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NEW REFINEMENTS OF ACZÉL–TYPE INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper we provide new refinements of Aczél-type inequality and give some applications. Furthermore, we show that two of three theorems in the work by J. Tian and M.-H. Ha (J. Math. Inequal. **12** (1) (2018), 175–189) are incorrect whereas the proof of the other is technically wrong. We establish an improvement of the correctly stated theorem with a simple proof and give counterexamples to the wrong ones.

1. Introduction

The famous Aczél's inequality states as follows.

THEOREM. *Let $n \in \mathbb{N}^+$, $n \geqslant 2$, and let a_i, b_i $(i = 1, \ldots, n)$ be real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 > 0$. Then*

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \leqslant \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2. \tag{1}$$

Inequality (1) was introduced by J. Aczél [1] in 1956. Since then it has had several applications in the theory of functional equations in non-Euclidean geometry and motivated a large number of research papers with various generalizations, refinements and applications (see [2, 4, 5, 6, 9, 10]). Among them, the work by Tian and Ha [6] provided some interesting properties and refinements of Aczél-type inequalities. Let us recall the first main result in [6].

THEOREM A. ([6, Theorem 2.2]) *Let $n, m \in \mathbb{N}^+$, $n \geqslant 2$, let $\lambda_1 \geqslant \cdots \geqslant \lambda_m > 0$ with $\sum_{j=1}^m \frac{1}{\lambda_j} \geqslant 1$, and let $a_{rj} > 0$ $(r = 1, \ldots, n;\ j = 1, \ldots, m)$ be such that $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$ $(j = 1, \ldots, m)$. Denote*

$$\Psi(n) = \prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}, \quad \Phi(n) = \left(\prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}\right)^2,$$

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and

$$V(n) = \Psi(n) - \Phi(n).$$

Then

$$V(n+1) \leqslant V(n) \leqslant 0.$$

This theorem is correctly stated, but its proof in [6] is too long (7 pages) and technically wrong. Thus, in that proof, the authors showed that $\Phi(n+1) - \Phi(n) \leqslant \Omega(n)$, where $\Omega(n)$ is the quantity in the right hand side of (13) in [6, p. 179]. It then follows that $\Phi(n+1) - \Phi(n) - \Psi(n) \geqslant \Omega(n) - \Psi(n)$, which is mathematically wrong. In this paper we will provide a similar result with a weaker assumption and prove it by a very short and simple proof.

Back to 1979, Vasić and Pečarić [7] presented an extension of Popoviciu's inequality [3], which is a generalization of Aczél's inequality:

THEOREM B. ([7]) *Let* $n, m \in \mathbb{N}^+$, $n \geqslant 2$ *and let* $\lambda_j > 0$ $(j = 2, \dots, m)$ *with* $\sum\limits_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, *and let* $a_{rj} > 0$ $(r = 1, \dots, n; j = 1, \dots, m)$ *be such that* $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1, \dots, m)$. *Then*

$$\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \leqslant \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj}. \tag{2}$$

In 2012, Tian [4] provided a reversed version of (2) stated as follows.

THEOREM C. ([4, Corollary 2.6]) *Let* $n, m \in \mathbb{N}^+$, $n \geqslant 2$ *and let* $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, \dots, m)$ *with* $\sum\limits_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$, *and let* $a_{rj} > 0$ $(r = 1, \dots, n;\ j = 1, \dots, m)$ *be such that* $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1, \dots, m)$. *Then*

$$\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \geqslant \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj}. \tag{3}$$

However, we will see in Proposition 1 below that the right hand side of inequality (3) is negative in the case of $\lambda_1 < 0$ and this inequality becomes trivial. In the present paper we will give refinements of Theorems B and C and their applications.

In addition, back to the work by Tian and M.-H. Ha [6], Theorems 2.3 and 2.4 are incorrect. As a consequence, all the corollaries of those theorems are also not true. Let us recall those theorems.

