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A GENERALIZED REFINEMENT OF YOUNG'S INEQUALITY

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Abstract. In this paper, we mainly give a generalized refinement of Young's inequality due to Yang and Wang [J. Math. Inequal., 17 (2023), 205–217]. More precisely, we show that

$$\frac{(a\nabla_v b)^m - K(h,2)^{mv}(a\sharp_v b)^m}{(a\nabla_\tau b)^m - K(h,2)^{m\tau}(a\sharp_\tau b)^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)},$$

where $0 < v \leqslant \tau < \frac{1}{2}$, $m \in \mathbb{N}^+$, $a > b > 0$, $K(h,2) = \frac{(h+1)^2}{4h}$ and $h = \frac{b}{a}$. As applications, we obtain some inequalities for operator, Hilbert-Schmidt norm and trace class norm.

1. Introduction

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $B(\mathscr{H})$ denote the algebra of all bounded linear operators acting on $\mathscr{H}$. A self adjoint operator A is said to be positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$, while it is said to be strictly positive if A is positive and invertible, denoted by $A > 0$. We say $A > B$ means $A - B > 0$ and $A \geqslant B$ implies $A - B \geqslant 0$, respectively.

In addition, $\mathbb{M}_n$ denotes the space of all $n \times n$ complex matrices. The unitarily invariance of the $\|\cdot\|_u$ on $\mathbb{M}_n$ means that $\|UAV\|_u = \|A\|_u$ for any $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. The singular values of A, that is, the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{\frac{1}{2}}$, is denoted by $s_j(A)$, $j = 1, 2, \cdots, n$, and arranged in a non-increasing order. For $A \in \mathbb{M}_n$, we define $||A||_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{\frac{1}{p}}$, then we call it as the trace norm and Hilbert-Schmidt norm of A when $p = 1$ and $p = 2$, respectively. It is well know that $\|\cdot\|_2$ is unitarily invariant.

As usual, we denote the v-weighted operator arithmetic mean and geometric mean by

$$A\nabla_v B = (1-v)A + vB \quad \text{and} \quad A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

respectively, where $A, B > 0$ and $v \in [0,1]$. Similarly, we define the v-weighted AM-GM means as $a\nabla_v b = (1-v)a + vb$ and $a\sharp_v b = a^{1-v}b^v$ for $a, b > 0$ and $0 \leqslant v \leqslant 1$.

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The Kantorovich constant and the Specht's ratio are defined by

$$K(h,2)=\frac{(h+1)^2}{4h}\quad \text{for } h>0$$

and

$$S(h)=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\log\left(h^{\frac{1}{h-1}}\right)} & \text{if } h\in(0,1)\cup(1,\infty),\\ 1 & \text{if } h=1.\end{cases}$$

The classical weighted arithmetic-geometric mean inequality reads

$$\prod_{i=1}^{n} a_i^{p_i} \leqslant \sum_{i=1}^{n} p_i a_i, \tag{1}$$

where $a_i, p_i \geqslant 0$ and $\sum_{i=1}^{n} p_i = 1$. Then we can get the famous Young's inequality by (1) when $n=2$,

$$a^{1-v}b^{v} \leqslant (1-v)a+vb, \tag{2}$$

where $a,b\geqslant 0$ and $v\in[0,1]$.

Zuo et al. [6] and Furuichi [1] improved (2) and Liao et al. [3] gave a reverse of (2) as follows

$$S(h^r)a\sharp_v b \leqslant K(h,2)^r a\sharp_v b \leqslant a\nabla_v b \leqslant K(h,2)^R a\sharp_v b, \tag{3}$$

where $a,b>0$, $0\leqslant v\leqslant 1$, $r=\min\{v,1-v\}$, $R=\max\{v,1-v\}$, $K(h,2)=\frac{(h+1)^2}{4h}$ and $h=\frac{b}{a}$.

