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SINGULAR VALUE INEQUALITIES OF MATRICES WITH CONCAVE FUNCTIONS

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Abstract. In this note, we present some singular value inequalities of products and direct sums of matrices involving concave functions, which can be regarded as complements of some recent results in [Ann. Funct. Anal, 14 (2023) doi:10.1007/s43034-022-00233-1].

1. Introduction

Let $\mathbb{M}_n$ be the set of all $n\times n$ complex matrices. The identity matrix of $\mathbb{M}_n$ is I_n. The conjugate transpose of A is denoted by A^*. Given Hermitian matrix $A\in\mathbb{M}_n$, we write $A\geqslant 0$ ($A>0$, resp.) to indicate that A is positive semidefinite (definite, resp.). If the eigenvalues of a square matrix $A\in\mathbb{M}_n$ are real, then we denote $\lambda_j(A)$ the j-th largest eigenvalue of A. The singular values of a complex matrix $A\in\mathbb{M}_n$ are the eigenvalues of $|A|:=(A^*A)^{1/2}$, and we denote $s_j(A):=\lambda_j(|A|)$, which are arranged in nonincreasing order and repeated according to multiplicity as $s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_n(A)$. Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\|=\|A\|$ for any $A\in\mathbb{M}_n$ and unitary matrices $U,V\in\mathbb{M}_n$. Some of the special examples of unitarily invariant norms are the spectral norm $\|A\|_\infty=s_1(A)$ and the Ky Fan k-norm $\|A\|_{(k)}=\sum_{j=1}^k s_j(A)$ for $k=1,2,\ldots,n$. We say that A is a contraction if $\|A\|_\infty\leqslant 1$, i.e., $A^*A\leqslant I$.

Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ be two elements of $\mathbb{R}^n$ arranged in nonincreasing order $x_1\geqslant x_2\geqslant\cdots\geqslant x_n$ and $y_1\geqslant y_2\geqslant\cdots\geqslant y_n$. If

$$\sum_{j=1}^k x_j\leqslant\sum_{j=1}^k y_j,\ \ k=1,\ldots,n,$$

we say that x is weakly majorized by y, denoted by $x\prec_\omega y$. Moreover, when $x_i,y_i\geqslant 0$ ($i=1,2,\ldots,n$), we say that x is weakly log-majorized by y, denoted by $x\prec_{\omega\log} y$, if

$$\prod_{j=1}^k x_j\leqslant\prod_{j=1}^k y_j,\ \ k=1,\ldots,n.$$

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It is known that weak log-majorization implies weak majorization (see, e.g., [6, p. 19]). A function $f:[0,\infty)\to\mathbb{R}$ is said to preserve weak log-majorization if $f(x)\prec_{\omega\log} f(y)$ whenever $x\prec_{\omega\log} y$, where $f(x)=(f(x_1),\ldots,f(x_n))$, $f(y)=(f(y_1),\ldots,f(y_n))$, see, e.g., [4] for more details. Also, f is called submultiplicative if $f(xy)\leqslant f(x)f(y)$ whenever $x,y\in[0,\infty)$.

Recently, Al-Natoor et al. [1, Theorem 2.4] proved the singular value inequalities with convex functions: Let $A,B,X\in\mathbb{M}_n$ be such that X is a positive semidefinite contraction and $f:[0,\infty)\to\mathbb{R}$ be a nonnegative increasing submultiplicative convex function that preserves weak log-majorization. Then,

$$\prod_{j=1}^{k} s_j(f(|AXB^*|^2))\leqslant \|f(C_{p,A,B})\|_\infty^k\|f(C_{q,A,B})\|_\infty^k\prod_{j=1}^{k} s_j^2(X),\quad k=1,2,\ldots,n, \tag{1}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $C_{p,A,B}=\frac{1}{p}A^*A+\frac{1}{q}B^*B$.

