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ON THE STABILITY OF CUBIC BI–DERIVATIONS ON BANACH ALGEBRAS

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Abstract. In this paper, using fixed point method, we investigate the stability and also the superstability of cubic bi-derivations on Banach algebras.

1. Introduction

Ulam is pioneer of the stability problem in functional equations (see [17]). In 1940, he discussed a number of important unsolved problems. Among these problems was the following question concerning the stability of homomorphism:

"Suppose $(\digamma_1, \bullet)$ *be a group,* $(\digamma_2, \circledast)$ *be a metric group with metric* $\Delta(\cdot,\cdot)$. *For* $\xi > 0$, *is there* $\nu > 0$ *such that if a mapping* $\phi : \digamma_1 \to \digamma_2$ *satisfies*

$$\Delta(\phi(st), \phi(s)\phi(t)) < \nu$$

for all $s,t \in \digamma_1$, *then there exists a homomorphism* $T : \digamma_1 \to \digamma_2$ *with*

$$\Delta(\phi(s), T(s)) < \xi$$

for all $s \in \digamma_1$*?"*

In 1941, Hyers gave the first confirmative answer to the question of Ulam as following theorem (see [9]):

Assume that Ω_1 is a normed vector space and Ω_2 is a Banach space. Let $p : \Omega_1 \to \Omega_2$ be a mapping satisfying the inequality

$$\|p(s+t) - p(s) - p(t)\| \leqslant \mu$$

for all $s \in \Omega_1$, where $\mu > 0$. Then

$$f(s) = \lim_{n\to\infty} \frac{p(2^n s)}{2^n}$$

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exists for all $s \in \Omega_1$ and $f : \Omega_1 \to \Omega_2$ is the unique additive mapping such that

$$\|p(s) - f(s)\| \leqslant \mu$$

for all $s \in \Omega_1$.

Subsequently, in 1978, Rassias proved the following theorem which is a generalized version of Hyers' theorem (see [16]):

Assume that B_1, B_2 are two Banach spaces and $p : B_1 \to B_2$ is a mapping. Suppose that there exist $\theta \geqslant 0$ and $\tau \in [0,1)$ such that

$$\|p(s+t) - p(s) - p(t)\| \leqslant \theta(\|s\|^{\tau} + \|t\|^{\tau})$$

for $s,t \in B_1$. Then the limit

$$L(s) = \lim_{n\to\infty} \frac{p(2^n s)}{2^n}$$

exists for all $s \in B_1$ and $L : B_1 \to B_2$ is the unique additive mapping which satisfies

$$\|p(s) - L(s)\| \leqslant \frac{2\theta}{2-2^p} \|s\|^{\tau}$$

for all $s \in B_1$. Additionally, if the mapping $p(us)$ is continuous for $u \in \mathbb{R}$ for each $s \in B_1$, then L is $\mathbb{R}$-linear.

In 1994, Găvruta provided a further generalization of the Th. M. Rassias' theorem as follows (see [7]).

Assume that $(G,+)$ is an abelian group, B is a Banach space and $\varphi : B \times B \to [0,\infty)$ is a function such that

$$\widetilde{\varphi}(s,t) := \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k s, 2^k t) < \infty$$

for all $s,t \in G$. Let $p : G \to B$ be such that

$$\|p(s+t) - p(s) - p(t)\| \leqslant \varphi(s,t)$$

for all $s,t \in G$. Then there exists a unique mapping $H : G \to B$ such that $H(s+t) = H(s) + H(t)$ for all $s,t \in G$ and $\|p(s) - H(s)\| \leqslant \frac{1}{2}\widetilde{\varphi}(s,s)$ for all $s \in G$.

In 2004, Cădariu and Radu applied fixed point method to the stability for the additive Cauchy functional equation (see [3]). Bae and Park proved the Hyers-Ulam stability of bi-homomorphisms and bi-derivations on C^*-ternary algebras (see [1]).

Jun and Kim obtained a general solution the following functional equation (see [10]):

$$p(2s+t) + p(2s-t) = 2p(s+t) + 2p(s-t) + 12p(s). \tag{1}$$

Also, they investigated the Hyers-Ulam stability of the functional equation (1). One can see that $p(s) = cs^3$ is a solution of (1). The functional equation (1) is called a cubic funtional equation and then every solution of (1) is called a cubic mapping.