Very recently, Yang and Wang [5] showed a new refinement and reverse of inequality (3): if $\frac{1}{2}<v\leqslant\tau<1$, $K(h,2)=\frac{(h+1)^2}{4h}$, $h=\frac{b}{a}$ and $a,b>0$, then

$$\frac{K(h,2)^{v}a\sharp_v b - a\nabla_v b}{K(h,2)^{\tau}a\sharp_\tau b - a\nabla_\tau b} \leqslant \frac{v}{\tau}. \tag{4}$$

Moreover, they [5] also presented that

$$\frac{(a\nabla_v b)^2-\left(a\sharp_v b\right)^2-v^2(a-b)^2}{(a\nabla_\tau b)^2-\left(a\sharp_\tau b\right)^2-\tau^2(a-b)^2} \geqslant \frac{v}{\tau}. \tag{5}$$

for $0<v\leqslant\tau<\frac{1}{2}$ and $a,b>0$.

In this short paper, we will give a refinement of inequality (4) and (5) when $0<v\leqslant\tau<\frac{1}{2}$, which can be regarded as some complement of Yang and Wang [5]. As applications, we obtain some inequalities for operator, Hilbert-Schmidt norm and trace class norm.

2. Main results

Firstly, we give the corresponding result of inequality (4) when $0 < v \leqslant \tau < \frac{1}{2}$. In fact, the following theorem can be obtained from ([5] Theorem 2.2). Here, we provide the details for the convenience of readers.

THEOREM 1. *Let* $0 < v \leqslant \tau < \frac{1}{2}$, $a, b > 0$ *and* $K(h,2) = \frac{(h+1)^2}{4h}$, $h = \frac{b}{a}$. *Then*

$$\frac{a\nabla_v b - K(h,2)^v a\sharp_v b}{a\nabla_\tau b - K(h,2)^\tau a\sharp_\tau b} \geqslant \frac{v}{\tau}.$$

Proof. Let $f(v) = \frac{(1-v+vx)-K(x,2)^v(x^v)}{v}$. Then $f'(v) = \frac{h(x)}{v^2}$, where

$$h(x) = \left[1 - 2v\ln\left(\frac{x+1}{2}\right)\right]\left(\frac{x+1}{2}\right)^{2v} - 1,$$

and then $h'(x) = -2v^2(\frac{x+1}{2})^{2v-1}\ln(\frac{x+1}{2})$. It is clearly that $h'(x) \leqslant 0$ for $x \in [1,\infty]$ and $h'(x) \geqslant 0$ for $x \in (0,1]$, so $h(x) \leqslant h(1) = 0$, and $f'(v) \leqslant 0$, which means $f(v) \geqslant f(\tau)$ when $0 < v \leqslant \tau < \frac{1}{2}$. Taking $x = \frac{b}{a}$, as desired. □

We now try to present a further improvement of Theorem 1.

THEOREM 2. *Let* $0 < v \leqslant \tau < \frac{1}{2}$. *If* $a > b > 0$, *then*

$$\frac{a\nabla_v b - K(h,2)^v a\sharp_v b}{a\nabla_\tau b - K(h,2)^\tau a\sharp_\tau b} \geqslant \frac{v(1-v)}{\tau(1-\tau)} \geqslant \frac{v}{\tau}, \tag{6}$$

where $K(h,2) = \frac{(h+1)^2}{4h}$ *and* $h = \frac{b}{a}$.

Proof. Let $f(v) = \frac{(1-v+vx)-K(x,2)^v(x^v)}{v(1-v)} = \frac{(1-v+vx)-(\frac{1+x}{2})^{2v}}{v(1-v)}$. Then $f'(v) = \frac{h(x)}{v^2(1-v)^2}$ for

$$h(x) = v(1-v)\left[x - 1 - 2\left(\frac{1+x}{2}\right)^{2v}\ln\frac{1+x}{2}\right] + (2v-1)\left[(1-v+vx) - \left(\frac{1+x}{2}\right)^{2v}\right],$$