THEOREM D. ([6, Theorem 2.3]) *Let* $n, m \in \mathbb{N}^+$, $n \geqslant 2$, *let* $\lambda_1 \leqslant \cdots \leqslant \lambda_m < 0$, *and let* $a_{rj} > 0$ $(r = 1, \dots, n;\ j = 1, \dots, m)$ *be such that* $a_{1j}^{\lambda_j} - \sum\limits_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j =$

$1,\dots,m)$. *If we denote*

$$\Psi(n)=\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}},\quad \Phi(n)=\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^{2},$$

and

$$V(n)=\Psi(n)-\Phi(n),$$

then

$$V(n+1)\geqslant V(n)\geqslant 0. \tag{4}$$

THEOREM E. ([6, Theorem 2.4]) *Let* $n,m\in\mathbb{N}^{+}$, $n\geqslant 2$, *let* $\lambda_1>0$, $\lambda_2\leqslant\cdots\leqslant\lambda_m\leqslant 0$ *with* $\sum_{j=1}^{m}\frac{1}{\lambda_j}\leqslant 1$, *and let* $a_{rj}>0$ $(r=1,\dots,n;\ j=1,\dots,m)$ *be such that* $a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}>0$ $(j=1,\dots,m)$. *If we denote*

$$\Psi(n)=\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}},\quad \Phi(n)=\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^{2}, \tag{5}$$

and

$$V(n)=\Psi(n)-\Phi(n),$$

then

$$V(n+1)\geqslant V(n)\geqslant 0. \tag{6}$$

It is worth mentioning that because the proof of Theorem A in [6] is wrong, it is impossible to prove Theorems D and E by the same argument as said in [6]. We will give counterexamples to these theorems in Section 3.

The paper is organized as follows. In Section 2 we first present a similar result to Theorem A with a weaker assumption and a simple proof. We then establish a reversed version of Theorem A. The corollaries following are refinements of Theorem B and C. Section 3 provides counterexamples to Theorems D and E.

2. New results

The first main result of this paper is the following theorem.

THEOREM 1. *Let* $n,\ m\in\mathbb{N}^{+}$, $n\geqslant 2$, *let* $\lambda_j>0$ $(j=1,\dots,m)$ *with* $\sum_{j=1}^{m}\frac{1}{\lambda_j}\geqslant 1$, *and let* $a_{rj}>0$ $(r=1,\dots,n;\ j=1,\dots,m)$ *be such that* $a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}>0$ $(j=1,\dots,m)$. *Denote*

$$\Psi(n)=\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}},\quad \Phi(n)=\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^{2},$$

and

$$V(n) = \Psi(n) - \Phi(n).$$

Then

$$V(n+1) \leqslant V(n) \leqslant 0. \tag{7}$$

Proof. First we show the second inequality of (7). For, using (2), we have

$$0 \leqslant \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \leqslant \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj}.$$

Hence

$$\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{2}{\lambda_j}} \leqslant \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right)^2,$$

which implies $V(n) \leqslant 0$.

Next, we prove the first inequality in (7). We write

$$
\begin{aligned}
&\Phi(n+1) - \Phi(n) - \Psi(n+1) \\
&= \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right)^2 - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right)^2 - \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n+1} a_{rj}^{\lambda_j} \right)^{\frac{2}{\lambda_j}} \\
&= \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right)^2 - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right)^2 \right] - \left[\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n+1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right]^2 \\
&= \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} - \prod_{j=1}^{m} a_{(n+1)j} \right) - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) \right] \\
&\quad \times \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right) + \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) \right] - \left[\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n+1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right]^2 \\
&= - \prod_{j=1}^{m} a_{(n+1)j} \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right) + \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) \right] \\
&\qquad\qquad - \left[\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n+1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right]^2 \\
&\geqslant - \prod_{j=1}^{m} a_{(n+1)j} \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right) + \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) \right] \\
&\qquad\qquad - \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n+1} \prod_{j=1}^{m} a_{rj} \right) \right]^2
\end{aligned} \tag{8}
$$

where we have used inequality (2) in (8). Set

$$A=\prod_{j=1}^{m}a_{(n+1)j} \quad \text{and} \quad B=\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n+1}\prod_{j=1}^{m}a_{rj}.$$

Then the right hand side of (8) is rewritten as

$$\begin{aligned}-A(2B+A)-B^2&=-(A^2+2AB+B^2)\\&=-(A+B)^2\\&=-\left(\prod_{j=1}^{m}a_{(n+1)j}-\prod_{j=1}^{m}a_{(n+1)j}\right)^2\\&=-\Psi(n).\end{aligned}$$

As a consequence,

$$\Phi(n+1)-\Phi(n)-\Psi(n+1)\leqslant-\Psi(n)$$

or

$$\Phi(n+1)-\Phi(n)\leqslant\Psi(n+1)-\Psi(n)$$

and hence

$$V(n+1)\leqslant V(n),$$

which completes the proof. □

REMARK 1. In the preceding theorem, we do not need the order $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m$ as in Theorem A.

