Journal of
Mathematical
Inequalities
Volume 18, Number 1 (2024), 1–24 doi:10.7153/jmi-2024-18-01

VARIABLE ANISOTROPIC HERZ–MORREY–HARDY SPACES AND THEIR APPLICATIONS

AITING WANG

(*Communicated by Y. Sawano*)

Abstract. Let A be an expansive dilation on $\mathbb{R}^n$ and let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. Also let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a variable exponent function satisfying the globally log-Hölder continuous condition. In this paper, the authors first introduce the variable anisotropic Herz-Morrey-Hardy spaces $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$, via the non-tangential grand maximal function, and then establish their atomic decompositions. As applications, the authors obtain the boundedness of some sublinear operators from $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ to $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ and from $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ to $MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$.

1. Introduction

The theory of Hardy spaces on the Euclidean space $\mathbb{R}^n$ plays an important role in various fields of analysis and partial differential equations; see [5, 10, 16]. It is well known that the Hardy space is a good substitution of $L^p(\mathbb{R}^n)$ when $p \in (0,1]$. Since some of the singular integrals (for example, the Riesz transform) are bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ when $p \in (0,1]$. The real-variable theory of Hardy spaces on the n-dimensional Euclidean space $\mathbb{R}^n$ was originally studied by Stein and Weiss [17] and systematically developed by Fefferman and Stein in a seminal paper [10].

In recent years, the theory of function spaces with variable exponents has been developed in the papers [6, 14, 15, 18], and applied in fluid dynamics [2], image processing [4], partial differential equations and variational calculus and harmonic analysis. In 2012, Almeida and Drihem [1] introduced the Herz spaces with two variable exponents and obtained the boundedness of some sublinear operators on those spaces. In the same year, Wang et al. [19] introduced the Herz-type Hardy spaces with variable exponents $H\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$, which are the generalization of classical Herz-type Hardy spaces. In 2015, Dong et al. [9] introduced the Herz-type Hardy spaces with two variable exponents $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$. In the same year, Xu et al. [21] also introduced the Herz-Morrey-Hardy spaces with variable exponents $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(\mathbb{R}^n)$, and obtained their atomic characterizations.

Mathematics subject classification (2020): Primary 42B30; Secondary 42B35, 46E30.
Keywords and phrases: Anisotropy, Herz space, Hardy space, variable exponent, atom.
This work is partially supported by the projects of university level planning in Qinghai Minzu University (Grant No. 2022GH25).

On the other hand, extending classic function spaces arising in harmonic analysis of Euclidean spaces to other domains and non-isotropic settings is an important topic. For example, in 2003, Bownik [3] introduced the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$. In 2008, Ding et al. [8] introduced the anisotropic Herz-type Hardy space $H\dot{K}_p^{\alpha,q}(A;\mathbb{R}^n)$ and $HK_p^{\alpha,q}(A;\mathbb{R}^n)$.

Inspired by previous papers, we would like to declare that the goal of this paper is to introduce new Herz-Morrey-Hardy spaces with variable exponents and give their applications.

Precisely, this article is organized as follows.

In Section 2, we first recall some notations and definitions concerning expansive dilations, variable exponent, variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the variable anisotropic Herz-Morrey spaces $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ and $MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$. Then, motivated by Xu et al. [21] and Ding et al. [8], we introduce anisotropic Herz-Morrey-Hardy spaces with variable exponents via non-tangential grand maximal function. The aim of Section 3 is to establish the atomic characterization of $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ and $HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ (see Theorem 3.2 below). As applications of the atomic characterization of $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ and $HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$, in Section 4, we obtain the boundedness of some sublinear operators from $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ to $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ and from $HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ to $MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ (see Theorem 4.3 below).

Finally, we make some conventions on notation. Let $\mathbb{N}:=\{1,2,\ldots\}$ and $\mathbb{Z}_+:=\{0\}\cup\mathbb{N}$. Denote by $\mathscr{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and $\mathscr{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). For any $\alpha:=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n:=(\mathbb{Z}_+)^n$, let $|\alpha|:=\alpha_1+\cdots+\alpha_n$ and $\partial^\alpha:=(\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $D\lesssim F$ means that $D\leqslant CF$. If $D\lesssim F$ and $F\lesssim D$, we then write $D\sim F$. For any $q\in[1,\infty]$, we denote by q' its conjugate index, namely, $1/q+1/q'=1$. We also use $C_{(\alpha,\beta,\ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha,\beta,\ldots$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*. If there are no special instructions, any space $\mathscr{X}(\mathbb{R}^n)$ is denoted simply by $\mathscr{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any $a\in\mathbb{R}$, $\lfloor a\rfloor$ denotes the *maximal integer* not larger than a.

2. Preliminaries

In this section, we introduce the definitions of the homogeneous anisotropic Herz-Morrey-Hardy space with variable exponents $HM\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ and the non-homogeneous anisotropic Herz-Morrey-Hardy space with variable exponents $HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$, via the non-tangential grand maximal function $M_N(f)$.

We begin with recalling the notion of an expansive dilation on $\mathbb{R}^n$; see [3, p. 5]. A real $n\times n$ matrix A is called an *expansive dilation*, shortly a *dilation*, if $\min_{\lambda\in\sigma(A)}|\lambda|>1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of A. Let λ_- and λ_+ be two *positive* *numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leqslant \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+. \tag{2.1}$$

In the case when A is diagonalizable over $\mathbb{C}$, we can even take $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}$. Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments.

By [3, Lemma 2.2], we have that, for a given dilation A, there exist a number $r \in (1,\infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where P is some non-degenerate $n \times n$ matrix, such that

$$\Delta \subset r\Delta \subset A\Delta,$$

and one can and do additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Let $B_k := A^k\Delta$ for $k \in \mathbb{Z}$. Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$, here and hereafter, $b := |\det A|$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Denote by $\mathfrak{B}$ the set of all such dilated balls, namely,

$$\mathfrak{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}. \tag{2.2}$$

Throughout the whole paper, let σ be the *smallest integer* such that $2B_0 \subset A^\sigma B_0$ and, for any subset E of $\mathbb{R}^n$, let $E^\complement := \mathbb{R}^n \setminus E$. Then, for all $k, j \in \mathbb{Z}$ with $k \leqslant j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \tag{2.3}$$

$$B_k + (B_{k+\sigma})^\complement \subset (B_k)^\complement, \tag{2.4}$$

where $E + F$ denotes the *algebraic sum* $\{x + y : x \in E, y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

DEFINITION 2.1. A *quasi-norm*, associated with a dilation A, is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0,\infty)$, for simplicity, denoted by ρ, satisfying

(i) $\rho(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, here and hereafter, $\vec{0}_n$ denotes the origin of $\mathbb{R}^n$;

(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where, as above, $b := |\det A|$;

(iii) $\rho(x+y) \leqslant C_A[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $C_A \in [1,\infty)$ is a constant independent of x and y.

In the standard dyadic case $A := 2\mathrm{I}_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of a homogeneous quasi-norm associated with A, here and hereafter, $\mathrm{I}_{n\times n}$ denotes the $n \times n$ *unit matrix*, $|\cdot|$ always denotes the *Euclidean norm* in $\mathbb{R}^n$.

It was proved, in [3, p. 6, Lemma 2.4], that all homogeneous quasi-norms associated with a given dilation A are equivalent. Therefore, for a given dilation A, in what follows, for simplicity, we always use the *step homogeneous quasi-norm* ρ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k\in\mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \text{ if } x \neq \vec{0}_n, \quad \text{or else} \quad \rho(\vec{0}_n) := 0.$$

By (2.3), we know that, for all $x,y\in\mathbb{R}^n$,

$$\rho(x+y)\leqslant b^{\sigma}\left(\max\{\rho(x),\rho(y)\}\right)\leqslant b^{\sigma}[\rho(x)+\rho(y)];$$

see [3, p. 8]. If we let λ_+ and λ_- be any numbers satisfying (2.1), then there exists a constant $C_2>0$ such that, for all $x\in\mathbb{R}^n$,

$$C_2^{-1}\rho(x)^{\ln\lambda_+/\ln b}\leqslant |x|\leqslant C_2\rho(x)^{\ln\lambda_-/\ln b}\ \text{for}\ \rho(x)\leqslant 1, \tag{2.5}$$

$$C_2^{-1}\rho(x)^{\ln\lambda_-/\ln b}\leqslant |x|\leqslant C_2\rho(x)^{\ln\lambda_+/\ln b}\ \text{for}\ \rho(x)\geqslant 1. \tag{2.6}$$

Now we recall that a measurable function $p(\cdot):\mathbb{R}^n\to(0,\infty)$ is called a *variable exponent*. For any variable exponent $p(\cdot)$, let

$$p_-:=\operatorname*{ess\,inf}_{x\in\mathbb{R}^n}p(x)\quad\text{and}\quad p_+:=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}p(x). \tag{2.7}$$

Denote by $\mathscr{P}$ the set of all variable exponents $p(\cdot)$ satisfying $p_->1$ and $p_+<\infty$.

