left\{\begin{array}{c} d(u,v)+|d(u,fu)-d(v,fv)|, \\ d(u,fu)+|d(u,v)-d(v,fv)|, \\ d(v,fv)+|d(u,v)-d(u,fu)| \end{array}\right\}.$$

Then, $Fix(f)=\{\omega\}$ *and* $\{f^n u_0\}$ *converges to* ω *for all* $u_0 \in X$.

Proof. Consider

$$\beta(t)=\begin{cases} \frac{1}{s+t}, \text{if } t>0 \\ \frac{1}{s+1}, \text{if } t=0. \end{cases}$$

Clearly, $\beta \in \mathscr{F}_s$. If $u \neq v$, $M(u,v) \neq 0$, hence (12) becomes

$$d(fu,fv) \leqslant \beta(M(u,v))M(u,v).$$

In the case $u=v$, we have $d(fu,fv)=M(u,v)=0$ and thus

$$d(fu,fv) \leqslant \beta(M(u,v))M(u,v)$$

holds trivially. By applying Corollary 1, the proof is completed. □

COROLLARY 3. *Let* (X,d,s) *be a complete* b*-metric space and a mapping* $f: X \to X$ *satisfying the condition*

$$d(fu,fv) \leqslant qM(u,v) \tag{13}$$

for all $u,v \in X$, *where* $q \in (0,\frac{1}{s})$ *and*

$$M(u,v)=\max\left\{\begin{array}{c} d(u,v)+|d(u,fu)-d(v,fv)|, \\ d(u,fu)+|d(u,v)-d(v,fv)|, \\ d(v,fv)+|d(u,v)-d(u,fu)| \end{array}\right\}.$$

Then, $Fix(f) = \{\omega\}$ *and* $\{f^n u_0\}$ *converges to* ω *for all* $u_0 \in X$*. Moreover, we have*

$$d(f^n u_0, \omega) \leqslant \gamma^{n-1} \frac{\lambda s}{1-\gamma} d(fu_0, u_0), \tag{14}$$

where

$$\gamma = \frac{2q}{1+q}, \quad \lambda = \sum_{n \geqslant 1} s^{2n} \gamma^{2^{n-1}}.$$

Proof. From (11) in Corollary 1, it suffices to consider for all nonnegative real number t, $\beta(t) = q$. Let $u_0 \in X$ and $u_n = f^n u_0$, thus (13) becomes

$$d(u_n, u_{n+1}) \leqslant qM(u_{n-1}, u_n), \tag{15}$$

where

$$M(u_{n-1}, u_n) = \max\left\{d(u_{n-1}, u_n) + |d(u_{n-1}, u_n) - d(u_n, u_{n+1})|, d(u_n, u_{n+1})\right\}.$$

Since $d(u_{n-1}, u_n) \geqslant d(u_n, u_{n+1})$, for all $n \geqslant 1$, then

$$M(u_{n-1}, u_n) = 2d(u_{n-1}, u_n) - d(u_n, u_{n+1}), \text{ for all } n \geqslant 1.$$

From (13), we obtain that

$$d(u_n, u_{n+1}) \leqslant \gamma d(u_{n-1}, u_n),$$

where $\gamma = \frac{2q}{1+q}$. Following the proof of Corollary 2.6 [3], we derive (14). □

4. Examples

In this section, we provide some examples where known results, in particular Theorem 1, in literature are not applicable.

EXAMPLE 2. Let $X = \mathbb{R}$ and $d(u,v) = (u-v)^2$ for all $u, v \in X$, then (X, d, s) is a complete b-metric space with a constant value of $s = 2$. We take $\beta(t) = \frac{1}{2}$ for all $t \geqslant 0$. Now, let's define $f\colon X \to X$ and $\alpha\colon X \times X \to \mathbb{R}$ as follows:

$$fu = \begin{cases} 2, & \text{if } u < 0 \\ u+2, & \text{if } 0 \leqslant u < 1 \\ \dfrac{-u+7}{2}, & \text{if } 1 \leqslant u \leqslant 3 \\ -4(u-3)^2+2, & \text{if } u > 3 \end{cases}, \qquad \alpha(u,v) = \begin{cases} 1, & \text{if } u, v \in [-3,3] \\ 0, & \text{otherwise.} \end{cases}$$