In 2007, Najati introduced the following functional equation

$$p(ms+t)+p(ms-t)=mp(s+t)+mp(s-t)+2(m^3-m)p(s) \tag{2}$$

where $m \in \mathbb{Z}^+$ and $m \geqslant 2$ (see [13]). If $m=2$, we have the functional equation (1).

In recent years, the stability of some type of cubic derivations associated with the functional equation (1) has been studied by a number of mathematicians. The stability, superstability and hyperstability of cubic Lie derivations and other related problems were studied (see [3, 4, 6, 8, 11, 12, 14, 15]). Motivated by these results, we study the Hyers-Ulam stability of cubic bi-derivations.

2. Preliminaries

Let B be a complex Banach algebra and X be a complex Banach B-bimodule. A mapping $p: B \to X$ is called a cubic homogeneous mapping if $p(\lambda s)=\lambda^3 p(s)$ for all $s \in B$ and $\lambda \in \mathbb{C}$. A cubic homogeneous mapping $\delta : B \to X$ is called a cubic derivation if $\delta(st)=\delta(s)t^3+s^3\delta(t)$ for all $s,t \in B$ (see [2]).

Now, we remember some elementary concepts in the fixed point theory.

DEFINITION 1. [5] Suppose that K is a nonempty set. A function $\Delta : K \times K \to [0,\infty]$ is called a generalized complete metric on K if Δ satisfies the following conditions:

(1) $\Delta(k,l)=0$ if and only if $k=l$;
(2) $\Delta(k,l)=\Delta(l,k)$ for all $k,l \in K$;
(3) $\Delta(k,m) \leqslant \Delta(k,l)+\Delta(l,m)$ for all $k,l,m \in K$;
(4) Every Δ-Cauchy sequence in K is Δ-convergent, i.e., $\lim_{n,m\to\infty} \Delta(k_n,k_m)=0$ for a sequence $\{k_n\}$ in K implies that there exists $k \in K$ with $\lim_{n\to\infty} \Delta(k,k_n)=0$.

EXAMPLE 1. Let $K := C(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$ and $\Delta : K \times K \to [0,\infty]$ be given by $\Delta(p,q) := \sup_{t\in\mathbb{R}} |p(t)-q(t)|$. Then the pair (K,Δ) is a generalized complete metric space.

DEFINITION 2. Let (K,Δ) be a generalized complete metric space. A mapping $S : K \to K$ is said to satisfy a Lipshitz condition with constant $L>0$ if

$$\Delta(S(k),S(l)) \leqslant L\Delta(k,l)$$

for all $k,l \in K$. If $L<1$, then S is called a strictly contractive operator.

We recall well known fixed point theorem which is significant for our aims.

THEOREM 1. [5] *Let (K,Δ) be a generalized complete metric space and $S : K \to K$ be a strictly contractive mapping with the Lipshitz constant L. Then for each given $k \in K$, either $\Delta(S^m k, S^{m+1}k)=\infty$ for all positive integers m or there exists a positive integer m_0 such that*

(1) *For each* $m \geqslant m_0$, $\Delta(S^m k, S^{m+1}k) < \infty$,
(2) *The sequence* $\{S^m k\}$ *converges to a fixed point* l^* *of* S,
(3) l^* *is the unique fixed point of* S *in the set* $\Omega = \{l \in K \mid \Delta(S^{m_0}k, l) < \infty\}$,
(4) $\Delta(l, l^*) \leqslant \frac{1}{1-L}\Delta(S(l), l)$ *for all* $l \in \Omega$.

DEFINITION 3. Let B be a Banach algebra and X be a Banach B-bimodule. A bi-cubic mapping $\delta : B \times B \to X$ is a cubic bi-derivation if it satisfies the following properties:

(1) δ is bi-cubic homogeneous, i.e.,

$$\delta(\lambda s, \mu t) = \lambda^3 \mu^3 \delta(s,t) \text{ for all } s,t \in B \text{ and all } \lambda, \mu \in \mathbb{C},$$

(2) $\delta(st, u) = s^3\delta(t,u) + \delta(s,u)t^3$ and $\delta(s, tu) = t^3\delta(s,u) + \delta(s,t)u^3$ for all $s,t,u \in B$.