Let f be a measurable function on $\mathbb{R}^n$ and $p(\cdot)\in\mathscr{P}$. Then the *modular function* (or, for simplicity, the *modular*) $\rho_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting

$$\rho_{p(\cdot)}(f):=\int_{\mathbb{R}^n}|f(x)|^{p(x)}\,dx$$

and the *Luxemburg* (also called *Luxemburg-Nakano*) *quasi-norm* $\|f\|_{L^{p(\cdot)}}$ by

$$\|f\|_{L^{p(\cdot)}}:=\inf\left\{\lambda\in(0,\infty):\rho_{p(\cdot)}(f/\lambda)\leqslant 1\right\}.$$

Moreover, the *variable Lebesgue space* $L^{p(\cdot)}$ is defined to the set of all measurable functions f satisfying that $\rho_{p(\cdot)}(f)<\infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}}$.

We recall the definition of *Hardy-Littlewood maximal function* $M_{\mathrm{HL}}(f)$. For any $f\in L^1_{\mathrm{loc}}$ and $x\in\mathbb{R}^n$,

$$M_{\mathrm{HL}}(f)(x):=\sup_{k\in\mathbb{Z}}\sup_{y\in x+B_k}\frac{1}{|B_k|}\int_{y+B_k}|f(z)|\,dz=\sup_{x\in B\in\mathfrak{B}}\frac{1}{|B|}\int_B|f(z)|\,dz, \tag{2.8}$$

where $\mathfrak{B}$ is as in (2.2).

Let $\mathscr{B}$ is the set of $p(\cdot)\in\mathscr{P}$ satisfying the condition that M_{HL} is bounded on $L^{p(\cdot)}$. It is well known that if $p(\cdot)\in\mathscr{P}$ and satisfies the following global log-Hölder continuous then $p(\cdot)\in\mathscr{B}$.

DEFINITION 2.2. Let $g(\cdot)$ be a real function on $\mathbb{R}^n$.

(1) $g(\cdot)$ is locally log-Hölder continuous, if there exists a constant $C>0$ such that

$$|g(x)-g(y)|\leqslant\frac{C}{\log(e+1/|x-y|)}$$

for any $x,y\in\mathbb{R}^n$ and $|x-y|<1/2$.

(2) $g(\cdot)$ is locally log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant $C>0$ such that

$$|g(x)-g(0)| \leqslant \frac{C}{\log(e+1/|x|)}$$

for any $x\in\mathbb{R}^n$.

(3) $g(\cdot)$ is locally log-Hölder continuous at infinity (or has a log decay at infinity), if there exist $g_\infty\in\mathbb{R}$ and a constant $C>0$ such that

$$|g(x)-g_\infty| \leqslant \frac{C}{\log(e+|x|)}$$

for any $x\in\mathbb{R}^n$.

If $g(\cdot)$ is both local log-Hölder continuous and log-Hölder continuous at infinity, then $g(\cdot)$ is said to be global log-Hölder continuous.

We denote by $\mathscr{P}_0^{log}$ and $\mathscr{P}_\infty^{log}$ the class of all variable exponents $p(\cdot)\in\mathscr{P}$, which are log-Hölder continuous at the origin and at infinity respectively. We call $p'(\cdot)$ the conjugate exponent to $p(\cdot)$, that is $p'(\cdot)=\frac{p(\cdot)}{p(\cdot)-1}$. We know that $p(\cdot)\in\mathscr{B}$ is equivalent to $p'(\cdot)\in\mathscr{B}$.

A C^∞ function φ is said to belong to the Schwartz class $\mathscr{S}$ if, for every integer $\ell\in\mathbb{Z}_+$ and multi-index α, $\|\varphi\|_{\alpha,\ell}:=\sup_{x\in\mathbb{R}^n}[\rho(x)]^\ell|\partial^\alpha\varphi(x)|<\infty$. The dual space of $\mathscr{S}$, namely, the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology, is denoted by $\mathscr{S}'$. For any $N\in\mathbb{Z}_+$, let

$$\mathscr{S}_N:=\left\{\varphi\in\mathscr{S}:\ \|\varphi\|_{\alpha,\ell}\leqslant 1,\ |\alpha|\leqslant N,\ \ \ell\leqslant N\right\};$$

equivalently,

$$\varphi\in\mathscr{S}_N \Longleftrightarrow \|\varphi\|_{\mathscr{S}_N}:=\sup_{|\alpha|\leqslant N}\sup_{x\in\mathbb{R}^n}\left[|\partial^\alpha\varphi(x)|\max\left\{1,[\rho(x)]^N\right\}\right]\leqslant 1.$$

In what follows, for $\varphi\in\mathscr{S}$, $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let

$$\varphi_k(x):=b^{-k}\varphi\left(A^{-k}x\right). \tag{2.9}$$

Let $f\in\mathscr{S}'$. The *non-tangential maximal function* $M_\varphi(f)$ with respect to φ is defined by setting, for any $x\in\mathbb{R}^n$,

$$M_\varphi(f)(x):=\sup_{y\in x+B_k,k\in\mathbb{Z}}\{|f*\varphi_k(y)|: x-y\in B_k, k\in\mathbb{Z}\}.$$

The *radial maximal function* $M_\varphi^0(f)$ with respect to φ is defined by setting, for any $x\in\mathbb{R}^n$,

$$M_\varphi^0(f)(x):=\sup_{k\in\mathbb{Z}}|f*\varphi_k(x)|.$$

Moreover, for any given $N\in\mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of $f\in\mathscr{S}'$ is defined by setting, for any $x\in\mathbb{R}^n$,

$$M_N(f)(x):=\sup_{\varphi\in\mathscr{S}_N} M_\varphi(f)(x).$$

The *radial grand maximal function* $M_N^0(f)$ of $f\in\mathscr{S}'$ is defined by setting, for any $x\in\mathbb{R}^n$,

$$M_N^0(f)(x):=\sup_{\varphi\in\mathscr{S}_N} M_\varphi^0(f)(x).$$

In this paper, we denote $C_k=B_k\setminus B_{k-1}$ and denote briefly the characteristic function $\chi_{(B_k\setminus B_{k-1})}$ by χ_k. The following definition is from [20].

DEFINITION 2.3. Let $0<q\leqslant\infty$, $0<\lambda\leqslant\infty$, $p(\cdot)\in\mathscr{P}$ and $\alpha(\cdot)\in L^\infty$. The *homogeneous variable anisotropic Herz-Morrey space* $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ and the *non-homogeneous variable anisotropic Herz-Morrey space* $MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$ are defined respectively by setting,

$$M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n):=\left\{f\in L_{\rm loc}^{p(\cdot)}:\|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)}<\infty\right\}$$

and

$$MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n):=\left\{f\in L_{\rm loc}^{p(\cdot)}:\|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)}<\infty\right\},$$

where

$$\|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)}:=\sup_{L\in\mathbb{Z}}2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\|b^{\alpha(\cdot)k}f\chi_k\|_{L^{p(\cdot)}}^q\right\}^{1/q}$$

and

$$\|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)}:=\sup_{L\in\mathbb{Z}}2^{-L\lambda}\left\{\sum_{k=0}^{L}\|b^{\alpha(\cdot)k}f\chi_k\|_{L^{p(\cdot)}}^q\right\}^{1/q}.$$

Here, there is the usual modification when $q=\infty$.

For $0<q<\infty$, we denote

$$N_q:=\begin{cases}[(1/q-1)\ln b/\ln\lambda_-]+2, & 0<q\leqslant 1,\\ 2, & q>1,\end{cases}$$

where λ_- is as in Page 2.

DEFINITION 2.4. Let $\alpha(\cdot) \in L^{\infty}$, $0 < \lambda \leqslant \infty$, $0 < q \leqslant \infty$, $p(\cdot) \in \mathscr{P}$ and $N > N_q$. The *homogeneous variable anisotropic Herz-Morrey-Hardy space* $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ and the *non-homogeneous variable anisotropic Herz-Morrey-Hardy space* $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ are defined respectively by setting,

$$HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n) := \left\{ f \in \mathscr{S}' : M_N(f) \in M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n) \right\}$$

and

$$HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n) := \left\{ f \in \mathscr{S}' : M_N(f) \in MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n) \right\},$$

where

$$\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)} = \|M_N(f)\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}$$

and

$$\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)} = \|M_N(f)\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}.$$

REMARK 2.5.