It should be noted that Theorem 1 is not applicable for any $\beta \in \mathscr{F}_s$. This is evident when we consider $u = 0$ and $v = 1$, we have $\alpha(0,1) = 1$ and

$$d(f0, f1) = 1 \leqslant \beta(E(0,1)) = \beta(E(0,1))E(0,1).$$

which is a contradiction. It can be easily proven that f is triangle α-orbital admissible. Additionally, f is continuous, we have $\alpha(0,f0)=\alpha(0,2)\geqslant 1$, for $u_0=0$. Next, we will show that f is $\alpha-\beta_M$-Geraghty type contraction mapping. By (1) and considering the case when $\alpha(u,v)\geqslant 1$, we require consideration of the following cases:

Case 1: $u,v\geqslant 0$ and $u>v$, we distinguish the following three subcases:

Subcase 1.1) $u,v\in[0,1)$, then

$$\begin{aligned} d(fu,fv) &= (u-v)^2 \leqslant 1 \leqslant \frac{1}{2}(4) \\ &\leqslant \frac{1}{2}\max\left\{\begin{matrix}(u-v)^2+|4-4|,\\ 4+|(u-v)^2-4|\end{matrix}\right\} \\ &= \beta(M(u,v))M(u,v). \end{aligned}$$

Subcase 1.2) $u,v\in[1,3]$, then

$$\begin{aligned} d(fu,fv) &= \left(\frac{u-v}{2}\right)^2 \leqslant \frac{1}{2}(u-v)^2 \\ &\leqslant \frac{1}{2}\max\left\{\begin{matrix}(u-v)^2+|(\frac{3u-7}{2})^2-(\frac{3v-7}{2})^2|,\\ (\frac{3u-7}{2})^2+|(u-v)^2-(\frac{3v-7}{2})^2|,\\ (\frac{3v-7}{2})^2+|(u-v)^2-(\frac{3u-7}{2})^2|\end{matrix}\right\} \\ &= \beta(M(u,v))M(u,v). \end{aligned}$$

Subcase 1.3) $u\in[1,3]$ and $v\in[0,1)$, then

$$\begin{aligned} d(fu,fv) &= \left(\frac{u+2v-3}{2}\right)^2 \leqslant 1 \leqslant \frac{1}{2}(4) \\ &\leqslant \frac{1}{2}\max\left\{\begin{matrix}(u-v)^2+|(\frac{3u-7}{2})^2-4|,\\ (\frac{3u-7}{2})^2+|(u-v)^2-4|,\\ 4+|(u-v)^2-(\frac{3u-7}{2})^2|\end{matrix}\right\} \\ &= \beta(M(u,v))M(u,v). \end{aligned}$$

Case 2: $u,v<0$, then $fu=fv=2$. We have

$$0=d(fu,fv)\leqslant \beta(M(u,v))M(u,v).$$

Case 3: $u\geqslant 0$ and $v<0$, we distinguish the following two subcases:

Subcase 3.1) $u\in[0,1)$ and $v<0$, then

$$\begin{aligned} d(fu,fv) &= u^2 \leqslant 1 \leqslant \frac{1}{2}(4) \\ &\leqslant \frac{1}{2}\max\left\{\begin{matrix}(u-v)^2+|4-(v-2)^2|,\\ 4+|(u-v)^2-(v-2)^2|,\\ (v-2)^2+|(u-v)^2-4|\end{matrix}\right\} \\ &= \beta(M(u,v))M(u,v). \end{aligned}$$

Subcase 3.2) $u \in [1,3]$ and $v < 0$, then

$$
\begin{aligned}
d(fu,fv) &= \left(\frac{3-u}{2}\right)^2 \leqslant 1 \leqslant \frac{1}{2}(v-2)^2 \\
&\leqslant \frac{1}{2}\max\left\{\begin{array}{l} (u-v)^2 + |(\frac{3u-7}{2})^2 - (v-2)^2|, \\ (\frac{3u-7}{2})^2 + |(u-v)^2 - (v-2)^2|, \\ (v-2)^2 + |(u-v)^2 - (\frac{3u-7}{2})^2| \end{array}\right\} \\
&= \beta(M(u,v))M(u,v).
\end{aligned}
$$

This means that the assumption (i) is satisfied for all $u,v \in X$ such that $\alpha(u,v) \geqslant 1$. Since all requirements of Theorem 2 are satisfied, thus f has a fixed point, which is $\omega = \dfrac{7}{3}$.