For the next main result, let us recall a well-known inequality of Vasić and Pečarić.

LEMMA 1. ([7]) *Let $n\geqslant 2,\ m\geqslant 2$ be integers and let $a_{rj}>0\ (r=1,2,\ldots,n;\ j=1,\ldots,m)$. If $\lambda_j<0\ (j=1,\ldots,m)$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}a_{rj}\geqslant\prod_{j=1}^{m}\left(\sum_{i=1}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \tag{9}$$

This result will be used to prove the following proposition.

PROPOSITION 1. *Let $n\geqslant 2,\ m\geqslant 2$ be integers and let $a_{rj}>0\ (r=1,\ldots,n;\ j=1,\ldots,m)$. If there are $\lambda_j<0\ (j=1,\ldots,m)$ such that $a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}>0\ (j=1,\ldots,m)$, then*

$$\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}<0. \tag{10}$$

Proof. Since $\lambda_j < 0$ and $a_{1j}^{\lambda_j} > \sum_{r=2}^{n} a_{rj}^{\lambda_j}$, $j = 1, \ldots, m$, we have

$$0 < a_{1j} < \left(\sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}, \quad j = 1, \ldots, m.$$

Hence

$$\prod_{j=1}^{m} a_{1j} < \prod_{j=1}^{m} \left(\sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}.$$

Together with Lemma 1 we obtain

$$\prod_{j=1}^{m} a_{1j} < \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj},$$

which is (10). □

REMARK 2. According to the previous lemma, Theorem C is trivial in the case of $\lambda_1 < 0$ since the right hand side of (3) is positive, and we do not need the assumption $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$.

LEMMA 2. *If a, b, c, d are positive numbers satisfying $a > b \geqslant c > d$, then*

$$ac > bd. \tag{11}$$

Proof. Since $a > b \geqslant c > d > 0$, we have

$$ac > bc \quad \text{and} \quad bc > bd,$$

which implies (11). □

The next main result of this paper is the following theorem, which can be seen as a reserved version of Theorem 1.

THEOREM 2. *Let $n \geqslant 2$, $m \geqslant 2$ be integers and $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, \ldots, m)$ such that $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$. Let $a_{rj} > 0$ $(r = 1, \ldots, n;\ j = 1, \ldots, m)$ be such that $a_{11} > \sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj}$, $\prod_{j=2}^{m} a_{1j} > \sum_{r=2}^{n} a_{r1}$, and $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1, \ldots, m)$. Denote*

$$\Psi(n) = \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{2}{\lambda_j}}, \quad \Phi(n) = \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right)^2,$$

and

$$V(n) = \Psi(n) - \Phi(n).$$

Then

$$V(n+1) \geqslant V(n) \geqslant 0. \tag{12}$$

We will use Theorem C to prove this theorem. First, we will see as below that the right hand side of (3) is positive, provided in addition that $\lambda_1 > 0$, $a_{11} > \sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj}$ and $\prod_{j=2}^{m} a_{1j} > \sum_{r=2}^{n} a_{r1}$.

LEMMA 3. *In the setting of Theorem 2, we have*

$$\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} > 0. \tag{13}$$

Proof. Due to $\lambda_j < 0$ and $a_{1j}^{\lambda_j} > \sum_{r=2}^{n} a_{rj}^{\lambda_j}$, $j = 2, \ldots, m$, it follows that

$$0 < a_{1j} < \left(\sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}, \quad j = 2, \ldots, m.$$

Thus

$$0 < \prod_{j=2}^{m} a_{1j} < \prod_{j=2}^{m} \left(\sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}$$

implies that

$$0 < \prod_{j=2}^{m} a_{1j} < \sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj}, \tag{14}$$

using (9). Due to hypothesis $a_{11} > \sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj}$ and $\prod_{j=2}^{m} a_{1j} > \sum_{r=2}^{n} a_{r1}$, it follows that

$$a_{11} > \sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj} > \prod_{j=2}^{m} a_{1j} > \sum_{r=2}^{n} a_{r1} > 0. \tag{15}$$

Applying Lemma 2 for $a = a_{11}$, $b = \sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj}$, $c = \prod_{j=2}^{m} a_{1j}$, $d = \sum_{r=2}^{n} a_{r1}$, we obtain

$$\prod_{j=1}^{m} a_{1j} > \left(\sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj} \right) \left(\sum_{r=2}^{n} a_{r1} \right) > \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj}.$$

This yields (13). □

We are in a position to prove Theorem 2.