(i) When the exponent functions $p(\cdot)$ and $\alpha(\cdot)$ are constant exponents p and α, these spaces are still new.

(ii) When the exponent functions $\alpha(\cdot) := \alpha$, $\lambda := 0$ and $A := 2\mathrm{I}_{n\times n}$, these spaces are the Herz-type Hardy spaces with variable exponents $H\dot{K}^{\alpha,q}_{p(\cdot)}$ and $HK^{\alpha,q}_{p(\cdot)}$ (see [19]).

(iii) When $A := 2\mathrm{I}_{n\times n}$, these spaces are the Herz-Morrey-Hardy spaces with variable exponents $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ (see [21]).

LEMMA 2.6. [11] *Let $p(\cdot) \in \mathscr{B}$. Then there exist $0 < \delta_1, \delta_2 < 1$ depending only on $p(\cdot)$ and n such that for all $B, S \in \mathfrak{B}$ and $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1} \quad \text{and} \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

LEMMA 2.7. [13] *Let $q \in (0,\infty)$, $p(\cdot) \in \mathscr{P}$, $\lambda \in [0,\infty)$ and $\alpha(\cdot) \in L^{\infty} \cap \mathscr{P}^{\log}_0 \cap \mathscr{P}^{\log}_{\infty}$. If $\alpha(\cdot)$ is log-Hölder continuous both at origin and at infinity, then for any measurable function f,*

$$\|f\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \leqslant C \max\left\{ \sup_{L<0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \|f\chi_k\|^q_{L^{p(\cdot)}}, \right.$$
$$\left. \sup_{L\geqslant 0, L\in\mathbb{Z}} \left[2^{-L\lambda q} \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \|f\chi_k\|^q_{L^{p(\cdot)}} + 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_{\infty}} \|f\chi_k\|^q_{L^{p(\cdot)}} \right] \right\}.$$

LEMMA 2.8. [12] *Let $p(\cdot)\in\mathscr{P}$. If $f\in L^{p(\cdot)}$ and $g\in L^{p'(\cdot)}$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n}|f(x)g(x)|\,dx\leqslant C_p\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}},$$

where $C_p=1+1/p_- - 1/p_+$.

LEMMA 2.9. [11] *Let $p(\cdot)\in\mathscr{B}$. Then there exists a positive constant $C>0$ such that for all $B\in\mathfrak{B}$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}}\|\chi_B\|_{L^{p'(\cdot)}}\leqslant C.$$

3. Atomic decomposition of $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$

In this section, we establish atomic decompositions of the variable anisotropic Herz-Morrey-Hardy spaces $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$. We first begin with the following notions of anisotropic $(\alpha(\cdot),p(\cdot),s)$-atoms.

DEFINITION 3.1. Let $p(\cdot)\in\mathscr{P}$, $\alpha(\cdot)\in L^\infty\cap\mathscr{P}_0^{\log}\cap\mathscr{P}_\infty^{\log}$ and a non-negative integer s satisfy $s\in[(\alpha_r-\delta_2)\ln b/\ln\lambda_-,\infty)$ with δ_2 as in Lemma 2.6. Here $\alpha_r=\alpha(0)$, if $r<0$ and $\alpha_r=\alpha_\infty$, if $r>0$.

(1) An *anisotropic central $(\alpha(\cdot),p(\cdot),s)$-atom* is a measurable function a on $\mathbb{R}^n$ satisfying

  (i) (support) $\operatorname{supp} a\subset B_r$, where $B_r\in\mathfrak{B}$ and $\mathfrak{B}$ is as in (2.2);

  (ii) (size) $\|a\|_{L^{p(\cdot)}}\leqslant|B_r|^{-\alpha_r}$;

  (iii) (vanishing moment) $\int_{\mathbb{R}^n}a(x)x^\beta\,dx=0$ for any $\beta\in\mathbb{Z}_+^n$ with $|\beta|\leqslant s$.

(2) An *anisotropic central $(\alpha(\cdot),p(\cdot),s)$-atom of restricted type* is a measurable function a on $\mathbb{R}^n$ satisfying

  (i) $\operatorname{supp} a\subset B_r$, $r\geqslant 0$, where $B_r\in\mathfrak{B}$ and $\mathfrak{B}$ is as in (2.2);

  (ii) $\|a\|_{L^{p(\cdot)}}\leqslant|B_r|^{-\alpha_\infty}$;

  (iii) $\int_{\mathbb{R}^n}a(x)x^\beta\,dx=0$ for any $\beta\in\mathbb{Z}_+^n$ with $|\beta|\leqslant s$.

THEOREM 3.2. *Let $p(\cdot)\in\mathscr{B}$, $0<q<\infty$, $0\leqslant\lambda<\infty$, $\alpha(\cdot)\in L^\infty\cap\mathscr{P}_0^{log}\cap\mathscr{P}_\infty^{log}$, $\alpha(\cdot)\geqslant 2\lambda$ and $\delta_2\leqslant\alpha(0)$, $\alpha_\infty<\infty$, where δ_2 is as in Lemma 2.6.*

(i) *$f\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ if and only if*

$$f=\sum_{j\in\mathbb{Z}}\lambda_j a_j\ \text{ in }\ \mathscr{S}',$$

where each a_j is a central $(\alpha(\cdot), p(\cdot), s)$-atom with support contained in B_j and

$$\sup_{L\in\mathbb{Z}} b^{-L\lambda}\left(\sum_{j=-\infty}^{L}|\lambda_j|^q\right)^{1/q}<\infty.$$

Moreover,

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}\sim\inf\sup_{L\in\mathbb{Z}} b^{-L\lambda}\left(\sum_{j=-\infty}^{L}|\lambda_j|^q\right)^{1/q},$$

where the infimum is taken over all above decompositions of f.

(ii) $f\in HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ *if and only if*

$$f=\sum_{j\in\mathbb{Z}_+}\lambda_j a_j \text{ in } \mathscr{S}',$$

where each a_j is a central $(\alpha(\cdot), p(\cdot), s)$-atom of restricted type with support contained in B_j and

$$\sup_{L\in\mathbb{Z}_+} b^{-L\lambda}\left(\sum_{j=0}^{L}|\lambda_j|^q\right)^{1/q}<\infty.$$

Moreover,

$$\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}\sim\inf\sup_{L\in\mathbb{Z}_+} b^{-L\lambda}\left(\sum_{j=0}^{L}|\lambda_j|^q\right)^{1/q},$$

where the infimum is taken over all above decompositions of f.

To prove Theorem 3.2, we need the following technical lemmas.

LEMMA 3.3. *Let $p(\cdot)$, $\alpha(\cdot)$, s be as in Definition 3.1, $j\in\mathbb{N}$ and a_j be a central $(\alpha(\cdot), p(\cdot), s)$-atom with support contained in B_j. Then we have, for any $x\in C_k$ with $k\geqslant j+\sigma+1$, $k\in\mathbb{Z}$, and $\varphi\in\mathscr{S}_N$,*

$$M_N(a_j)(x)\lesssim b^{-j\alpha_j-j}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\left(b\lambda_-^{s+1}\right)^{-m}, \tag{3.1}$$

where $m=k-j-\sigma-1$.

Proof. For any $x\in C_k$, $\varphi\in\mathscr{S}_N$, $j,r\in\mathbb{Z}$ and a polynomial P_s of degree $\leqslant s$, by the vanishing moment of a_j, we have

$$\begin{aligned}|a_j*\varphi_r(x)|&=b^{-r}\left|\int_{\mathbb{R}^n}a_j(y)\varphi\left(A^{-r}(x-y)\right)dy\right|\\&=b^{-r}\left|\int_{B_j}a_j(y)\left[\varphi\left(A^{-r}(x-y)\right)-P_s\left(A^{-r}(x-y)\right)\right]dy\right|\\&\leqslant b^{-r}\int_{B_j}|a_j(y)|dy\sup_{y\in A^{-r}x+B_{j-r}}|\varphi(y)-P_s(y)|.\end{aligned}$$

Since $x \in C_k$ with $k \geqslant j+\sigma+1$, then $x \in B_{j+\sigma+m+1}/B_{j+\sigma+m}$, where $m = k-j-\sigma-1 \geqslant 0$. Therefore,

$$\begin{aligned} A^{-r}x + B_{j-r} &\subseteq A^{-r}\left(B_{j+\sigma+m+1}/B_{j+\sigma+m}\right) + B_{j-r} \\ &= A^{j-r}\left[(B_{\sigma+m+1}/B_{\sigma+m}) + B_0\right] \\ &\subseteq A^{j-r}(B_m)^{\complement} = (B_{m+j-r})^{\complement}. \end{aligned}$$