so we have

$$\begin{aligned} h'(x) = v(1-v)&\left[1 - 2v\left(\frac{1+x}{2}\right)^{2v-1}\ln\frac{1+x}{2} - \left(\frac{1+x}{2}\right)^{2v-1}\right] \\ &+ (2v-1)\left[v - v\left(\frac{1+x}{2}\right)^{2v-1}\right], \end{aligned}$$

and

$$h''(x) = v(1-v)\left[-2v(2v-1)\frac{1}{2}\left(\frac{1+x}{2}\right)^{2v-2}\ln\frac{1+x}{2} - 2v\frac{1}{2}\left(\frac{1+x}{2}\right)^{2v-2} - (2v-1)\frac{1}{2}\left(\frac{1+x}{2}\right)^{2v-2}\right] + (2v-1)\left[-v(2v-1)\frac{1}{2}\left(\frac{1+x}{2}\right)^{2v-2}\right]$$
$$= v\left(\frac{1+x}{2}\right)^{2v-2}\left[v(v-1)(2v-1)\ln\frac{1+x}{2} - \frac{v}{2}\right].$$

We have $h''(x) \leqslant 0$ for $v \in (0, \frac{1}{2}]$ and $x \in (0,1)$, which implies $h'(x) \geqslant h'(1) = 0$, and then $h(x) \leqslant h(1) = 0$, it means $f'(v) \leqslant 0$. So $f(v) \geqslant f(\tau)$ when $0 < v \leqslant \tau < \frac{1}{2}$. We complete the proof by putting $x = \frac{b}{a}$. □

Next, we give a generalization of Theorem 2.

THEOREM 3. *Let $0 < v \leqslant \tau < \frac{1}{2}$ and $m \in \mathbb{N}^+$. If $a > b > 0$, then*

$$\frac{(a\nabla_v b)^m - K(h,2)^{mv}(a\sharp_v b)^m}{(a\nabla_\tau b)^m - K(h,2)^{m\tau}(a\sharp_\tau b)^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)}, \tag{7}$$

where $K(h,2) = \frac{(h+1)^2}{4h}$ and $h = \frac{b}{a}$.

Proof. Letting $f(v) = (1-v+vx)^m - \left((\frac{1+x}{2})^{2v}\right)^m$. Then $f(v) = \left((1-v+vx) - (\frac{1+x}{2})^{2v}\right)h(v)$, where $h(v) = \sum_{k=1}^{m}(1-v+vx)^{m-k}\left((\frac{1+x}{2})^{2v}\right)^{k-1}$. So we have

$$h'(v) = \sum_{k=1}^{m}(m-k)(x-1)(1-v+vx)^{m-k-1}\left(\left(\frac{1+x}{2}\right)^{2v}\right)^{k-1} + \sum_{k=1}^{m}2(k-1)(1-v+vx)^{m-k}\left(\left(\frac{1+x}{2}\right)^{2v}\right)^{k-1}\ln\frac{1+x}{2}.$$

It is easy to see that $h'(v) \leqslant 0$ when $x \in (0,1)$, which means $h(v) \geqslant h(\tau)$ when $0 < v \leqslant \tau < \frac{1}{2}$. Therefore,

$$\frac{f(v)}{f(\tau)} = \frac{(1-v+vx)^m - \left((\frac{1+x}{2})^{2v}\right)^m}{(1-\tau+\tau x)^m - \left((\frac{1+x}{2})^{2\tau}\right)^m}$$
$$= \frac{\left((1-v+vx) - (\frac{1+x}{2})^{2v}\right)h(v)}{\left((1-\tau+\tau x) - (\frac{1+x}{2})^{2\tau}\right)h(\tau)}$$
$$\geqslant \frac{(1-v+vx) - (\frac{1+x}{2})^{2v}}{(1-\tau+\tau x) - (\frac{1+x}{2})^{2\tau}}$$
$$\geqslant \frac{v(1-v)}{\tau(1-\tau)} \quad \text{(by (6))}.$$

Taking $x = \frac{b}{a}$, we get the desired results. $\square$

Motivated by the idea of Theorem 2, we now give a further improvement of (5).