Since weak log-majorization implies weak majorization, by the Fan dominance theorem [2, p. 93], (1) means that

$$\|f(|AXB^*|^2)\|\leqslant\|f(C_{p,A,B})\|_\infty\|f(C_{q,A,B})\|_\infty\||X|^2\|. \tag{2}$$

If $p=q=2$ in (2), a stronger version inequality [1, Theorem 2.6] is obtained

$$s_j(f(|AXB^*|))\leqslant\left\|f\left(\frac{A^*A+B^*B}{2}\right)\right\|_\infty s_j(X),\quad j=1,2,\ldots,n. \tag{3}$$

In particular, letting $f(t)=t$, (3) becomes

$$s_j(|AXB^*|)\leqslant\left\|\frac{A^*A+B^*B}{2}\right\|_\infty s_j(X),\quad j=1,2,\ldots,n. \tag{4}$$

As an application of (1), Al-Natoor et al. [1, Corollary 3.8] also gave the following result: Let $A,B,X,Y\in\mathbb{M}_n$ be such that X,Y are positive semidefinite contractions and $f:[0,\infty)\to\mathbb{R}$ be a nonnegative increasing submultiplicative convex function that preserves weak log-majorization. Then for $k=1,2,\ldots,n$,

$$\begin{aligned}\prod_{j=1}^{k} s_j(f(|AX+YB^*|^2))&\\ &\leqslant 2\max(\|f(C_{p,A,I_n})\|_\infty^k,\|f(C_{q,B,I_n})\|_\infty^k)\\ &\quad\times\max(\|f(C_{q,A,I_n})\|_\infty^k,\|f(C_{p,B,I_n})\|_\infty^k)\prod_{j=1}^{k} s_j^2(X\oplus Y),\end{aligned} \tag{5}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $C_{p,A,B}=\frac{1}{p}A^*A+\frac{1}{q}B^*B$.

When $p=q=2$ in (5), the authors in [1, Corollary 3.9] proved that

$$s_j(f(|AX+YB^*|))\leqslant 2\max\left(f\left(\frac{\|A\|_\infty^2+1}{2}\right),f\left(\frac{\|B\|_\infty^2+1}{2}\right)\right)s_j(X\oplus Y) \tag{6}$$

for $j=1,2,\ldots,n$.

Inspired by the inequalities (1)–(6) with convex functions, we would like to present more singular value inequalities of matrices involving concave functions.

2. Main results

For presenting and proving our results, we begin this section with the following several lemmas. For the first lemma, see, e.g., [5]. The second and third lemmas are standard. We list a consequence of Corollary 2.5 in [3] as Lemma 2.4 for convenience.

LEMMA 2.1. [5] *If $X, Y \in \mathbb{M}_n$, then*

$$s_j(X+Y) \leqslant 2s_j(X \oplus Y), \quad j=1,2,\ldots,n. \tag{7}$$

LEMMA 2.2. [2, p. 291] *Let $A \in \mathbb{M}_n$ and $f:[0,\infty) \to \mathbb{R}$ be a nonnegative increasing function. Then,*

$$s_j(f(|A|)) = f(s_j(A)), \quad j=1,2,\ldots,n.$$

LEMMA 2.3. (See [7, p. 275]) *Let $A, B, X \in \mathbb{M}_n$. Then,*

$$s_j(AXB) \geqslant s_n(A)s_n(B)s_j(X), \quad j=1,\ldots,n.$$

LEMMA 2.4. [3, Corollary 2.5] *Let $A, X \in \mathbb{M}_n$ such that A is positive semidefinite and X is contraction. If $f:[0,\infty) \to \mathbb{R}$ is a nonnegative increasing concave function, then*

$$s_j(f(X^*AX)) \geqslant s_j(X^*f(A)X)), \quad j=1,\ldots,n.$$

Now we will present several singular value inequalities in Theorem 2.5, Corollary 2.7 and Theorem 2.9 which can be considered as complements of inequalities (1)–(4).