If $j \geqslant r$, then we choose $P_s \equiv 0$, and

$$\sup_{y \in A^{-r}x+B_{j-r}} |\varphi(y) - P_s(y)| \lesssim \sup_{y \in (B_{m+j-r})^{\complement}} \min\left(1, \rho(y)^{-N}\right) \lesssim b^{-N(m+j-r)}.$$

If $j < r$, then we choose P_s to be the Taylor expansion of φ at the point $A^{-r}x$ of order s. Therefore, by (2.5), we obtain

$$\begin{aligned} \sup_{y \in A^{-r}x+B_{j-r}} |\varphi(y) - P_s(y)| &\lesssim \sup_{z \in B_{j-r}} \sup_{\theta \in (0,1)} \sup_{|\alpha|=s+1} \left|\partial^{\alpha}\varphi\left(A^{-r}x + \theta z\right)\right| |z|^{s+1} \\ &\lesssim \lambda_-^{(s+1)(j-r)} \sup_{y \in A^{-r}x+B_{j-r}} \min\left(1, \rho(y)^{-N}\right) \\ &\lesssim \lambda_-^{(s+1)(j-r)} \min\left(1, b^{-N(m+j-r)}\right). \end{aligned}$$

Combining the above two estimates and [3, Proposition 3.10], for any $x \in B_{j+\sigma+m+1} \setminus B_{j+\sigma+m}$, we have

$$\begin{aligned} M_N(a_j)(x) &= \sup_{\varphi \in \mathscr{S}_N} \sup_{r \in \mathbb{Z}} |(a_j * \varphi_r)(x)| \\ &\lesssim b^{-j\alpha_j - j} \|\chi_{B_j}\|_{L^{p'(\cdot)}} \max\left[\sup_{r \in \mathbb{Z}, r \leqslant j} b^{(j-r)} b^{-N(m+j-r)},\right. \\ &\qquad \left. C \sup_{r \in \mathbb{Z}, r > j} b^{(j-r)} \lambda_-^{(s+1)(j-r)} \min\left(1, b^{-N(m+j-r)}\right)\right]. \end{aligned}$$

We find that, when $r = j$, the supremum over $r \leqslant j$ is attained, when $j-r+m=0$, the supremum over $r > j$ is attained. Since $b\lambda_-^{s+1} \leqslant b^N$ with $N \geqslant s+2$, it suffices to check the maximum value for $j < r \leqslant j+m$ and $j \geqslant r+m$. For any $x \in B_{j+\sigma+m+1}/B_{j+\sigma+m}$ with $m \geqslant 0$, we have

$$\begin{aligned} M_N(a_j) &\lesssim b^{-j\alpha_j - j} \|\chi_{B_j}\|_{L^{p'(\cdot)}} \max\left[b^{-Nm}, C\left(b\lambda_-^{s+1}\right)^{-m}\right] \\ &\lesssim b^{-j\alpha_j - j} \|\chi_{B_j}\|_{L^{p'(\cdot)}} \left(b\lambda_-^{s+1}\right)^{-m}. \quad \square \end{aligned}$$

Proof of Theorem 3.2. We only need to prove (i). (ii) can be proved in the similar way. The proof is divided into 2 steps.

Step 1. In this step, we show the sufficiency of Theorem 3.2. We assume that $f = \sum_{j \in \mathbb{Z}} \lambda_j a_j$ in $\mathscr{S}'$, where each a_j is a central $(\alpha(\cdot), p(\cdot), s)$-atom with support contained in B_j and

$$\sup_{L\in\mathbb{Z}} b^{-L\lambda}\left(\sum_{j=-\infty}^{L}|\lambda_j|^q\right)^{1/q}<\infty.$$

By Lemma 2.7, we have

$$\begin{aligned}
&\|M_N(f)\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\\
&\leqslant C\max\Bigg\{\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\|M_N(f)\chi_k\|^q_{L^{p(\cdot)}},\\
&\sup_{L\in\mathbb{Z}_+}\left[b^{-L\lambda q}\sum_{k=-\infty}^{-1} b^{kq\alpha(0)}\|M_N(f)\chi_k\|^q_{L^{p(\cdot)}}+b^{-L\lambda q}\sum_{k=0}^{L} b^{kq\alpha_\infty}\|M_N(f)\chi_k\|^q_{L^{p(\cdot)}}\right]\Bigg\}\\
&=: C\max\{\mathrm{I},\mathrm{J}+\mathrm{K}\}.
\end{aligned}$$

For I, J and K, by the boundedness of M_N on $L^{p(\cdot)}$ and $f=\sum_{j\in\mathbb{Z}}\lambda_j a_j$ in $\mathscr{S}'$, we obtain

$$\begin{aligned}
\mathrm{I}&\leqslant C\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\left(\sum_{j=k-\sigma}^{\infty}|\lambda_j|\left\|a_j\right\|_{L^{p(\cdot)}}\right)^q\\
&\quad+C\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\left(\sum_{j=-\infty}^{k-\sigma-1}|\lambda_j|\left\|M_N(a_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&=: \mathrm{I}_1+\mathrm{I}_2,
\end{aligned}$$

$$\begin{aligned}
\mathrm{J}&\leqslant C\sup_{L\geqslant 0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{-1} b^{kq\alpha(0)}\left(\sum_{j=k-\sigma}^{\infty}|\lambda_j|\left\|a_j\right\|_{L^{p(\cdot)}}\right)^q\\
&\quad+C\sup_{L\geqslant 0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{-1} b^{kq\alpha(0)}\left(\sum_{j=-\infty}^{k-\sigma-1}|\lambda_j|\left\|M_N(a_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&=: \mathrm{J}_1+\mathrm{J}_2
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{K}&\leqslant C\sup_{L\geqslant 0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=0}^{L} b^{kq\alpha_\infty}\left(\sum_{j=k-\sigma}^{+\infty}|\lambda_j|\left\|a_j\right\|_{L^{p(\cdot)}}\right)^q\\
&\quad+C\sup_{L\geqslant 0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=0}^{L} b^{kq\alpha_\infty}\left(\sum_{j=-\infty}^{k-\sigma-1}|\lambda_j|\left\|M_N(a_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&=: \mathrm{K}_1+\mathrm{K}_2.
\end{aligned}$$

To deal with I, J and K, we consider two cases: $0<q\leqslant 1$ and $1<q<\infty$.

Case 1. When $0<q\leqslant 1$, by the size condition of a_j and the fact that $\alpha_j=\alpha(0)$, if $j<0$ and $\alpha_j=\alpha_\infty$, if $j>0$, we have

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\left(\sum_{j=k-\sigma}^{\infty}|\lambda_j|b^{-j\alpha_j}\right)^q\\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\left(\sum_{j=k-\sigma}^{-1}|\lambda_j|^q b^{-jq\alpha(0)}+\sum_{j=0}^{\infty}|\lambda_j|^q b^{-jq\alpha_\infty}\right)\\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=k-\sigma}^{-1}|\lambda_j|^q b^{(k-j)q\alpha(0)}\\
&\quad+\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{j=0}^{\infty}|\lambda_j|^q b^{kq\alpha(0)}b^{-jq\alpha_\infty}\\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=-\infty}^{-1}|\lambda_j|^q\sum_{k=-\infty}^{j+\sigma} b^{(k-j)q\alpha(0)}\\
&\quad+\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=0}^{\infty}|\lambda_j|^q\sum_{k=-\infty}^{L} b^{kq\alpha(0)}b^{-jq\alpha_\infty}.
\end{aligned}$$

From

$$\sum_{k=-\infty}^{j+\sigma} b^{(k-j)q\alpha(0)}\sim 1,$$

we further deduce that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q+\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=L}^{-1}|\lambda_j|^q\sum_{k=-\infty}^{j+\sigma} b^{(k-j)q\alpha(0)}\\
&\quad+\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=0}^{\infty}|\lambda_j|^q\sum_{k=-\infty}^{L} b^{kq\alpha(0)}b^{-jq\alpha_\infty}\\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q+\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=L}^{-1}|\lambda_j|^q\sum_{k=-\infty}^{j+\sigma} b^{(k-j)q\alpha(0)}\\
&\quad+\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q\sup_{L<0,L\in\mathbb{Z}}\sum_{j=0}^{\infty} b^{(\lambda-\alpha_\infty)jq}\sum_{k=-\infty}^{L} b^{kq\alpha(0)-L\lambda q}\\
&\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q}\sum_{j=-\infty}^{L}|\lambda_j|^q.
\end{aligned}$$

For any $j<0$, using the same estimate of (3.1), we have

$$\|M_N(a_j)\chi_k\|^q_{L^{p(\cdot)}}\lesssim b^{-jq\alpha(0)-jq}(b\lambda_-^{s+1})^{(j+\sigma+1-k)q}\|\chi_{B_j}\|^q_{L^{p'(\cdot)}}\|\chi_{B_k}\|^q_{L^{p(\cdot)}}. \tag{3.2}$$