THEOREM 4. *Let $0 < v \leqslant \tau < \frac{1}{2}$. If $a > b > 0$, then*

$$\frac{(a\nabla_v b)^2 - \left(a\sharp_v b\right)^2 - v^2(a-b)^2}{(a\nabla_\tau b)^2 - \left(a\sharp_\tau b\right)^2 - \tau^2(a-b)^2} \geqslant \frac{v(1-v)}{\tau(1-\tau)}. \tag{8}$$

Proof. Let $f(v) = \frac{(1-v+vx)^2 - x^{2v} - v^2(x-1)^2}{v(1-v)}$. Then $f'(v) = \frac{h(x)}{v^2(1-v)^2}$ for

$$h(x) = (1-v+vx)(-1+v+vx) + x^{2v}\left[(1-2v) + 2v(v-1)\ln x\right] - v^2(x-1)^2,$$

so we have

$$h'(x) = 2v^2x + 2vx^{2v-1}\left[1-2v+2v(v-1)\ln x\right] + 2v(v-1)x^{2v-1} - 2(x-1)v^2$$

and

$$h''(x) = x^{2v-2}\left[4(2v-1)(v-1)\ln x - 2\right]v^2.$$

We have $h''(x) \leqslant 0$ for $v \in (0, \frac{1}{2}]$ and $x \in (0,1)$, which implies $h'(x) \geqslant h'(1) = 0$, and then $h(x) \leqslant h(1) = 0$, it means $f'(v) \leqslant 0$. So $f(v) \geqslant f(\tau)$ when $0 < v \leqslant \tau < \frac{1}{2}$. We complete the proof by putting $x = \frac{b}{a}$. $\square$

Hirzallah and Kittaneh [2] showed a quadratic refinements of Young's inequality

$$(a^{1-v}b^v)^2 + \min\{v, 1-v\}^2(a-b)^2 \leqslant \left((1-v)a + vb\right)^2 \tag{9}$$

for $a, b > 0$ and $0 \leqslant v \leqslant 1$. Our inequality (8) is a refinement and reverse of (9) when $0 < v \leqslant \frac{1}{2}$.

We do not get the same generalization as (6) for (8) for the time being. Interested readers could have a try.

Next, we give some inequalities for operator, Hilbert-Schmidt norm and trace class norm as promised.

LEMMA 5. ([4]) *Let $X \in B(\mathscr{H})$ be self-adjoint and f and g be continuous real functions such that $f(t) \geqslant g(t)$ for all $t \in Sp(X)$ (the spectrum of X). Then $f(X) \geqslant g(X)$.*

THEOREM 6. *Let $A, B \in B(\mathscr{H})$, $0 < v \leqslant \tau < \frac{1}{2}$. If $0 < hA \leqslant B \leqslant h'A$, then we have*

$$A\nabla_v B \geqslant \frac{v(1-v)}{\tau(1-\tau)}\left(A\nabla_\tau B - K(h,2)^\tau(A\sharp_\tau B)\right) + K(h',2)^v(A\sharp_v B), \tag{10}$$

where $h' = \frac{m'}{M'}$ and $h = \frac{m}{M}$.

Proof. Taking $a=1$ in inequality (6), then we obtain

$$1\nabla_v b - K(b,2)^v(1\sharp_v b) \geqslant \frac{v(1-v)}{\tau(1-\tau)}\left(1\nabla_\tau b - K(b,2)^\tau(1\sharp_\tau b)\right). \tag{11}$$