EXAMPLE 3. Let $X = \{0,1,2,3\}$ and $d(u,v) = (u-v)^2$ for all $u,v \in X$, then (X,d,s) is a complete b-metric space with a constant value of $s = 2$. For all $t > 0$, let $\beta(t) = \frac{2}{4+t}$ and $\beta(0) = \frac{1}{4}$. Consider the mapping $f\colon X \to X$ given by

$$f0 = f2 = f3 = 2, \text{ and } f1 = 3.$$

Then, $d(f0,f2) = d(f0,f3) = d(f2,f3) = 0$ is trivial. Moreover, we have that

$$d(f0,f1) = 1 \leqslant \frac{14}{11} = \frac{2}{4+7}\max\left\{\begin{array}{l} 1+|4-4|, \\ 4+|1-4| \end{array}\right\} = \beta(M(0,1))M(0,1).$$

$$d(f1,f2) = 1 \leqslant \frac{10}{9} = \frac{2}{4+5}\max\left\{\begin{array}{l} 1+|4-0|, \\ 4+|1-0|, \\ 0+|1-4| \end{array}\right\} = \beta(M(1,2))M(1,2).$$

$$d(f1,f3) = 1 \leqslant \frac{14}{11} = \frac{2}{4+7}\max\left\{\begin{array}{l} 4+|4-1|, \\ 1+|4-4| \end{array}\right\} = \beta(M(1,3))M(1,3).$$

Hence, we obtain, $d(fu,fv) \leqslant \beta(M(u,v))M(u,v)$ for all $u,v \in X$. As a result, by satisfying all the hypotheses of Corollary 1, thus f has a unique fixed point, which is $\omega = 2$.

5. Application on matrix equations

In the real world, it is widely acknowledged that various problems can be presented as mathematical models. In order to obtain a solution to these problems, the equations must be solved. Certain studies have suggested using fixed point theory to propose solutions to such problems. As a result, we are interested in applying our findings to some of these problems. In fact, we utilize Corollary 3 to study the existence of a unique Hermitian positive definite solution of the nonlinear matrix equation

$$U^N = \left(AU^{-\frac{1}{M}}A^* + B\right)^{\frac{1}{L}} + C, \tag{16}$$

where A is a nonsingular $n\times n$ matrix, A^* represents the conjugate transpose of the matrix A, matrices B and C are $n\times n$ positive semidefinite and N,M,L are positive integer numbers.

Indication that the equation (16) is equivalent to

$$U=f(U)=\left[\left(AU^{-\frac{1}{M}}A^*+B\right)^{\frac{1}{L}}+C\right]^{\frac{1}{N}}, \tag{17}$$

in other words U is a fixed point of the mapping f.

In this investigation, we use the Thompson metric proposed by Thompson [21] for study solutions of nonlinear matrix equations related to contraction mappings on b-metric spaces. We first review the Thompson metric on the open convex cone $\mathscr{P}_n$ for $n\geqslant 2$, the set of all $n\times n$ Hermitian positive definite matrices. Let $A,B\in\mathscr{P}_n$, the Thompson metric for A and B is defined by

$$d(A,B)=\max\left\{\log\left(M\left(\frac{A}{B}\right)\right),\log\left(M\left(\frac{B}{A}\right)\right)\right\},$$

where $M\left(\frac{A}{B}\right)=\inf\{\lambda_i>0: A\leqslant\lambda_i B, i=1,\cdots,n\}=\lambda_{\max}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)$, maximum eigenvalues of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$. Here $A\leqslant B$ means that $B-A$ is positive semidefinite and $A<B$ means that $B-A$ is positive definite. Thompson [21] (also discussed in [15, 16]) has demonstrated the completeness of $\mathscr{P}_n$ as a metric space under the Thompson metric b and we have