From this, Lemmas 2.9 and 2.6 and the fact that $\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2} < 1$, we conclude that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} b^{kq\alpha(0)} \sum_{j=-\infty}^{k-\sigma-1} |\lambda_j|^q b^{-jq\alpha(0)-jq} \left(b\lambda_-^{s+1}\right)^{(j+\sigma+1-k)q} \\
&\quad \times \|\chi_{B_j}\|_{L^{p'(\cdot)}}^q \|\chi_{B_k}\|_{L^{p(\cdot)}}^q \\
&\lesssim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=-\infty}^{k-\sigma-1} |\lambda_j|^q \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2}\right)^{(k-j)q} \\
&\lesssim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L-\sigma-1} \sum_{k=j+\sigma+1}^{L} |\lambda_j|^q \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2}\right)^{(k-j)q} \\
&\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q.
\end{aligned}$$

By the size condition of a_j and the fact that $\alpha_j = \alpha(0)$, if $j<0$ and $\alpha_j = \alpha_\infty$, if $j>0$, we obtain that

$$\begin{aligned}
\mathrm{J}_1 &\sim \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{-1} b^{kq\alpha(0)} \left(\sum_{j=k-\sigma}^{\infty} |\lambda_j| \|a_j\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{-1} b^{kq\alpha(0)} \left(\sum_{j=k-\sigma}^{-1} |\lambda_j|^q b^{-jq\alpha(0)} + \sum_{j=0}^{\infty} |\lambda_j|^q b^{-jq\alpha_\infty} \right) \\
&\sim \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \left[\sum_{k=-\infty}^{-1} \sum_{j=k-\sigma}^{-1} b^{(k-j)q\alpha(0)} |\lambda_j|^q + \sum_{k=-\infty}^{-1} \sum_{j=0}^{\infty} b^{kq\alpha(0)} b^{-jq\alpha_\infty} |\lambda_j|^q \right] \\
&\lesssim \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j+\sigma} b^{(k-j)q\alpha(0)} + \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=0}^{\infty} |\lambda_j|^q \sum_{k=-\infty}^{-1} b^{kq\alpha(0)} \\
&\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q.
\end{aligned}$$

From (3.2), Lemmas 2.9 and 2.6, we obtain

$$\begin{aligned}
\mathrm{J}_2 &\lesssim \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{-1} b^{kq\alpha(0)} \sum_{j=-\infty}^{k-\sigma-1} |\lambda_j|^q b^{-jq\alpha(0)-jq} \left(b\lambda_-^{s+1}\right)^{(j+\sigma+1-k)q} \\
&\quad \times \|\chi_{B_j}\|_{L^{p'(\cdot)}}^q \|\chi_{B_k}\|_{L^{p(\cdot)}}^q \\
&\lesssim \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{-1} \sum_{j=-\infty}^{k-\sigma-1} |\lambda_j|^q \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2}\right)^{(k-j)q} \\
&\lesssim \sup_{L\geqslant 0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{-\sigma-2} |\lambda_j|^q \sum_{k=j+\sigma+1}^{-1} \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2}\right)^{(k-j)q} \\
&\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q.
\end{aligned}$$

By a similar method of J_1 and J_2, respectively, we can obtain

$$\mathrm{K}_1 \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q \text{ and } \mathrm{K}_2 \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q.$$

Case 2. When $1 < q \leqslant \infty$, by the size condition of a_j and the fact that $\alpha_j = \alpha(0)$, if $j < 0$ and $\alpha_j = \alpha_\infty$, if $j > 0$, the Hölder inequality, we have

$$\begin{aligned}
\mathrm{I}_1 &\sim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} b^{kq\alpha(0)} \left(\sum_{j=k-\sigma}^{\infty} |\lambda_j| \, \|a_j\|_{L^{p(\cdot)}} \right)^q \\
&\sim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} b^{kq\alpha(0)} \left(\sum_{j=k-\sigma}^{-1} |\lambda_j| b^{-j\alpha(0)} + \sum_{j=0}^{\infty} |\lambda_j| b^{-j\alpha_\infty} \right)^q \\
&\lesssim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=k-\sigma}^{-1} |\lambda_j|^q b^{(k-j)\alpha(0)q/2} \right) \times \left(\sum_{j=k-\sigma}^{-1} b^{(k-j)\alpha(0)q'/2} \right)^{q/q'} \\
&\quad + \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} b^{kq\alpha(0)} \left(\sum_{j=0}^{\infty} |\lambda_j|^q b^{-j\alpha_\infty q/2} \right) \times \left(\sum_{j=0}^{\infty} b^{-j\alpha_\infty q'/2} \right)^{q/q'} \\
&\lesssim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=k-\sigma}^{-1} |\lambda_j|^q b^{(k-j)\alpha(0)q/2} \\
&\quad + \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=0}^{\infty} |\lambda_j|^q b^{-j\alpha_\infty q/2} b^{kq\alpha(0)} \\
&\lesssim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{-1} |\lambda_j|^q \sum_{k=-\infty}^{j+\sigma} b^{(k-j)\alpha(0)q/2} \\
&\quad + \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=0}^{\infty} b^{-j\lambda q} |\lambda_j|^q b^{(\lambda-\alpha_\infty/2)jq} \sum_{k=-\infty}^{L} b^{kq\alpha(0)} \\
&\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q.
\end{aligned}$$

From (3.2) and the Hölder inequality, we conclude that

$$\begin{aligned}
\mathrm{I}_2 &\sim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} b^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-\sigma-1} |\lambda_j| \|M_N a_j \chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L<0, L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} \left[\sum_{j=-\infty}^{k-\sigma-1} |\lambda_j| \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2} \right)^{(k-j)} \right]^q
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} \left[\sum_{j=-\infty}^{k-\sigma-1} |\lambda_j|^q \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2} \right)^{(k-j)q/2} \right] \\
&\quad \times \left(\sum_{j=-\infty}^{k-\sigma-1} \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2} \right)^{(k-j)q'/2} \right)^{q/q'} \\
&\lesssim \sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L-\sigma-1} |\lambda_j|^q \sum_{k=j+\sigma+1}^{L} \left(\lambda_-^{-(s+1)} b^{\alpha(0)-\delta_2} \right)^{(k-j)q/2} \\
&\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q.
\end{aligned}$$

From (3.1) and a similar proof of I_1 and I_2, we deduce that

$$\mathrm{J}_1 \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q \quad \mathrm{J}_2 \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q$$

and

$$\mathrm{K}_1 \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q \quad \mathrm{K}_2 \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q.$$

This establishes the estimate we wanted.

Step 2. In this step, we prove the necessity of Theorem 3.2. Choosing $\phi \in \mathscr{S}$ such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. For any $f \in HMK_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)$, set $f^{(k)} := f * \phi_k$, where $\phi_k(\cdot) := b^{-k}\phi\left(A^{-k}\cdot\right)$. From [3, Lemma 3.8], we obtain that $f^{(k)} \to f$ in $\mathscr{S}'$. Now we divide Step 2 into two substeps.

Substep 1. We show that, for any $x \in \mathbb{R}^n$,

$$f^{(i)}(x) = \sum_{j\in\mathbb{Z}} \lambda_j a_j^{(i)}(x),$$

where $a_j^{(i)}$ is a $(\alpha(\cdot), p(\cdot), s)$-atom with $\operatorname{supp} a_j^{(i)} \subset B_{k+2}$, λ_j is independent of i and

$$\sup_{L\in\mathbb{Z}} b^{-L\lambda} \left(\sum_{j=-\infty}^{L} |\lambda_j|^q \right)^{1/q} \lesssim \|M_N f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}(A;\mathbb{R}^n)}.$$

Let $\psi \in C_0^\infty$ such that $0 \leqslant \psi \leqslant 1$, $\operatorname{supp}\psi \subset C_0' := C_{-1} \cup C_0 \cup C_1$ and $\psi(x) = 1$ if $x \in C_0$. Let $\psi_{(k)}(\cdot) = \psi(A^{-k}\cdot)$ for $k \in \mathbb{Z}$. Then we observe that

$$\operatorname{supp}\psi_{(k)} \subset C_k' := C_{k-1} \cup C_k \cup C_{k+1}.$$

Let

$$\Phi_k(x) := \begin{cases} \frac{\psi_{(k)}(x)}{\sum_{j\in\mathbb{Z}} \psi_{(j)}(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{3.3}$$

Then we obtain, for any $x \neq 0$

$$\Phi_k \in C_0^\infty,\ \mathrm{supp}\Phi_k \subset C_k',\ 0 \leqslant \Phi_k(x) \leqslant 1 \text{ and } \sum_{k\in\mathbb{Z}} \Phi_k(x) = 1.$$