Under our conditions, we can get $I \geqslant h'I = \frac{m'}{M'}I \geqslant X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geqslant hI = \frac{m}{M}I$, and then $Sp(X) \subseteq [h,h'] \subseteq (0,1)$. The operator X has a positive spectrum, then by Lemma 5 and the inequality (11), we have

$$I\nabla_v X \geqslant \frac{v(I-v)}{\tau(I-\tau)}\left(I\nabla_\tau X - \max_{h\leqslant x\leqslant h'} K(x,2)^\tau(I\sharp_\tau X)\right) + \min_{h\leqslant x\leqslant h'} K(x,2)^v(I\sharp_v X). \tag{12}$$

Since the Kantorovich constant $K(t,2) = \frac{(t+1)^2}{4t}$ is a decreasing function on $(0,1)$, then

$$I\nabla_v X \geqslant \frac{v(I-v)}{\tau(I-\tau)}\left(I\nabla_\tau X - K(h,2)^\tau(I\sharp_\tau X)\right) + K(h',2)^v\left(I\sharp_v X\right), \tag{13}$$

Multiplying $A^{\frac{1}{2}}$ on both left and right sides of the inequality (13), we can get (10) directly. $\square$

THEOREM 7. *Let $X \in \mathbb{M}_n$ and $A,B \in \mathbb{M}_n$ be positive for $0 < v \leqslant \tau < \frac{1}{2}$. If $A > B$, then we have*

$$\begin{aligned}&||(1-v)AX + vXB||_2^2 \\ &\geqslant \frac{v(1-v)}{\tau(1-\tau)}\left[||(1-\tau)AX + \tau XB||_2^2 - K_2^{2\tau}||A^{1-\tau}XB^\tau||_2^2\right] + K_1^{2v}||A^{1-v}XB^v||_2^2,\end{aligned}$$

where $K_1 := \min_{1\leqslant i,l\leqslant n} K\left(\frac{\lambda_i}{x_l},2\right)$, $K_2 := \max_{1\leqslant i,l\leqslant n} K\left(\frac{\lambda_i}{x_l},2\right)$ and λ_i, x_l are eigenvalues of A,B respectively.

Proof. Since A,B are positive definite matrices, it follows by spectral theorem that there exist unitary matrices $U,V \in \mathbb{M}_n$ such that $A = U\Lambda_1 U^*$ and $B = V\Lambda_2 V^*$, where $\Lambda_1 = \operatorname{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$ and $\Lambda_2 = \operatorname{diag}(x_1,x_2,\cdots,x_n)$ for λ_i, x_i are eigenvalues of A,B respectively, so $\lambda_i, x_i > 0$, $i = 1,2,\cdots,n$. Let $Y = U^*XV = [y_{il}]$. Then

$$\begin{aligned}(1-v)AX + vXB &= U\left[(1-v)\Lambda_1 Y + vY\Lambda_2\right]V^* \\ &= U\left[\left((1-v)\lambda_i + vx_l\right)y_{il}\right]V^*\end{aligned}$$