THEOREM 2.5. *Let $A, B, X \in \mathbb{M}_n$, $s_n(B) \leqslant 1$ and $f:[0,\infty) \to \mathbb{R}$ be a nonnegative increasing concave function that preserves weak log-majorization.*

(a) *If X is a positive semidefinite contraction, then*

$$\prod_{j=1}^{k} s_j(f(|AXB^*|^2)) \geqslant s_n^k(X)s_n^k(f(|A|^2))s_n^{2k}(B)\prod_{j=1}^{k} s_j(X) \tag{8}$$

for $k=1,2,\ldots,n$.

(b) *If X is a nonzero positive semidefinite matrix, then*

$$\prod_{j=1}^{k} s_j\left(f\left(\frac{|AXB^*|^2}{\|X\|_\infty^2}\right)\right) \geqslant \frac{s_n^k(X)s_n^k(f(|A|^2))s_n^{2k}(B)}{\|X\|_\infty^{2k}}\prod_{j=1}^{k} s_j(X)$$

for $k=1,2,\ldots,n$. In particular, letting $f(t)=t$, the result becomes

$$\prod_{j=1}^{k} s_j^2(AXB^*) \geqslant s_n^k(X)s_n^k(|A|^2)s_n^{2k}(B)\prod_{j=1}^{k} s_j(X)$$

for $k=1,2,\ldots,n$.

Proof. Suppose that X is contraction and $\mathcal{A} = AX^{1/2}$, $\mathcal{B} = BX^{1/2}$. Then for $k = 1,2,\dots,n$, we have

$$\begin{aligned}\prod_{j=1}^{k} s_j(|AXB^*|^2) &= \prod_{j=1}^{k} s_j^2(\mathcal{A}\mathcal{B}^*) \\ &\geqslant \prod_{j=1}^{k} s_j^2(\mathcal{A}) s_n^2(\mathcal{B}^*) \qquad \text{(by Lemma 2.3)}\end{aligned}$$

which means that

$$(s_j^2(\mathcal{A}) s_n^2(\mathcal{B}^*))_{j=1}^n \prec_{\omega \log} (s_j(|AXB^*|^2))_{j=1}^n.$$

Since f preserves weak log-majorization, we have

$$(f(s_j^2(\mathcal{A}) s_n^2(\mathcal{B}^*)))_{j=1}^n \prec_{\omega \log} (f(s_j(|AXB^*|^2)))_{j=1}^n. \tag{9}$$

Therefore, for $k = 1,\dots,n$,

$$\begin{aligned}\prod_{j=1}^{k} s_j(f(|AXB^*|^2)) &= \prod_{j=1}^{k} f(s_j(|AXB^*|^2)) \quad \text{(by Lemma 2.2)} \\ &\geqslant \prod_{j=1}^{k} f(s_j^2(\mathcal{A}) s_n^2(\mathcal{B}^*)) \quad \text{(by (9))} \\ &\geqslant \prod_{j=1}^{k} s_n^2(\mathcal{B}^*) f(s_j^2(\mathcal{A})) \ \text{(by concavity of the function)} \\ &= \prod_{j=1}^{k} s_n^2(\mathcal{B}^*) s_j(f(X^{1/2}A^*AX^{1/2})) \\ &\geqslant \prod_{j=1}^{k} s_n^2(\mathcal{B}^*) s_j(X^{1/2} f(A^*A) X^{1/2}) \quad \text{(by Lemma 2.4)} \\ &\geqslant \prod_{j=1}^{k} s_n^2(X^{1/2}) s_n^2(B^*) s_j(f^{1/2}(A^*A) X f^{1/2}(A^*A)) \\ &\qquad\qquad\qquad \text{(by Lemma 2.3)} \\ &\geqslant \prod_{j=1}^{k} s_n(X) s_n^2(B) s_n^2(f^{1/2}(A^*A)) s_j(X) \\ &\qquad\qquad\qquad \text{(by Lemma 2.3)} \\ &= s_n^k(X) s_n^k(f(|A|^2)) s_n^{2k}(B) \prod_{j=1}^{k} s_j(X),\end{aligned}$$

which completes the proof. Similarly, part (b) follows by applying part (a) to the contraction matrix $\frac{X}{\|X\|_\infty}$. $\square$

REMARK 2.6. Obviously, (8) is a complement of (1).