Let $\nu_k(x) = |C_k'|^{-1}\chi_{C_k'}(x)$. Then we have

$$\begin{aligned} f^{(i)}(x) &= f^{(i)}(x) \sum_{k\in\mathbb{Z}} \Phi_k(x) \\ &= \sum_{k\in\mathbb{Z}} \left[f^{(i)}(x)\Phi_k(x) - \left(\int_{\mathbb{R}^n} f^{(i)}(y)\Phi_k(y)\,dy \right) \nu_k(x) \right] \\ &\quad + \sum_{k\in\mathbb{Z}} \left(\int_{\mathbb{R}^n} f^{(i)}(y)\Phi_k(y)\,dy \right) \nu_k(x) \\ &=: \mathtt{I}_1^{(i)} + \mathtt{I}_2^{(i)}. \end{aligned}$$

Let us deal with $\mathtt{I}_1^{(i)}$. Let

$$g_k^{(i)}(x) := f^{(i)}(x)\Phi_k(x) - \left(\int_{\mathbb{R}^n} f^{(i)}(y)\Phi_k(y)\,dy \right) \nu_k(x)$$

and

$$a_{1,k}^{(i)}(x) = \frac{g_k^{(i)}(x)}{\lambda_{1,k}},\ \lambda_{1,k} = C_1 b^{\alpha_{k+1}(k+1)} \sum_{j=k-1}^{k+1} \|M_N f \chi_j\|_{L^{p(\cdot)}},$$

where C_1 is a constant which will be chosen later. Then we know that

$$\mathrm{supp} a_{1,k}^{(i)} \subset B_{k+1},\ \int_{\mathbb{R}^n} a_{1,k}^{(i)}(x)\,dx = 0.$$

Moreover,

$$\mathtt{I}_1^{(i)} = \sum_{k\in\mathbb{Z}} \lambda_{1,k} a_{1,k}^{(i)}(x).$$

From the Hölder inequality, we conclude that

$$\|g_k^{(i)}\|_{L^{p(\cdot)}} \lesssim \|f^{(i)}\Phi_k\|_{L^{p(\cdot)}} \leqslant C_2 \sum_{j=k-1}^{j=k+1} \|M_N f\|_{L^{p(\cdot)}}.$$

Choose $C_1 = C_2$; then we obtain that

$$\|a_{1,k}^{(i)}\|_{L^{p(\cdot)}} \leqslant |B_{k+1}|^{-\alpha_{k+1}}$$

and $a_{1,k}^{(i)}$ is a $(\alpha(\cdot), p(\cdot), s)$-atom with $\mathrm{supp} a_{1,k}^{(i)} \subset B_{k+1}$. Therefore,

$$\begin{aligned} \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_k|^q &\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} |B_{k+1}|^{q\alpha_{k+1}} \left(\sum_{j=k-1}^{j=k+1} \|M_N f \chi_j\|_{L^{p(\cdot)}} \right)^q \\ &\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} |B_{k+1}|^{q\alpha_{k+1}} \|M_N f \chi_j\|_{L^{p(\cdot)}}^q. \end{aligned}$$

If $L \leqslant 0$, then

$$\sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_k|^q \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} b^{kq\alpha(0)} \|M_N f \chi_j\|_{L^{p(\cdot)}}^q$$
$$\lesssim \|M_N f\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

If $L > 0$, then

$$\sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_k|^q \lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{-2} b^{(k+1)q\alpha(0)} \|M_N f \chi_j\|_{L^{p(\cdot)}}^q$$
$$+ \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-1}^{L} b^{(k+1)q\alpha_\infty} \|M_N f \chi_j\|_{L^{p(\cdot)}}^q$$
$$\lesssim \|M_N f\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}.$$

Next we deal with $\mathtt{I}_2^{(i)}$,

$$\mathtt{I}_2^{(i)} = \sum_{k\in\mathbb{Z}} \left(\sum_{j=-\infty}^{k} \int_{\mathbb{R}^n} f^{(i)}(y)\Phi_j(y)\,dy \right) (\nu_k(x) - \nu_{k+1}(x))$$
$$=: \sum_{k\in\mathbb{Z}} h_k^{(i)}(x).$$

Let $a_{2,k}^{(i)} = h_k^{(i)}/\lambda_{2,k}$, where $\lambda_{2,k} = C_3 b^{(k+2)\alpha_{k+2}} \sum_{j=k-1}^{k+2} \|M_N f \chi_j\|_{L^{p(\cdot)}}$, C_3 is a constant to be determined later. Then we have

$$\mathrm{supp} a_{2,k}^{(i)} \subset B_{k+2}, \ \int_{\mathbb{R}^n} a_{2,k}^{(i)}(x)\,dx = 0.$$

Moreover,

$$\mathtt{I}_2^{(i)} = \sum_{k\in\mathbb{Z}} \lambda_{2,k} a_{2,k}^{(i)}(x).$$

Denote $\varphi(x) := \Sigma_{j=-\infty}^{-2}\Phi_j(x)$, where Φ_j is as in (3.3). From $\mathrm{supp}\Phi_j \subset C_j'$ and $\{C_j'\}_{j=-\infty}^{-2}$ has bounded overlapping, i.e., $\sum_{j=-\infty}^{-2} \chi_{C_j'} \leqslant C$, we know that $\varphi \in C_0^\infty$ and $\varphi \in \mathscr{S}$. Notice that

$$\sum_{j=-\infty}^{k} \Phi_j(x) = \varphi(A^{-k-2}x) = b^{k+2}\varphi_{k+2}(x),$$

where φ_{k+2} is as in (2.9). By [3, Lemma 6.6], we conclude that, for any $x \in B_{k+2}$,

$$\left| \sum_{j=-\infty}^{k} \int_{\mathbb{R}^n} f^{(i)}(y)\Phi_j(y)\,dy \right| = b^{k+2} \left| \int_{B_{k+2}} f^{(i)}(y)\Phi_j(y)\,dy \right|$$
$$\leqslant b^{k+2} \|\widetilde{\varphi}\|_{S_{N+2}} M_{N+2}(f^{(i)})(x)$$
$$\leqslant C b^{k+2} M_N f(x),$$

where $\widetilde{\varphi}(y) = \varphi(-y)$ and C is a constant dependent of N.

It is obvious that, for any $x \in \mathbb{R}^n$

$$|\nu_k(x) - \nu_{k+1}(x)| \lesssim b^{-k-2} \sum_{j=k-1}^{k+2} \chi_j(x).$$

Thus we obtain

$$\|h_k^{(i)}\|_{L^{p(\cdot)}} \leqslant C_4 \sum_{j=k-1}^{k+2} \|M_N f \chi_j\|_{L^{p(\cdot)}}.$$

Choose $C_3 = C_4$; we know that $a_{2,k}^{(i)}$ is a $(\alpha(\cdot), p(\cdot), s)$-atom with $\operatorname{supp} a_{2,k}^{(i)} \subset B_{k+2}$. Moreover,

$$\begin{aligned}\sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} |\lambda_{2,k}|^q &\lesssim \sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{k=-\infty}^{L} |B_{k+2}|^{q\alpha_{k+1}} \left(\sum_{j=k-1}^{j=k+1} \|M_N f \chi_j\|_{L^{p(\cdot)}}\right)^q \\ &\lesssim \|M_N f\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}.\end{aligned}$$

From this, we further conclude that, for any $x \in \mathbb{R}^n$

$$f^{(i)}(x) = \sum_{j\in\mathbb{Z}} \lambda_j a_j^{(i)}(x),$$

where $a_j^{(i)}$ is a $(\alpha(\cdot), p(\cdot), s)$-atom with $\operatorname{supp} a_j^{(i)} \subset B_{k+2}$, λ_j is independent of i and

$$\sup_{L\in\mathbb{Z}} b^{-L\lambda} \left(\sum_{j=-\infty}^{L} |\lambda_j|^q\right)^{1/q} \lesssim \|M_N f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}.$$

Notice that

$$\sup_{i\in\mathbb{N}} \|a_0^{(i)}\|_{L^{p(\cdot)}} \leqslant |B_2|^{-\alpha_2}.$$

Combining the Banach-Alaoglu theorem, we obtain a subsequence $\{a_0^{(i_{n_0})}\}$ of $\{a_0^{(i)}\}$ converging in the w* topology of $L^{p(\cdot)}$ to some $a_0 \in L^{p(\cdot)}$. It is obvious that a_0 is a central $(\alpha(\cdot), p(\cdot), s)$-atom with $\operatorname{supp} a_0 \subset B_2$. Next, since

$$\sup_{i_{n_0}\in\mathbb{N}} \|a_0^{(i_{n_0})}\|_{L^{p(\cdot)}} \leqslant |B_3|^{-\alpha_3},$$

applying Banach-Alaoglu theorem, we obtain that there exists a subsequent $\{a_1^{(i_{n_1})}\}$ of $\{a_1^{(i_{n_0})}\}$ converging in the w* topology of $L^{p(\cdot)}$ to a central $(\alpha(\cdot), p(\cdot), s)$-atom a_1 with $\operatorname{supp} a_1 \subset B_3$. Repeating the above procedure for any $j \in \mathbb{Z}$, we can find a subsequence $\{a_j^{(i_{n_j})}\}$ of $\{a_j^{(i)}\}$ converging in the w* topology of $L^{p(\cdot)}$ to a central

$(\alpha(\cdot), p(\cdot), s)$-atom a_j with $\mathrm{supp}\, a_j \subset B_{j+2}$. By usual diagonal method we get a subsequence $\{i_\nu\}$ of $\mathbb{N}$ such that for any $j \in \mathbb{N}$, $\lim_{\nu\to\infty} a_j^{(i_\nu)} = a_j$ in the w* topology of $L^{p(\cdot)}$ and therefore in $\mathscr{S}'$.