and

$$A^{1-v}XB^v = U\left[(\lambda_i^{1-v}x_l^v)y_{il}\right]V^*.$$

By (7) and the unitarily invariance of the Hilbert-Schmidt norm, we have

$$
\begin{aligned}
&||(1-v)AX+vXB||_2^2-K_1^{2v}||A^{1-v}XB^v||_2^2\\
&=\sum_{i,l=1}^{n}\left((1-v)\lambda_i+vx_l\right)^2|y_{il}|^2-\sum_{i,l=1}^{n}\min K\left(\frac{\lambda_i}{x_l},2\right)^{2v}\left(\lambda_i^{1-v}x_l^{v}\right)^2|y_{il}|^2\\
&=\sum_{i,l=1}^{n}\left[\left((1-v)\lambda_i+vx_l\right)^2-\min K\left(\frac{\lambda_i}{x_l},2\right)^{2v}(\lambda_i^{1-v}x_l^{v})^2\right]|y_{il}|^2\\
&\geqslant\sum_{i,l=1}^{n}\left[\left((1-v)\lambda_i+vx_l\right)^2-K\left(\frac{\lambda_i}{x_l},2\right)^{2v}(\lambda_i^{1-v}x_l^{v})^2\right]|y_{il}|^2\\
&\geqslant\sum_{i,l=1}^{n}\frac{v(1-v)}{\tau(1-\tau)}\left[\left((1-\tau)\lambda_i+\tau x_l\right)^2-K\left(\frac{\lambda_i}{x_l},2\right)^{2\tau}(\lambda_i^{1-\tau}x_l^{\tau})^2\right]|y_{il}|^2\\
&\geqslant\sum_{i,l=1}^{n}\frac{v(1-v)}{\tau(1-\tau)}\left[\left((1-\tau)\lambda_i+\tau x_l\right)^2-\max K\left(\frac{\lambda_i}{x_l},2\right)^{2\tau}(\lambda_i^{1-\tau}x_l^{\tau})^2\right]|y_{il}|^2\\
&=\frac{v(1-v)}{\tau(1-\tau)}\left[\sum_{i,l=1}^{n}\left((1-\tau)\lambda_i+\tau x_l\right)^2|y_{il}|^2-\sum_{i,l=1}^{n}\max K\left(\frac{\lambda_i}{x_l},2\right)^{2\tau}(\lambda_i^{1-\tau}x_l^{\tau})^2|y_{il}|^2\right]\\
&=\frac{v(1-v)}{\tau(1-\tau)}\left[||(1-\tau)AX+\tau XB||_2^2-K_2^{2\tau}||A^{1-\tau}XB^{\tau}||_2^2\right].\quad\square
\end{aligned}
$$

THEOREM 8. *Let $X\in\mathbb{M}_n$ and $A,B\in\mathbb{M}_n$ be positive for $0<v\leqslant\tau<\frac{1}{2}$. If $A>B$, then we have*

$$
\begin{aligned}
&||(1-v)AX+vXB||_2^2-||A^{1-v}XB^v||_2^2-v^2||AX+XB||_2^2\\
&\geqslant\frac{v(1-v)}{\tau(1-\tau)}\left[||(1-\tau)AX+\tau XB||_2^2-||A^{1-\tau}XB^{\tau}||_2^2-\tau^2||AX+XB||_2^2\right].
\end{aligned}
$$

Proof. Combination inequality (8) and Theorem 7, we can get the proof easily, so we omit it. $\square$

THEOREM 9. *Let $A,B\in\mathbb{M}_n$ be positive and $0<v\leqslant\tau<\frac{1}{2}$. If $A\geqslant B$, then we have*

$$
\frac{||(1-v)A+vB||_1-K(h,2)^v||A||_1^{1-v}||B||_1^v}{||(1-\tau)A+\tau B||_1-K(h,2)^{\tau}||A||_1^{1-\tau}||B||_1^{\tau}}\geqslant\frac{v(1-v)}{\tau(1-\tau)}.
$$

where $K(h,2)=\frac{(h+1)^2}{4h}$ and $h=\frac{\mathrm{tr}B}{\mathrm{tr}A}$.

Proof. By the inequality (6), we have

$$\begin{aligned}
&||(1-v)A+vB||_1 = \mathrm{tr}\big((1-v)A+vB\big) = (1-v)\mathrm{tr}(A)+v\mathrm{tr}(B)\\
&\geqslant \frac{v(1-v)}{\tau(1-\tau)}\big((1-\tau)\mathrm{tr}(A)+\tau\mathrm{tr}(B)-K(h,2)^{\tau}\mathrm{tr}(A)^{1-\tau}\mathrm{tr}(B)^{\tau}\big)+K(h,2)^{v}\mathrm{tr}(A)^{1-v}\mathrm{tr}(B)^{v}\\
&= \frac{v(1-v)}{\tau(1-\tau)}\big(||(1-\tau)A+\tau B||_1-K(h,2)^{\tau}||A||_1^{1-\tau}||B||_1^{\tau}\big)+K(h,2)^{v}||A||_1^{1-v}||B||_1^{v}. \quad \square
\end{aligned}$$

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