Since weak log-majorization implies weak majorization, by the Ky Fan dominance theorem, we have the following corollary.

COROLLARY 2.7. *Let $A,B,X \in \mathbb{M}_n$, $s_n(B) \leqslant 1$ and $f:[0,\infty)\to\mathbb{R}$ be a nonnegative increasing concave function that preserves weak log-majorization.*

(a) *If X is a positive semidefinite contraction, then*

$$\left|\left|f(|AXB^*|^2)\right|\right| \geqslant s_n(X)s_n(f(|A|^2))s_n^2(B)\,||X|| \tag{10}$$

for any unitarily invariant norm.

(b) *If X is a nonzero positive semidefinite matrix, then*

$$\left|\left|f\left(\frac{|AXB^*|^2}{\|X\|_\infty^2}\right)\right|\right| \geqslant \frac{s_n(X)s_n(f(|A|^2))s_n^2(B)}{\|X\|_\infty^2}\,||X||$$

for any unitarily invariant norm. In particular, letting $f(t)=t$, the result becomes

$$\left|\left||AXB^*|^2\right|\right| \geqslant s_n(X)s_n(|A|^2)s_n^2(B)\,||X||$$

for any unitarily invariant norm.

REMARK 2.8. It is easy to see that (10) is a complement of (2).

The following theorem on singular value inequalities of products of matrices can also be considered as a complement of inequalities (3)–(4).

THEOREM 2.9. *Let $A,B,X \in \mathbb{M}_n$, $s_n(B) \leqslant 1$ and $f:[0,\infty)\to\mathbb{R}$ be a nonnegative increasing concave function.*

(a) *If X is a positive semidefinite contraction, then*

$$s_j(f(|AXB^*|)) \geqslant s_n(X)s_n(B)s_j(f(A))$$

for $j=1,2,\ldots,n$.

(b) *If X is a nonzero positive semidefinite matrix, then*

$$s_j\left(f\left(\frac{|AXB^*|}{\|X\|_\infty}\right)\right) \geqslant \frac{s_n(X)s_n(B)}{\|X\|_\infty}s_j(f(A))$$

for $j=1,2,\ldots,n$. In particular, letting $f(t)=t$, the result becomes

$$s_j(AXB^*) \geqslant s_n(X)s_n(B)s_j(A)$$

for $j=1,2,\ldots,n$.

Proof. Suppose that X is contraction and $\mathcal{A} = AX^{1/2}$, $\mathcal{B} = BX^{1/2}$. Then for $j = 1,2,\dots n$, we have

$$\begin{aligned}
s_j(f(|AXB^*|)) &= f(s_j(AXB^*)) \quad \text{(by Lemma 2.2)}\\
&= f(s_j(\mathcal{A}\mathcal{B}^*))\\
&\geqslant f(s_j(\mathcal{A})s_n(\mathcal{B}^*)) \ \ \text{(by Lemma 2.3)}\\
&\geqslant s_n(\mathcal{B}^*)f(s_j(\mathcal{A})) \quad \text{(by concavity of the function)}\\
&\geqslant s_n(X^{1/2})s_n(B)f(s_j(A)s_n(X^{1/2})) \quad \text{(by Lemma 2.3)}\\
&\geqslant s_n(X^{1/2})s_n(B)s_n(X^{1/2})f(s_j(A))\\
&\qquad\qquad\qquad\qquad \text{(by concavity of the function)}\\
&= s_n(B)s_j(f(A))s_n(X),
\end{aligned}$$

which completes the proof. Similarly, part (b) follows by applying part (a) to the contraction matrix $\frac{X}{\|X\|_\infty}$. $\square$

Next, we give more general results on direct sums of matrices involving concave functions when X and Y are positive semidefinite matrices. Although we can not give the singular value inequalities of general sums of matrices analogous to (5)–(6), as complements of the results, the lower bounds of singular value inequalities of direct sums are obtained as follows.