EXAMPLE 2. Let B be a Banach algebra. Consider

$$X := \left\{ \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

for all $b_1, b_2, \cdots, b_6 \in B$. Then X is a Banach algebra with the usual matrix operations and the following norm:

$$\left\| \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\| = \sum_{i=1}^{6} \|b_i\| \quad (b_i \in B).$$

It is known that

$$X^* = \left\{ \begin{pmatrix} 0 & p_1^* & p_2^* & p_3^* \\ 0 & 0 & p_4^* & p_5^* \\ 0 & 0 & 0 & p_6^* \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}$$

where $p_i \in B^*$, is the dual of X equipped with the following norm:

$$\left\| \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ 0 & 0 & p_4 & p_5 \\ 0 & 0 & 0 & p_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\| = \max\{\|p_i\| : 0 \leqslant i \leqslant 6\} \quad (p_i \in B^*).$$

Assume that $C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ 0 & 0 & c_4 & c_5 \\ 0 & 0 & 0 & c_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & d_1 & d_2 & d_3 \\ 0 & 0 & d_4 & d_5 \\ 0 & 0 & 0 & d_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in X$ and $P = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ 0 & 0 & p_4 & p_5 \\ 0 & 0 & 0 & p_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\in X^*$, where $p_i \in B^*$ and $c_i, d_i \in B$ $(0 \leqslant i \leqslant 6)$.

Consider the module actions of X on X^* as follows:

$$\langle P\cdot C,D\rangle=\sum_{i=1}^{6}p(c_id_i),\ \langle C\cdot P,D\rangle=\sum_{i=1}^{6}p(d_ic_i).$$

Then X^* is a Banach X-bimodule. Let $P=\begin{pmatrix}0&p_1&p_2&p_3\\0&0&p_4&p_5\\0&0&0&p_6\\0&0&0&0\end{pmatrix}\in X^*$. We define $\delta:X\times X\to X^*$ by $\delta(C,D)=P\cdot C+D\cdot P$ for all $C,D\in X$. It is easy to see that δ is a bi-cubic homogeneous mapping. Since $X^4=\{0\}$, we have $\delta(CD,E)=C^3\delta(D,E)+\delta(C,E)D^3=0$ for all $C,D,E\in X$. Therefore, δ is a cubic bi-derivation.

In this paper, we investigate the Hyers-Ulam stability of cubic bi-derivations associated with the general cubic functional equation in Banach algebras:

$$p(st,u)=s^3p(t,u)+p(s,u)t^3,\ p(s,tu)=t^3p(s,u)+p(s,t)u^3,$$

$$p(ks,kt+u)+p(ks,kt-u)=k^4\left[p(s,t+u)+p(s,t-u)\right]+2k^3(k^3-k)p(s,t). \quad (3)$$

Furthermore, we investigate the superstability of (3).

In the following theorem, we present the general solution of (3).

THEOREM 2. *Let B be a Banach algebra and X be a Banach B-bimodule. Then every solution of the functional equation (3) with the positive integer $k\geqslant 2$ is a bi-cubic homogeneous mapping. Also, if a bi-cubic homogeneous mapping $p:B\times B\to X$ satisfies $p(s,0)=0$ for all $s\in B$, then p satisfies the functional equation (3).*

Proof. Suppose that $p:B^2\to X$ satisfies (3). Letting $u=0$ in (3), we get

$$2p(ks,kt)=2k^4p(s,t)+2k^3(k^3-k)p(s,t)$$

for all $s,t\in B$ and $k\in\mathbb{Z}^+$ with $k\geqslant 2$. Thus we have

$$p(ks,kt)=k^6p(s,t)$$

for all $s,t\in B$ and $k\in\mathbb{Z}^+$ with $k\geqslant 2$. Hence $p:B^2\to X$ is bi-cubic homogeneous.