Substep 2. In this substep, we prove

$$f = \sum_{j\in\mathbb{Z}} \lambda_j a_j \text{ in } \mathscr{S}'. \tag{3.4}$$

For any $\phi \in \mathscr{S}$, observe that

$$\mathrm{supp}\, a_j^{(i_\nu)} \subset C_{j-1} \cup C_j \cup C_{j+1} \cup C_{j+2}.$$

From this, we have

$$\langle f, \phi\rangle = \lim_{\nu\to\infty} \sum_{j\in\mathbb{Z}} \lambda_j \int_{\mathbb{R}^n} a_j^{(i_\nu)}(x)\phi(x)\,dx.$$

If $j+1 \leqslant 0$, then, by Lemma 2.8, the size condition of $a_j^{(i_\nu)}$, Lemmas 2.9 and 2.6, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} a_j^{(i_\nu)}(x)\phi(x)\,dx\right| &= \left|\int_{\mathbb{R}^n} a_j^{(i_\nu)}(x)(\phi(x)-\phi(0))\,dx\right| \\
&\lesssim \sup_{y\in B_{j+2}} \sup_{|\beta|=1} |\partial^\beta \phi(y)| \int_{B_{j+2}} \left|a_j^{(i_\nu)}(x)\right| |x|\,dx \\
&\lesssim b^{(j+1)\ln\lambda_-/\ln b} \int_{B_{j+2}} \left|a_j^{(i_\nu)}(x)\right| dx \\
&\lesssim b^{(j+1)\ln\lambda_-/\ln b} \left\|a_j^{(i_\nu)}\right\|_{L^{p(\cdot)}} \|\chi_{B_{j+2}}\|_{L^{p'(\cdot)}} \\
&\lesssim b^{(j+1)(\ln\lambda_-/\ln b-\alpha_{j+2})} \left(\frac{|B_{j+2}|}{|B_2|}\right)^{\delta_2} \|\chi_{B_2}\|_{L^{p'(\cdot)}} \\
&\lesssim b^{(j+1)(\ln\lambda_-/\ln b+\delta_2-\alpha_{j+2})} \frac{|B_2|}{|B_0|} \|\chi_{B_0}\|_{L^{p'(\cdot)}} \\
&\lesssim b^{(j+1)(\ln\lambda_-/\ln b+\delta_2-\alpha_{j+2})} \inf\left\{\gamma>0 : \int_{B_0} \gamma^{-p'(x)} \leqslant 1\right\} \\
&\lesssim b^{(j+1)(\ln\lambda_-/\ln b+\delta_2-\alpha_{j+2})} \inf\left\{0<\gamma\leqslant 1 : \int_{B_0} \gamma^{-p'_+} \leqslant 1\right\} \\
&\lesssim b^{(j+1)(\ln\lambda_-/\ln b+\delta_2-\alpha_{j+2})}.
\end{aligned}$$

If $j+1>0$, choose $k_0 \in \mathbb{Z}_+$ such that $\min\{k_0+\alpha_0-1, k_0+\alpha_\infty-1\}>0$, then by a similar proof of the above, we get

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} a_j^{(i_\nu)}(x)\phi(x)\,dx\right| &\lesssim \int_{\mathbb{R}^n} \left|a_j^{(i_\nu)}(x)\right| (\rho(x))^{-k_0}\,dx \\
&\lesssim b^{-jk_0} \left\|a_j^{(i_\nu)}\right\|_{L^{p(\cdot)}} \|\chi_{B_{j+2}}\|_{L^{p'(\cdot)}}
\end{aligned}$$

$$\lesssim b^{-j(k_0+\alpha_{j+2})} \left\| \chi_{B_{j+2}} \right\|_{L^{p'(\cdot)}}$$
$$\lesssim b^{-j(k_0+\alpha_{j+2}-1)}.$$

Let

$$\mu_j := \begin{cases} \left|\lambda_j\right| b^{(j+1)(\ln\lambda_-/\ln b+\delta_2-\alpha_{j+2})}, & j+1 \leqslant 0, \\ |\lambda_j| b^{-j(k_0+\alpha_{j+2}-1)}, & j+1 > 0. \end{cases}$$

By the Hölder inequality, we obtain

$$\sup_{L\in\mathbb{Z}} b^{-L\lambda} \sum_{j=-\infty}^{L} |\mu_j| \lesssim \left(\sup_{L\in\mathbb{Z}} b^{-L\lambda q} \sum_{j=-\infty}^{L} |\lambda_j|^q \right)^{1/q} \lesssim \|M_N f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}$$

and

$$|\lambda_j| \left| \int_{\mathbb{R}^n} a_j^{(i_\nu)}(x)\phi(x)\,dx \right| \lesssim |\mu_j|.$$

From the dominated convergence theorem, we further conclude that

$$\langle f, \phi \rangle = \sum_{j\in\mathbb{Z}} \lim_{\nu\to\infty} \lambda_j \int_{\mathbb{R}^n} a_j^{(i_\nu)}(x)\phi(x)\,dx = \sum_{j\in\mathbb{Z}} \lambda_j \int_{\mathbb{R}^n} a_j(x)\phi(x)\,dx,$$

which implies that (3.4) holds true. This finishes the proof of Theorem 3.2. □

4. Applications

In this section, as an application of the atomic characterization of $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ in Theorem 3.2, we obtain the boundedness of some sublinear operators from $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ to $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ and from $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ to $MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$.

DEFINITION 4.1. For $s \in \mathbb{Z}_+$, let $\mathbf{D}(\mathbb{R}^n)$ be the space of infinitely differentiable complex-valued functions with compact supported in $\mathbb{R}^n$.

$$\mathbf{D}_s(\mathbb{R}^n) = \left\{ f \in \mathbf{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)x^\beta\,dx = 0, \text{ for all } |\beta| \leqslant s \right\}$$

and

$$\dot{\mathbf{D}}_s(\mathbb{R}^n) = \{ f \in \mathbf{D}_s(\mathbb{R}^n), 0 \notin \operatorname{supp} f \}.$$

The following lemma is very important in this section. Its proof is similar to [22, Lemma 3.2]. The concrete details are omitted.

LEMMA 4.2. *Let* $p(\cdot) \in \mathscr{B}$, $0 < q < \infty$, $\alpha(\cdot) \in L^\infty \cap \mathscr{P}_0^{log} \cap \mathscr{P}_\infty^{log}$ *such that* $\max\{n\delta_1, n\delta_2\} \leqslant \alpha(0)$, $\alpha_\infty < \infty$, *where* δ_1 *and* δ_2 *are as in Lemma 2.6.* $0 \leqslant \lambda \leqslant 1/2\min\{\alpha(0), \alpha_\infty\}$. *Let* s *be a non-negative integer such that* $s \geqslant [\max\{\alpha(0), \alpha_\infty\} - \min\{n\delta_1, n\delta_2\}]$. *Then*

(i) $\dot{\mathbf{D}}_s(\mathbb{R}^n)$ *is dense in* $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$*;*

(ii) $\mathbf{D}_s(\mathbb{R}^n)$ *is dense in* $HK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$.