THEOREM 2.10. *Let $A,B,X,Y \in \mathbb{M}_n$, $s_n(B) \leqslant 1$ and $f : [0,\infty) \to \mathbb{R}$ be a nonnegative increasing concave function that preserves weak log-majorization.*

(a) *If X and Y are positive semidefinite contractions, then*

$$\begin{aligned}
&\prod_{j=1}^{k} s_j(f(|AX \oplus YB^*|^2))\\
&\qquad \geqslant \frac{s_n^k(X+Y)s_n^{2k}(I_n+B)s_n^k(f(|A|^2+I_n))}{2^{5k}} \prod_{j=1}^{k} s_j(X+Y)
\end{aligned} \tag{11}$$

for $k = 1,2,\dots,n$.

(b) *If X and Y are nonzero positive semidefinite matrices, then*

$$\begin{aligned}
&\prod_{j=1}^{k} s_j\left(f\left(\frac{|AX \oplus YB^*|^2}{\min(\|X\|_\infty^2, \|Y\|_\infty^2)}\right)\right)\\
&\qquad \geqslant \frac{s_n^k(X+Y)s_n^{2k}(I_n+B)s_n^k(f(|A|^2+I_n))}{2^{5k}\max(\|X\|_\infty^{2k}, \|Y\|_\infty^{2k})} \prod_{j=1}^{k} s_j(X+Y)
\end{aligned}$$

for $k = 1,2,\dots,n$.

Proof. Let $\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & I_n \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} I_n & 0 \\ 0 & B \end{bmatrix}$, and $\mathcal{X} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$. Observe that $\mathcal{X}$ is positive semidefinite. Then we have

$$s_j(f(|AX \oplus YB^*|)) = s_j(f(|\mathcal{A}\mathcal{X}\mathcal{B}^*|)). \tag{12}$$

Applying Theorem 2.5 to the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{X}$, we have

$$\prod_{j=1}^{k} s_j(f(|\mathcal{A}\mathcal{X}\mathcal{B}^*|^2)) \geqslant s_n^k(\mathcal{X}) s_n^k(f(|\mathcal{A}|^2)) s_n^{2k}(\mathcal{B}) \prod_{j=1}^{k} s_j(\mathcal{X}).$$

Therefore,

$$\begin{aligned}
&\prod_{j=1}^{k} s_j(f(|AX \oplus YB^*|^2)) \\
&\quad \geqslant s_n^k(\mathcal{X}) s_n^k(f(|\mathcal{A}|^2)) s_n^{2k}(\mathcal{B}) \prod_{j=1}^{k} s_j(\mathcal{X}) \qquad \text{(by (12))} \\
&\quad = s_n^k(X \oplus Y) s_n^{2k}(I_n \oplus B) s_n^k(f(|A|^2 \oplus I_n)) \prod_{j=1}^{k} s_j(X \oplus Y) \\
&\quad \geqslant s_n^k\left(\frac{X+Y}{2}\right) s_n^{2k}\left(\frac{I_n+B}{2}\right) s_n^k\left(f\left(\frac{|A|^2+I_n}{2}\right)\right) \prod_{j=1}^{k} s_j\left(\frac{X+Y}{2}\right) \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.1)} \\
&\quad \geqslant \frac{s_n^k(X+Y) s_n^{2k}(I_n+B) s_n^k(f(|A|^2+I_n))}{2^{5k}} \prod_{j=1}^{k} s_j(X+Y) \\
&\qquad\qquad\qquad\qquad \text{(by concavity of the function)}
\end{aligned}$$

for $k = 1,2,\ldots,n$, which completes the proof. Similarly, part (b) follows by applying part (a) to the contraction matrices $\frac{X}{\|X\|_\infty}$ and $\frac{Y}{\|Y\|_\infty}$. $\square$

REMARK 2.11. It is obvious that (11) is a complement of the inequality (5).