Let $p:B^2\to X$ be a bi-cubic homogeneous mapping with $p(s,0)=0$ for all $s\in B$. Letting $u=t$ in (3) and using $p(s,0)=0$, we obtain that

$$p(ks,(k+1)t)+p(ks,(k-1)t)=k^4p(s,2t)+2k^3(k^3-k)p(s,t)$$

for all $s,t\in B$. Since p is a bi-cubic homogeneous mapping, we have

$$k^3[(k+1)^3+(k-1)^3]p(s,t)=8k^4p(s,t)+2k^3(k^3-k)p(s,t)$$

for all $s,t\in B$. Therefore, p satisfies the functional equation (3). $\square$

3. Stability of cubic bi-derivations

Throughout this paper, let B be a Banach algebra, X be a Banach B-bimodule and $\Upsilon = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. For a given mapping $p : B \times B \to X$ we define

$$\begin{aligned}\Delta_{\lambda,\mu} p(s;t,u) := {}& p(\lambda ks, \mu kt + \mu u) + p(\lambda ks, \mu kt - \mu u) \\ & - \lambda^3\mu^3 k^4 [p(s,t+u) + p(s,t-u)] - 2\lambda^3\mu^3 k^3(k^3 - k)p(s,t)\end{aligned}$$

for all $s,t,u \in B$ and all $\lambda, \mu \in \Upsilon$, where $k \in \mathbb{Z}^+$ with $k \geqslant 2$.

THEOREM 3. *Let $\varphi : B^3 \to [0,\infty)$ be a function and $p : B \times B \to X$ be a mapping with $p(0,0) = 0$ satisfying*

$$\left\|\Delta_{\lambda,\mu} p(s;t,u)\right\| \leqslant \varphi(s,t,u), \tag{4}$$

$$\left\|p(st,u) - s^3 p(t,u) - p(s,u)t^3\right\| + \left\|p(s,tu) - t^3 p(s,u) - p(s,t)u^3\right\| \leqslant \varphi(s,t,u), \tag{5}$$

$$\varphi(k^n s, k^n t, k^n u) \leqslant k^{6n} L\varphi(s,t,u) \tag{6}$$

for all $s,t,u \in B$ and all $\lambda, \mu \in \Upsilon$ and $L < 1$. If for each fixed $a,b \in B$, the mapping $\gamma \mapsto p(\gamma a, \gamma b)$ from $\mathbb{R}$ to X is continuous in $\gamma \in \mathbb{R}$, then there exists a unique cubic bi-derivation $\delta : B \times B \to X$ which satisfies the functional equation (3) and the following inequality

$$\|p(s,t) - \delta(s,t)\| \leqslant \frac{1}{2k^6(1-L)}\varphi(s,t,0) \tag{7}$$

for all $s,t \in B$, $k \in \mathbb{Z}^+$ with $k \geqslant 2$.

Proof. Putting $\lambda = \mu = 1$ and $u = 0$ in (4), we have

$$\left\|p(s,t) - \frac{p(ks,kt)}{k^6}\right\| \leqslant \frac{1}{2k^6}\varphi(s,t,0) \tag{8}$$

for all $s,t \in B$. Consider $\Psi := \{q : B \times B \to X \mid q(0,0) = 0\}$ and a generalized metric on Ψ as follows: $\Delta(q,r) = \inf\{\beta \in \mathbb{R}^+ \mid \|q(s,t) - r(s,t)\| \leqslant \beta\varphi(s,t,0),\ \forall s,t \in B\}$.

Let $\{q_n\}$ be a Cauchy sequence in Ψ with $q_n(0,0) = 0$. Then for each $(s,t) \in B \times B$, $\{q_n(s,t)\}$ is a Cauchy squence in X. So there exists $q(s,t) \in X$ with $q(0,0) = 0$ such that $\{q_n(s,t)\}$ converges to $q(s,t)$ for each $(s,t) \in B \times B$, since X is complete. For a given $\varepsilon > 0$, there exists a positive integer N such that $\Delta(q_n, q_m) < \frac{\varepsilon}{4}$ for all $n,m \geqslant N$, since $\{q_n\}$ is a Cauchy sequence. So $\|q_n(s,t) - q_m(s,t)\| \leqslant \frac{\varepsilon}{2}\varphi(s,t,0)$ for all $n,m \geqslant N$ and all $(s,t) \in B \times B$. Thus

$$\begin{aligned}\|q_n(s,t) - q(s,t)\| &= \lim_{m\to\infty} \|q_n(s,t) - q_m(s,t)\| \\ &\leqslant \frac{\varepsilon}{2}\varphi(s,t,0)\end{aligned}$$

for all $n \geqslant N$ and all $(s,t) \in B \times B$. Hence

$$\Delta(q_n, q) \leqslant \frac{\varepsilon}{2} < \varepsilon$$

for all $n \geqslant N$. So (Ψ, Δ) is complete.