Proof of Theorem 2. We first show $V(n) \geqslant 0$. Apply Theorem C and Lemma 3 to get

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}=\left[\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]^2\geqslant\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^2,$$

which yields $V(n) \geqslant 0$.

Next, we prove the first inequality in (12), that is $V(n+1) \geqslant V(n)$. For, analogously to the proof of Theorem 1, we write

$$\begin{aligned}
&\Phi(n+1)-\Phi(n)-\Psi(n+1)\\
&=\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n+1}\prod_{j=1}^{m}a_{rj}\right)^2-\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^2-\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n+1}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}\\
&=-\prod_{j=1}^{m}a_{(n+1)j}\left[\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n+1}\prod_{j=1}^{m}a_{rj}\right)+\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)\right]\\
&\qquad\qquad-\left[\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n+1}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]^2\\
&\leqslant-\prod_{j=1}^{m}a_{(n+1)j}\left[\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n+1}\prod_{j=1}^{m}a_{rj}\right)+\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)\right]\\
&\qquad\qquad-\left[\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n+1}\prod_{j=1}^{m}a_{rj}\right)\right]^2 \qquad (16)\\
&=-\Psi(n),
\end{aligned}$$

where we have used

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}}\geqslant\left(\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\right)^2$$

obtained from (2) and (13) in inequality (16). Hence

$$\Psi(n+1)-\Phi(n+1)\geqslant\Psi(n)-\Phi(n),$$

or equivalently,

$$V(n+1)\geqslant V(n), \qquad (17)$$

completing the proof. □

REMARK 3. In Theorem 2, we have added the assumptions

$$a_{11} > \sum_{r=2}^{n}\prod_{j=2}^{m} a_{rj}, \quad \prod_{j=2}^{m} a_{1j} > \sum_{r=2}^{n} a_{r1}$$

in comparison to Theorem E which will be shown to be wrong in the next section.

An application of Theorem 1 gives us the following corollary which is a better result in comparison to [6, Corollary 11]. This is a refinement of Theorem B.

COROLLARY 1. *Let* $n,\ m \in \mathbb{N}^+,\ n \geqslant 2$, *let* $\lambda_j > 0\ (j = 1, \ldots, m)$ *with* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, *and let* $a_{rj} > 0\ (r = 1, \ldots, n;\ j = 1, \ldots, m)$ *be such that* $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0\ (j = 1, \ldots, m)$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leqslant \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right)\left[1 + \frac{V(2)}{(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj})^2}\right]$$
$$\leqslant \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj},$$

where

$$V(2) = \prod_{j=1}^{m}(a_{1j}^{\lambda_j} - a_{2j}^{\lambda_j})^{\frac{2}{\lambda_j}} - \left(\prod_{j=1}^{m} a_{1j} - \prod_{j=1}^{m} a_{2j}\right)^2 \leqslant 0.$$

Proof. From Theorem 1, it follows that

$$V(n) \leqslant V(2) \leqslant 0.$$

Then

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{2}{\lambda_j}} - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right)^2 \leqslant V(2)$$

implies that

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leqslant \left[\left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right)^2 + V(2)\right]^{\frac{1}{2}}$$
$$\leqslant \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right)\left[1 + \frac{V(2)}{(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj})^2}\right]^{\frac{1}{2}}$$
$$\leqslant \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj},$$

owing to $V(2) \leqslant 0$, as was to be shown. □

The same argument applies to yield Corollary 2, which is a refinement of Theorem C.

COROLLARY 2. *Let $n \geqslant 2$, $m \geqslant 2$ be integers and $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, \dots, m)$ with $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$. If $a_{rj} > 0$ $(r = 1, \dots, n;\ j = 1, \dots, m)$ are such that $a_{11} > \sum_{r=2}^{n} \prod_{j=2}^{m} a_{rj}$, $\prod_{j=2}^{m} a_{1j} > \sum_{r=2}^{n} a_{r1}$, and $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1, \dots, m)$, then*

$$\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \geqslant \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right) \left[1 + \frac{V(2)}{(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj})^2} \right]^{\frac{1}{2}}$$
$$\geqslant \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} > 0,$$

where

$$V(2) = \prod_{j=1}^{m} (a_{1j}^{\lambda_j} - a_{2j}^{\lambda_j})^{\frac{2}{\lambda_j}} - (\prod_{j=1}^{m} a_{1j} - \prod_{j=1}^{m} a_{2j})^2 \geqslant 0.$$

Setting $m = 2$, $a_{r1} = a_r$, $a_{r2} = b_r$ $(r = 1, \dots, n)$ in Theorem 2, we have the following corollary.