$$d(A,B)=\left\|\log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right\|,$$

where $\|.\|$ represent the spectral norm. The Thompson metric exists on any open normal convex cone in real Banach spaces [16, 21]. It is particularly applicable to the open convex cone of positive definite operators within a Hilbert space. Notably, the Thompson metric remains invariant under both matrix inversion and congruence transformations, for any nonsingular matrix M, we derive

$$d(A,B)=d(A^{-1},B^{-1})=d(MAM^*,MBM^*),$$

An important outcome is the property of nonpositive curvature of the Thompson metric, we derive,

$$d(A^r,B^r)\leqslant rd(A,B),\quad 0\leqslant r\leqslant 1. \tag{18}$$

By utilizing the invariant properties of the metric, we can observe that

$$d(MA^rM^*,MB^rM^*)\leqslant |r|d(A,B),\quad -1\leqslant r\leqslant 1, \tag{19}$$

for all $A,B\in\mathscr{P}_n$ and nonsingular matrix M.

LEMMA 1. [14] *For every $A,B,C,D\in\mathscr{P}_n$, we obtain*

$$d(A+B,C+D)\leqslant\max\{d(A,C),d(B,D)\}.$$

Furthermore, for all positive semidefinite A and $B,C \in \mathscr{P}_n$,

$$d(A+B,A+C) \leqslant d(B,C). \tag{20}$$

Consider the b-metric $d_i : \mathscr{P}_n \times \mathscr{P}_n \to [0,\infty)$ defined as follows

$$d_i(U,V) = d^i(U,V), \tag{21}$$

where i is a positive integer.

THEOREM 5. *For numbers L,N,M,i are positive integer and $s \geqslant 1$, the problem (16) has a unique solution $U \in \mathscr{P}_n$. Furthermore, for all $U(0) \in \mathscr{P}_n$, the sequence $U(k)_{k\geqslant 0}$ defined by*

$$U(k+1) = \left[\left(AU(k)^{-\frac{1}{M}}A^* + B\right)^{\frac{1}{L}} + C\right]^{\frac{1}{N}}, \tag{22}$$

converges to U and the error estimation is

$$d_i(U(k),U) \leqslant \left(\frac{(NLM)^i+1}{(NLM)^i-1}\right)\left(\frac{2}{(NLM)^i+1}\right)^{k-1} \lambda E_i(U(1)U(0)), \tag{23}$$

where

$$\lambda = \sum_{n\geqslant 1} s^{2n}\left(\frac{2}{(NLM)^i+1}\right)^{2^{n-1}}.$$

Proof. First, to show that the problem (16) has a unique solution and the iteration (22) converges to $U \in \mathscr{P}_n$. Let $U,V \in \mathscr{P}_n$, from (17) we have

$$d_i(f(U),f(V)) = d^i\left(\left[\left(AU^{-\frac{1}{M}}A^*+B\right)^{\frac{1}{L}}+C\right]^{\frac{1}{N}}, \left[\left(AV^{-\frac{1}{M}}A^*+B\right)^{\frac{1}{L}}+C\right]^{\frac{1}{N}}\right),$$

by using (21) we get

$$d_i(f(U),f(V)) = d^i\left(\left[\left(AU^{-\frac{1}{M}}A^*+B\right)^{\frac{1}{L}}+C\right]^{\frac{1}{N}}, \left[\left(AV^{-\frac{1}{M}}A^*+B\right)^{\frac{1}{L}}+C\right]^{\frac{1}{N}}\right).$$

From equations (18), (19) and (20), the result is

$$\begin{aligned} d_i(f(U),f(V)) &\leqslant \frac{1}{(NLM)^i} d^i(U,V) \\ &= \frac{1}{(NLM)^i} d_i(U,V). \end{aligned}$$

Since $d_i(U,V) \leqslant E_i(U,V)$ for all positive integer i, where $E_i(U,V)$ is defined in Definition 1.10 by substituting $d_i(U,V)$ into $d(U,V)$. It is obtain that

$$d_i(f(U),f(V)) \leqslant \frac{1}{(NLM)^i} E_i(U,V) = qE_i(U,V),$$

where $q = \dfrac{1}{(NLM)^i}$.