The following theorem involving direct sums of singular values can be regarded as a complement of the inequality (6).

THEOREM 2.12. *Let $A,B,X,Y \in \mathbb{M}_n$, $s_n(B) \leqslant 1$ and $f : [0,\infty) \to \mathbb{R}$ be a nonnegative increasing concave function.*

(a) *If X and Y are positive semidefinite contractions, then*

$$s_j(f(|AX \oplus YB^*|)) \geqslant \frac{s_n(X+Y)s_n(I_n+B)}{8} s_j(f(I_n+A))$$

for $j = 1,2,\ldots,n$.

(b) *If X and Y are nonzero positive semidefinite matrices, then*

$$s_j\left(f\left(\frac{|AX \oplus YB^*|}{\min(\|X\|_\infty, \|Y\|_\infty)}\right)\right) \geqslant \frac{s_n(X+Y)s_n(I_n+B)}{8\max(\|X\|_\infty, \|Y\|_\infty)} s_j(f(I_n+A))$$

for $j = 1,2,\ldots,n$.

Proof. Applying Theorem 2.9 to the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{X}$ yields the following inequality

$$s_j(f(|\mathcal{A}\mathcal{X}\mathcal{B}^*|)) \geqslant s_n(\mathcal{X})s_n(\mathcal{B})s_j(f(\mathcal{A})), \quad j=1,2,\ldots,n.$$

So we have

$$\begin{aligned} s_j(f(|AX\oplus YB^*|)) &\geqslant s_n(\mathcal{X})s_n(\mathcal{B})s_j(f(\mathcal{A})) \qquad \text{(by (12))}\\ &= s_n(X\oplus Y)s_n(I_n\oplus B)f(s_j(I_n\oplus A))\\ &\geqslant s_n\left(\frac{X+Y}{2}\right)s_n\left(\frac{I_n+B}{2}\right)f\left(\frac{s_j(I_n+A)}{2}\right)\\ &\qquad\qquad \text{(by Lemma 2.1)}\\ &\geqslant \frac{s_n(X+Y)s_n(I_n+B)}{8}s_j(f(I_n+A))\\ &\qquad\qquad \text{(by concavity of the function)} \end{aligned}$$

for $j=1,2,\ldots,n$, which completes the proof. Similarly, part (b) follows by applying part (a) to the contraction matrices $\frac{X}{\|X\|_\infty}$ and $\frac{Y}{\|Y\|_\infty}$. $\square$

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REFERENCES

[1] A. AL-NATOOR, O. HIRZALLAH AND F. KITTANEH, *Singular value inequalities for convex functions of positive semidefinite matrices*, Ann. Funct. Anal, **14** (2023), doi:10.1007/s43034-022-00233-1.

[2] R. BHATIA, *Matrix Analysis*, GTM, vol. 169, Springer-Verlag, New York, 1997.

[3] J.-C. BOURIN AND E.-Y. LEE, *Unitary orbits of Hermitian operators with convex or concave functions*, Bull. London Math. Soc. **44** (2012) 1085–1102.

[4] I. H. GUMUS, O. HIRZALLAH AND F. KITTANEH, *Norm inequalities involving accretive-dissipative* 2×2 *block matrices*, Linear Algebra Appl. **528** (2017) 76–93.

[5] O. HIRZALLAH AND F. KITTANEH, *Singular values, norms, and commutators*, Linear Algebra Appl. **432** (2010) 1322–1336.

[6] X. ZHAN, *Matrix Inequalities*, Springer-Verlag, Berlin, 2002.
[7] F. ZHANG, *Matrix Theory: Basic Results and Techniques*, second edition, Springer, New York, 2011.

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