COROLLARY 3. *Let $n \geqslant 2$ be an integer and $\lambda_1 > 0 > \lambda_2$ such that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \leqslant 1$. Let a_r, b_r $(r = 1, \dots, n)$ be positive numbers such that $a_1 > \sum_{r=2}^{n} b_r$, $b_1 > \sum_{r=2}^{n} a_r$, $a_1^{\lambda_1} - \sum_{r=2}^{n} a_r^{\lambda_1} > 0$, and $b_1^{\lambda_2} - \sum_{r=2}^{n} b_r^{\lambda_2} > 0$. Denote*

$$V(n) = \left(a_1^{\lambda_1} - \sum_{r=2}^{n} a_r^{\lambda_1} \right)^{\frac{2}{\lambda_1}} \left(b_1^{\lambda_2} - \sum_{r=2}^{n} b_r^{\lambda_2} \right)^{\frac{2}{\lambda_2}} - \left(a_1 b_1 - \sum_{r=2}^{n} a_r b_r \right)^2.$$

Then

$$V(n+1) \geqslant V(n) \geqslant 0.$$

Due to the right hand side of inequality (3) in Theorem C is negative in the case of $\lambda_1 < 0$ (see Proposition 1), the right hand side of (22) in [4, Theorem 3.1] is nonpositive, and hence [4, Theorem 3.1] is trivial for $\lambda_1 < 0$. The next result is an improvement of that theorem.

COROLLARY 4. *Let $m \geqslant 2$ be an integer and $\lambda_1 > 0$ and $\lambda_j < 0$ $(j = 2, \dots, m)$ with $\sum_{j=1}^{m} \frac{1}{\lambda_j} = 1$. Let $a_j > 0$ and $f_j : [a, b] \to (0, \infty)$ be Riemann integrable functions such that $a_1 > \int_a^b \prod_{j=2}^{m} f_j(x)dx$, $\prod_{j=2}^{m} a_j > \int_a^b f_1(x)dx$, $a_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)dx > 0$ $(j = 1, \dots, m)$. Then*

$$\prod_{j=1}^{m} \left(a_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)dx \right)^{\frac{1}{\lambda_j}} \geqslant \prod_{j=1}^{m} a_j - \int_a^b \prod_{j=1}^{m} f_j(x)dx \geqslant 0.$$

Proof. The proof is similar to that of [4, Theorem 3.1] by using Corollary 2. □

3. Counterexamples to Theorem D and Theorem E

COUNTEREXAMPLE 1. Consider $n=3$, $m=2$, $\lambda_1=\lambda_2=-1$, and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

We have

$$a_{11}^{\lambda_1} - a_{21}^{\lambda_1} - a_{31}^{\lambda_1} = 1^{-1} - 2^{-1} - 4^{-1} > 0,$$
$$a_{12}^{\lambda_2} - a_{22}^{\lambda_2} - a_{32}^{\lambda_2} = 1^{-1} - 3^{-1} - 5^{-1} > 0.$$

Therefore, λ_i and a_{ij} $(i=1,2; j=1,2,3)$ satisfy the assumption of Theorem D. However,

$$\begin{aligned} V(2) = \Psi(2) - \Phi(2) &= \left(a_{11}^{\lambda_1} - a_{21}^{\lambda_1}\right)^{\frac{2}{\lambda_1}} \left(a_{12}^{\lambda_2} - a_{22}^{\lambda_2}\right)^{\frac{2}{\lambda_2}} - (a_{11}a_{12} - a_{21}a_{22})^2 \\ &= \left(1^{-1} - 2^{-1}\right)^{\frac{2}{-1}} \left(1^{-1} - 3^{-1}\right)^{\frac{2}{-1}} - (1\cdot 1 - 2\cdot 3)^2 \\ &= 3^2 - 5^2 = -16 < 0, \end{aligned}$$