Applying Corollary 3, the mapping f has a unique fixed point $U \in \mathscr{P}_n$. Consequently, the problem (17) has a unique fixed point, indicating that the nonlinear matrix equation (16) has a unique solution in $\mathscr{P}_n$. Moreover, the iteration (22) converges to $U \in \mathscr{P}_n$.

Next, we will show that the error estimation satisfies (23). Since $q = \dfrac{1}{(NLM)^i}$, by Corollary 3 the following result is obtained

$$\gamma = \frac{2q}{q+1} = \frac{2}{(NLM)^i+1},$$

$$\frac{1}{1-\gamma} = \frac{(NLM)^i+1}{(NLM)^i-1},$$

$$\lambda = \sum_{n\geqslant 1} s^{2n}\gamma^{2^{n-1}} = \sum_{n\geqslant 1} s^{2n}\left(\frac{2}{(NLM)^i+1}\right)^{2^{n-1}}.$$

Hence

$$\begin{aligned}
d_i(U(k),U) &\leqslant \frac{\gamma^{k-1}}{1-\gamma} s\lambda d_i\left(U(1),U(0)\right) \\
&\leqslant \frac{\gamma^{k-1}}{1-\gamma} s\lambda E_i\left(U(1),U(0)\right) \\
&\leqslant \left(\frac{(NLM)^i+1}{(NLM)^i-1}\right)\left(\frac{2}{(NLM)^i+1}\right)^{k-1} s\lambda E_i\left(U(1),U(0)\right).
\end{aligned}$$

The proof is completed. □

EXAMPLE 4. [3] Put $N = M = 2$ and $L = 3$, the nonlinear equation (16) reduced to

$$U^2 = \left(AU^{-\frac{1}{2}}A^* + B\right)^{\frac{1}{3}} + C. \tag{24}$$

If $i = s = 2$, then the error estimation (23) becomes

$$d_2(U(k),U) \leqslant \left(\frac{290}{143}\right)\left(\frac{2}{145}\right)^{k-1} \lambda E_2\left(U(1)U(0)\right),$$

where

$$\lambda = \sum_{n\geqslant 1} 4^n \left(\frac{2}{145}\right)^{2^{n-1}}.$$

EXAMPLE 5. Put $N=3$ and $M=L=2$, the nonlinear equation (16) reduced to

$$U^3=\left(AU^{-\frac{1}{2}}A^*+B\right)^{\frac{1}{2}}+C. \tag{25}$$

The sequence $U(k)_{k\geqslant 0}$ of (25) defined by

$$U(k+1)=\left[\left(AU(k)^{-\frac{1}{2}}A^*+B\right)^{\frac{1}{2}}+C\right]^{\frac{1}{3}}. \tag{26}$$

If $i=s=1$, then the error estimation (23) becomes

$$d(U(k),U)\leqslant\left(\frac{13}{11}\right)\left(\frac{2}{13}\right)^{k-1}\lambda E\left(U(1),U(0)\right),$$

where

$$\lambda=\sum_{n\geqslant 1}\left(\frac{2}{13}\right)^{2^{n-1}}.$$

Next, we investigate a numerical example to demonstrate our findings by using the iteration (26) to solve the problem (25). Consider the nonsingular matrix A, the positive semidefinite B and $C\in\mathscr{P}_n$ defined by

$$A=\begin{bmatrix}1&1&-1&0\\0&2&-1&1\\1&0&-1&3\\0&1&4&2\end{bmatrix},\quad B=\begin{bmatrix}1&0&2&0\\0&2&1&0\\2&1&5&0\\0&0&0&0\end{bmatrix}\quad\text{and}\quad C=\begin{bmatrix}1&-1&0&0\\-1&3&1&0\\0&1&4&0\\0&0&0&1\end{bmatrix},$$

note that spectrums of B and C are $spec(B)=\{0,0.0885,1.8705,6.0410\}$ and $spec(C)=\{0.5395,1.0000,2.7609,4.6996\}$ respectively. First, we select the diagonal positive definite matrix