We define a mapping $S : \Psi \to \Psi$ such that

$$Sq(s,t) := \frac{1}{k^6} q(ks, kt)$$

for all $s,t \in B$ and all $q \in \Psi$. Let $q, r \in \Psi$ and $\beta \in [0, \infty)$ be an arbitrary constant with $\Delta(q,r) \leqslant \beta$, i.e.,

$$\|q(s,t) - r(s,t)\| \leqslant \beta \varphi(s,t,0)$$

for all $s,t \in B$. Hence we have

$$\begin{aligned} \|Sq(s,t) - Sr(s,t)\| &= \frac{1}{k^6} \|q(ks,kt) - r(ks,kt)\| \\ &\leqslant \frac{1}{k^6} \beta \varphi(ks,kt,0) \\ &\leqslant \beta L \varphi(s,t,0) \end{aligned}$$

for all $s,t \in B$ and all $q, r \in \Psi$. Therefore, we see that $\Delta(Sq, Sr) \leqslant L\Delta(q,r)$. This means that S is a strictly contractive operator on Ψ. It follows from (8) that $\Delta(p, Sp) \leqslant \frac{1}{2k^6}$. By Theorem 1, there exists a unique mapping $\delta : B \times B \to X$ satisfying the following conditions:

(i) δ is a unique fixed point of S, that is, $\delta(ks, kt) = k^6 \delta(s,t)$ for all $s,t \in B$. The δ is a unique fixed point of S in the set $\Omega = \{q \in \Psi \mid \Delta(p,q) < \infty\}$ in which there exists $\beta \in [0, \infty)$ such that for all $s,t \in B$,

$$\|p(s,t) - \delta(s,t)\| \leqslant \beta \varphi(s,t,0).$$

(ii) Since $\Delta(S^n, \delta) \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} S^n p(s,t) = \lim_{n\to\infty} \frac{p(k^n s, k^n t)}{k^{6n}} = \delta(s,t) \tag{9}$$

for all $s,t \in B$.

(iii) By (8),

$$\Delta(p, \delta) \leqslant \frac{1}{1-L} \Delta(p, Sp) \leqslant \frac{1}{2k^6(1-L)} \varphi(s,t,0),$$

which means the inequality (7) is valid. It follows from (4), (6) and (9) that

$$\begin{aligned} \left\|\Delta_{\lambda,\mu} \delta(s;t,u)\right\| &\leqslant \lim_{n\to\infty} \frac{1}{k^{6n}} \left\|\Delta_{\lambda,\mu} p(k^n s, k^n t, k^n u)\right\| \\ &\leqslant \lim_{n\to\infty} \frac{1}{k^{6n}} \varphi(k^n s, k^n t, k^n u) \\ &= 0 \end{aligned} \tag{10}$$

for all $s,t,u \in B$ and all $\lambda, \mu \in \Upsilon$. Therefore, there exists a bi-cubic mapping $\delta : B \times B \to X$ satisfying (7).

Now, it follows from (10) that $\Delta_{\lambda,\mu}\delta(s;t,u)=0$. If we put $\lambda=\mu=1$ and $u=0$ in (10), then we have $\delta(ks,kt)=k^6\delta(s,t)$ for all $s,t\in B$ and all $\lambda,\mu\in\Upsilon$. From the assumption that $p(\gamma a,\gamma b)$ is continuous in $\gamma\in\mathbb{R}$ for each fixed $a,b\in B$, by the same reasoning as in [1, Lemma 2.1],

$$\begin{aligned}\delta(\lambda s,\mu t)&=\delta\left(\frac{\lambda}{|\lambda|}|\lambda|s,\frac{\mu}{|\mu|}|\mu|t\right)\\&=\frac{\lambda^3\mu^3}{|\lambda|^3|\mu|^3}\delta(|\lambda|s,|\mu|t)\\&=\lambda^3\mu^3\delta(s,t)\end{aligned}$$

for all $s,t\in$ and all $\lambda,\mu\in\mathbb{C}$. Hence, δ is bi-cubic homogeneous. It follows from (5) and (9) that