and

$$\begin{aligned} V(3) &= \Psi(3) - \Phi(3) \\ &= \left(a_{11}^{\lambda_1} - a_{21}^{\lambda_1} - a_{31}^{\lambda_1}\right)^{\frac{2}{\lambda_1}} \left(a_{12}^{\lambda_2} - a_{22}^{\lambda_2} - a_{32}^{\lambda_2}\right)^{\frac{2}{\lambda_2}} - (a_{11}a_{12} - a_{21}a_{22} - a_{31}a_{32})^2 \\ &= \left(1^{-1} - 2^{-1} - 4^{-1}\right)^{\frac{2}{-1}} \left(1^{-1} - 3^{-1} - 5^{-1}\right)^{\frac{2}{-1}} - (1\cdot 1 - 2\cdot 3 - 4\cdot 5)^2 \\ &= \left(\frac{1}{4}\right)^{-2} \left(\frac{7}{15}\right)^{-2} - (1-6-20)^2 \\ &= \left(\frac{7}{60}\right)^{-2} - 25^2 = -\frac{27025}{49} < -16 = V(2). \end{aligned}$$

Hence

$$V(3) < V(2) < 0.$$

This means that Theorem D is incorrect.

COUNTEREXAMPLE 2. Consider $n=3$, $m=2$, $\lambda_1=1$, $\lambda_2=-1$, and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

Then

$$a_{11}^{\lambda_1}-a_{21}^{\lambda_1}-a_{31}^{\lambda_1}=4-2-1>0,$$
$$a_{12}^{\lambda_2}-a_{22}^{\lambda_2}-a_{32}^{\lambda_2}=1^{-1}-4^{-1}-2^{-1}>0.$$

This implies that λ_i and a_{ij} $(i=1,2;j=1,2,3)$ satisfy the assumption of Theorem E. However,

$$\begin{aligned} V(2)=\Psi(2)-\Phi(2) &= \left(a_{11}^{\lambda_1}-a_{21}^{\lambda_1}\right)^{\frac{2}{\lambda_1}}\left(a_{12}^{\lambda_2}-a_{22}^{\lambda_2}\right)^{\frac{2}{\lambda_2}}-(a_{11}a_{12}-a_{21}a_{22})^2\\ &=(4-2)^2\left(1^{-1}-4^{-1}\right)^{-2}-(4\cdot 1-2\cdot 4)^2\\ &=-\frac{80}{9}<0 \end{aligned}$$

and

$$\begin{aligned} V(3)&=\Psi(3)-\Phi(3)\\ &=\left(a_{11}^{\lambda_1}-a_{21}^{\lambda_1}-a_{31}^{\lambda_1}\right)^{\frac{2}{\lambda_1}}\left(a_{12}^{\lambda_2}-a_{22}^{\lambda_2}-a_{32}^{\lambda_2}\right)^{\frac{2}{\lambda_2}}-(a_{11}a_{12}-a_{21}a_{22}-a_{31}a_{32})^2\\ &=(4-2-1)^2\left(1-4^{-1}-2^{-1}\right)^{-2}-(4\cdot 1-2\cdot 4-2)^2\\ &=-20 \end{aligned}$$

yield

$$V(3)<V(2)<0,$$

which contradicts Theorem E. This means that Theorem E is incorrect.

REMARK 4. (a) In order to prove Theorems D and E, it is necessary that

$$\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}\geqslant 0. \tag{18}$$

The matter of fact is that the assumptions in Theorems D and E are not sufficient to guarantee that (18). In fact, with the assumption in Theorem D, we have the reversed inequality

$$\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}<0,$$

according to Proposition 1. If we suppose, in addition, that $\lambda_1>0$ and $a_{11}>\sum_{r=2}^{n}\prod_{j=2}^{m}a_{rj}$, then (18) holds (see Lemma 3), and the conclusion in Theorem E is obtained; see Theorem 2.

(b) From (a), Theorems D and E can not be proved by the same method as that of Theorem A (or Theorem 1) because (18) does not hold.

(c) Since Theorems D and E are incorrect, so are the results in [6] which follow from them, including Corollaries 2.5, 2.7, 2.9, 2.10, 2.12, and 2.14.

(d) We can use Counterexample 1 to show directly that Corollaries 2.7, 2.9, 2.14 are incorrect, whereas Counterexample 2 also shows that Corollaries 2.5, 2.10, 2.12 are not true.

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