THEOREM 4.3. *Let* $p(\cdot)\in\mathscr{P}$, $0<q<\infty$, $0\leqslant\lambda<\infty$, $\alpha(\cdot)\in L^\infty\cap\mathscr{P}^{log}_0\cap\mathscr{P}^{log}_\infty$, $\alpha(\cdot)\geqslant 2\lambda$ *and* $\delta_2\leqslant\alpha(0)$, $\alpha_\infty<\delta_2+\ln\lambda_-/\ln b$, *where* δ_2 *is as in Lemma 2.6. If a sublinear operator* T *satisfies that*

(i) T *is bounded on* $L^{p(\cdot)}$*;*

(ii) *For any* $f\in L^{p(\cdot)}$ *with* $\operatorname{supp} f\subset B_j$ *and*

$$\int_{B_j} f(x)\,dx=0,$$

$T(f)$ satisfies the size condition

$$|T(f)(x)|\lesssim\frac{b^k\|f\|_{L^1}}{(\rho(x))^2},\ \text{if}\ \inf_{y\in B_j}\rho(x-y)\geqslant b^{-\sigma}\left(1-\frac{1}{b}\right)\rho(x).$$

Then there exists a positive constant C *independent of* f *such that, for any* $f\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ *and* $f\in HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$*, respectively,*

$$\|T(f)\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}\leqslant C\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}$$

and

$$\|T(f)\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}\leqslant C\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}.$$

Proof of Theorem 4.3. We only need to prove the homogeneous case. The non-homogeneous case can be proved in the similar way. Let $f\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$. From Theorem 3.2, we know that there exist $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ and a sequence of central $(\alpha(\cdot),p(\cdot),s)$-atoms, $\{a_j\}_{j\in\mathbb{Z}}$, supported, respectively, on $\{B_j\}_{j\in\mathbb{Z}}\subset\mathfrak{B}$ such that

$$f=\sum_{j\in\mathbb{Z}}\lambda_j a_j\ \text{ in }\ \mathscr{S}'$$

and

$$\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)}\sim\inf\sup_{L\in\mathbb{Z}}b^{-L\lambda}\left(\sum_{j=-\infty}^{L}|\lambda_j|^q\right)^{1/q}, \tag{4.1}$$

where the infimum is taken over all the decompositions of f as above.

By Lemma 2.7, we obtain

$$
\begin{aligned}
&\|T(f)\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}\\
&\leqslant C\max\Big\{\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\|T(f)\chi_k\|^q_{L^{p(\cdot)}},\\
&\sup_{L\geqslant 0,L\in\mathbb{Z}}\Big[b^{-L\lambda q}\sum_{k=-\infty}^{-1} b^{kq\alpha(0)}\|T(f)\chi_k\|^q_{L^{p(\cdot)}}+b^{-L\lambda q}\sum_{k=0}^{L} b^{kq\alpha_\infty}\|T(f)\chi_k\|^q_{L^{p(\cdot)}}\Big]\Big\}\\
&=: C\max\{\mathrm{I}',\mathrm{J}'+\mathrm{K}'\}.
\end{aligned}
$$

For I', J' and K', by the boundedness of T on $L^{p(\cdot)}$, we have

$$
\begin{aligned}
\mathrm{I}' &\leqslant C\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\Big(\sum_{j=k-\sigma}^{\infty}|\lambda_j|\,\|a_j\|_{L^{p(\cdot)}}\Big)^q\\
&\quad + C\sup_{L<0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=-\infty}^{L} b^{kq\alpha(0)}\Big(\sum_{j=-\infty}^{k-\sigma-1}|\lambda_j|\,\|T(a_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&=: \mathrm{I}'_1+\mathrm{I}'_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{J}' &\leqslant C\sum_{k=-\infty}^{-1} b^{kq\alpha(0)}\Big(\sum_{j=k-\sigma}^{\infty}|\lambda_j|\,\|a_j\|_{L^{p(\cdot)}}\Big)^q\\
&\quad + C\sum_{k=-\infty}^{-1} b^{kq\alpha(0)}\Big(\sum_{j=-\infty}^{k-\sigma-1}|\lambda_j|\,\|T(a_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&=: \mathrm{J}'_1+\mathrm{J}'_2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{K}' &\leqslant C\sup_{L\geqslant 0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=0}^{L} b^{kq\alpha(0)}\Big(\sum_{j=k-\sigma}^{L}|\lambda_j|\,\|a_j\|_{L^{p(\cdot)}}\Big)^q\\
&\quad + C\sup_{L\geqslant 0,L\in\mathbb{Z}} b^{-L\lambda q}\sum_{k=0}^{L} b^{kq\alpha(0)}\Big(\sum_{j=-\infty}^{k-\sigma-1}|\lambda_j|\,\|T(a_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&=: \mathrm{K}'_1+\mathrm{K}'_2.
\end{aligned}
$$

When $j\leqslant k-\sigma-1$, $x\in C_k$ and $y\in B_j$, we have

$$
\rho(x-y)\geqslant b^{-\sigma}\rho(x)-\rho(y)\geqslant b^{-\sigma}\rho(x)-b^{-\sigma-1}\rho(x)=b^{-\sigma}(1-1/b)\rho(x).
$$

From this, Lemma 2.8 and the size condition of a_j, we conclude that

$$
|Ta_j|\lesssim\frac{b^j\|a_j\|_{L^1}}{(\rho(x))^2}\lesssim b^{j+2-2k}\|a_j\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\lesssim b^{j+2-2k-j\alpha_j}\|\chi_{B_j}\|_{L^{p'(\cdot)}}.
$$

Combining the above estimate, Lemma 2.9 and Lemma 2.6, we have

$$\begin{aligned}\|Ta_j\chi_k\|_{L^{p(\cdot)}} &\lesssim b^{j+2-2k-j\alpha_j}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}} \\ &\lesssim b^{j+2-2k-j\alpha_j}|B_k|\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}} \\ &\lesssim b^{j+2-k-j\alpha_j}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}} \\ &\lesssim b^{j+2-k-j\alpha_j}b^{(j-k)\delta_2}.\end{aligned} \tag{4.2}$$

By this, the density of $\dot{\mathbf{D}}_s(\mathbb{R}^n)$ in $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}(A;\mathbb{R}^n)$ and a similar method of Theorem 3.2, we can easily complete the proof of Theorem 4.3. We omit its details. □

REFERENCES

[1] A. ALMEDIDA AND D. DRIHEM, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl., **394** (2012), 781–795.

[2] E. ACERBI AND G. MINGIONE, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal., **164** (2002), 213–259.

[3] M. BOWNIK, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc., **164** (2003), vi+122 pp.

[4] Y. CHEN, S. LEVINE AND M. RAO, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., **66** (2006), 1383–1406.

[5] R. R. COIFMAN, P.-L. LIONS, Y. MEYER AND S. SEMMES, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl., (9) **72** (1993), 247–286.

[6] D. V. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Heidelberg, 2013. x+312 pp.

[7] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RUŽIČKA, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, 2017, Springer, Heidelberg, 2011. x+509 pp.

[8] Y. DING, S. LAN AND S. LU, *New Hardy spaces associated with some anisotropic Herz spaces and their applications*, Acta Math. Sin., **24** (2008), 1449–1470.

[9] B. DONG AND J. XU, *Variable exponent Herz-type Hardy spaces with and their applications*, Anal. Theory Appl., **31** (2015), 321–353.

[10] C. FEFFERMAN AND E. M. STEIN, *H^p spaces of several variables*, Acta Math, **129** (1972), 137–193.

[11] M. IZUKI, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math., **36** (2010), 33–50.

[12] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., **41** (1991), 592–618.

[13] Y. LU AND Y. ZHU, *Boundedness of some sublinear operators and commutators on Morrey-Herz spaces with variable exponents*, Czech. Math. J. **64** (1991), 592–618.

[14] E. NAKAI AND Y. SAWANO, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal., **262** (2012), 3665–3748.

[15] Y. SAWANO, *Atomic decompositions of Hardy space with variable exponent and its application to bounded linear operators*, Integral Equations Operator Theory, **77** (2013), 123–148.

[16] E. M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N. J., 1993.

[17] E. M. STEIN AND G. WEISS, *On the theory of harmonic functions of several variables. I. The theory of H^p-spaces*, Acta Math., **103** (1960), 25–62.

[18] J. TAN, *Atomic decompositions of localized Hardy spaces with variable exponents and applications*, J. Geom. Anal., **29** (2019), 799–827.

[19] H. WANG AND Z. LIU, *The Herz-type Hardy spaces with variable exponent and their applications*, Taiwanese J. Math., **16** (2012), 1363–1389.

[20] H. WANG, *Anisotropic Herz spaces with variable exponents*, Commun. Math. Anal., **18** (2015), 1–14.
[21] J. XU AND X. YANG, *Herz-Morrey-Hardy spaces with variable exponents and their applications*, J. Function Spaces, **2015** (2015), 1–19.
[22] J. XU AND X. YANG, *The molecular decomposition of Herz-Morrey-Hardy spaces with variable exponents and its application*, J. Math. Inequal., **10** (2016), 977–1008.

(Received January 20, 2021)

Aiting Wang
School of Mathematics and Statistics
Qinghai Minzu University
Xining 810000, Qinghai, China
e-mail: 2358063796@qq.com