$$U_1(0)=\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix},$$

mention that $spec(U_1(0))=\{1,2,3,4\}$. Using the iterative algorithm (26), after 10 iterations, we derive the unique positive definite solution

$$U_1(10)=\begin{bmatrix}1.3011&-0.0424&0.0253&-0.0050\\-0.1879&1.7371&0.1516&0.0011\\0.0668&0.1575&1.9631&0.0380\\0.0909&0.0395&0.0302&1.6512\end{bmatrix},$$

note that $spec(U_1(10))=\{1.2899,2.0455,1.6858,1.6313\}$. The residual error is

$$R(U_1(10))=\left\|U_1(10)-\left[\left(AU_1(10)^{-\frac{1}{2}}A^*+B\right)^{\frac{1}{2}}+C\right]^{\frac{1}{3}}\right\|=3.1592\times 10^{-13}.$$

Next, we choose the different positive definite matrix

$$U_2(0)=\begin{bmatrix} 4 & 1 & 1 & -2\\ 1 & 3 & -1 & -1\\ 1 & -1 & 3 & -1\\ -2 & -1 & -1 & 4\end{bmatrix},$$

remark that $spec(U_2(0))=\{1.1716,2,4,6.8284\}$. We use the iterative algorithm (26), after 10 iterations, we obtian the unique positive definite solution

$$U_2(10)=\begin{bmatrix} 1.3011 & -0.0424 & 0.0253 & -0.0050\\ -0.1879 & 1.7371 & 0.1516 & 0.0011\\ 0.0668 & 0.1575 & 1.9631 & 0.0380\\ 0.0909 & 0.0395 & 0.0302 & 1.6512\end{bmatrix},$$

and its residual error

$$R(U_2(10))=\left\|U_2(10)-\left[\left(AU_2(10)^{-\frac{1}{2}}A^*+B\right)^{\frac{1}{2}}+C\right]^{\frac{1}{3}}\right\|=3.9364\times 10^{-13}.$$

The algorithm's convergence history for different initial values of $U(0)$ is illustrated in Figure (1). It is evident that $U_1(k)$ is close to $U_2(k)$, establishing support for Theorem (5).

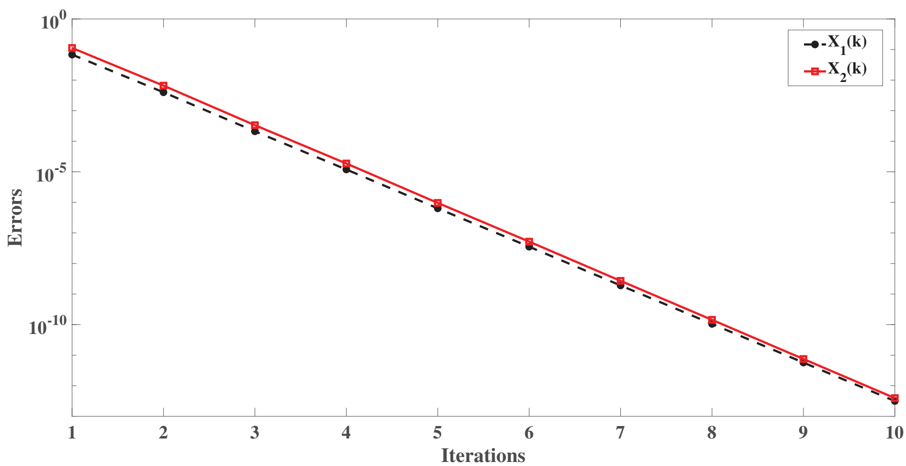

Figure 1: *Convergence curve for* (26).

6. Conclusions

The present study reveals a compelling condition for the existence and uniqueness of the $\alpha-\beta_M$-Geraghty type contraction mapping that encompasses $\alpha-\beta_E$-Geraghty contraction. Furthermore, the outcomes garnered from this investigation were implemented in the development of a theorem pertaining to a nonlinear matrix equation on a Hermitian positive definite matrix, in addition to examining examples that corresponded to the aforementioned theorem utilizing a numerical method.

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