$$\left\|p(st,u)-s^3p(t,u)-p(s,u)t^3\right\|\leqslant\varphi(s,t,u) \tag{$*$}$$

$$\left\|p(s,tu)-t^3p(s,u)-p(s,t)u^3\right\|\leqslant\varphi(s,t,u) \tag{$**$}$$

for all $s,t,u\in B$. By the inequality $(*)$, we get

$$\begin{aligned}&\left\|\delta(st,u)-s^3\delta(t,u)-\delta(s,u)t^3\right\|\\&=\lim_{n\to\infty}\frac{1}{k^{9n}}\left[p(k^{2n}st,k^nu)-k^{3n}s^3p(k^nt,k^nu)-p(k^ns,k^nu)k^{3n}t^3\right]\\&\leqslant\lim_{n\to\infty}\frac{1}{k^{9n}}\varphi(k^ns,k^nt,k^nu)=0\end{aligned}$$

for all $s,t,u\in B$. Thus we obtain $\delta(st,u)=s^3\delta(t,u)+\delta(s,u)t^3$ for all $s,t,u\in B$. Following the similar argument by $(**)$, we have $\delta(s,tu)=t^3\delta(s,u)+\delta(s,t)u^3$ for all $s,t,u\in B$. Thus δ is a cubic bi-derivation on B and δ satisfies (7). Hence the proof is complete. $\square$

COROLLARY 1. *Let $\varepsilon\in\mathbb{R}^+$ and $p:B\times B\to X$ with $p(0,0)=0$ be a mapping fulfilling*

$$\left\|p(st,u)-s^3p(t,u)-p(s,u)t^3\right\|+\left\|p(s,tu)-t^3p(s,u)-p(s,t)u^3\right\|\leqslant\varepsilon$$

and

$$\left\|\Delta_{\lambda,\mu}p(s;t,u)\right\|\leqslant\varepsilon$$

for all $s,t,u\in B$ and all $\lambda,\mu\in\Upsilon$ and $L<1$. If for each fixed $a,b\in B$, the mapping $\gamma\mapsto p(\gamma a,\gamma b)$ from $\mathbb{R}$ to X is continuous in $\gamma\in\mathbb{R}$, then there exists a unique cubic bi-derivation $\delta:B\times B\to X$ satisfying

$$\|p(s,t)-\delta(s,t)\|\leqslant\frac{\varepsilon}{k^6(1-L)}$$

for all $s,t\in B$ and $k\in\mathbb{Z}^+$ with $k\geqslant 2$.

THEOREM 4. *Let $\varphi : B^3 \to [0,\infty)$ be a mapping such that there exists $L<1$ with*

$$\varphi\left(\frac{s}{k^n},\frac{t}{k^n},\frac{u}{k^n}\right) \leqslant \frac{L}{k^{6n}}\varphi(s,t,u)$$

for all $s,t,u \in B$. Suppose that $p : B\times B \to X$ with $p(0,0)=0$ is a mapping satisfying (4) and (5). If for each fixed $a,b \in B$ the mapping $\gamma \mapsto p(\gamma a,\gamma b)$ from $\mathbb{R}$ to X is continuous in $\gamma \in \mathbb{R}$, then there exists a unique cubic bi-derivation $\delta : B\times B \to X$ satisfying the functional equation (3) and the following inequality

$$\|p(s,t)-\delta(s,t)\| \leqslant \frac{L}{2k^6(1-L)}\varphi(s,t,0)$$

for all $s,t \in B$ and $k \in \mathbb{Z}^+$ with $k \geqslant 2$.

Proof. Let (Ψ,Δ) be a generalized complete metric space as in proof of Theorem 3. Consider the mapping $S : \Psi \to \Psi$ defined by

$$Sq(s,t) := k^6 q\left(\frac{s}{k},\frac{t}{k}\right)$$

for all $s,t \in B$ and all $q \in \Psi$. Then we have $\Delta(Sq,Sr) \leqslant L\Delta(q,r)$ for all $q,r \in \Psi$. By (8)

$$\left\|p(s,t)-k^6 p\left(\frac{s}{k},\frac{t}{k}\right)\right\| \leqslant \frac{L}{2k^6}\varphi(s,t,0)$$

for all $s,t \in B$ and $k \in \mathbb{Z}^+$ with $k \geqslant 2$. Hence we obtain that $\Delta(Sp,p) \leqslant \frac{L}{2k^6} < \infty$. From Theorem 3, there exists a unique mapping δ which is a unique fixed point of S in the set $\Omega = \{q \in \Psi \mid \Delta(p,q) < \infty\}$ such that $\delta\left(\frac{s}{k},\frac{t}{k}\right) = k^6\delta(s,t)$ for all $s,t \in B$. Thus we get

$$\Delta(p,\delta) \leqslant \frac{1}{1-L}\Delta(p,Sp) \leqslant \frac{L}{2k^6(1-L)}\varphi(s,t,0).$$

The rest of the proof is similar to the proof of Theorem 3. □

Now by Theorem 4, we can consider the superstability of cubic bi-derivations as follows:

THEOREM 5. *Let $\tau,\ \theta \in \mathbb{R}^+$ with $\tau < 3$ and suppose that a function $\varphi : B^3 \to [0,\infty)$ satisfies*

$$\lim_{n\to\infty} \frac{1}{k^{9n}}\varphi(k^n s,k^n t,k^n u) = 0$$

for all $s,t,u \in B$. Assume that a mapping $p : B\times B \to X$ with $p(0,0)=0$ satisfies the following conditions:

$$\left\|p(st,u)-s^3p(t,u)-p(s,u)t^3\right\| + \left\|p(s,tu)-t^3p(s,u)-p(s,t)u^3\right\| \leqslant \varphi(s;t,u),$$

$$\left\|\Delta_{\lambda,\mu}p(s,t,u)\right\| \leqslant \theta\,\|s\|^\tau\,\|t\|^\tau \tag{11}$$

for all $s,t,u \in B$ and all $\lambda,\mu \in \Upsilon$, then p is a cubic bi-derivation.

Proof. If we put $\lambda = \mu = 1$ and $u = 0$ in (11), then we have

$$p(ks, kt) = k^6 p(s,t)$$

for all $s,t \in B$ and $k \in \mathbb{Z}^+$ with $k \geqslant 2$. From induction, we have

$$p(k^n s, k^n t) = k^{6n} p(s,t) \tag{12}$$

for all $s,t \in B$ and $n \in \mathbb{N}$. By Theorem 4, the mapping $\delta : B \times B \to X$ defined by

$$\delta(s,t) = \lim_{n\to\infty} \frac{1}{k^{6n}} p(k^n s, k^n t)$$

is a unique cubic bi-derivation. Hence it follows from (12) that $\delta(s,t) = p(s,t)$ for all $s,t \in B$. Thus p is a cubic bi-derivation. □

COROLLARY 2. *Suppose that τ_1, τ_2, θ are positive real numbers and $\tau_1 + \tau_2 < 6$. Let $p : B \times B \to X$ be a mapping and $\varphi : B^3 \to \mathbb{R}^+$ be a function such that $p(0,0) = 0$ and*

$$\left\|p(st,u) - s^3 p(t,u) - p(s,u)t^3\right\| + \left\|p(s,tu) - t^3 p(s,u) - p(s,t)u^3\right\| \leqslant \varphi(s,t,u),$$

$$\left\|\Delta_{\lambda,\mu} p(s,t,u)\right\| \leqslant \theta \|s\|^{\tau_1} \|t\|^{\tau_1} \|u\|^{\tau_2}$$

for all $s,t,u \in B$. Then p is a cubic bi-derivation.

Proof. If $\tau_2 = 0$, then we have the desired result by Theorem 5. □

COROLLARY 3. *Assume $\tau < 3$ and θ be positive real numbers. Suppose that a mapping $p : B \times B \to X$ with $p(0,0) = 0$ satisfies*

$$\left\|p(st,u) - s^3 p(t,u) - p(s,u)t^3\right\| + \left\|p(s,tu) - t^3 p(s,u) - p(s,t)u^3\right\| \leqslant \theta \|s\|^{\tau} \|t\|^{\tau},$$

$$\left\|\Delta_{\lambda,\mu} p(s;t,u)\right\| \leqslant \theta \|s\|^{\tau} \|t\|^{\tau}$$

for all $s,t,u \in B$. Then p is a cubic bi-derivation.

Proof. Let $\varphi(s,t,u) = \theta \|s\|^{\tau} \|t\|^{\tau}$. Then by Theorem 5 we obtain the result. □

4. Conclusion

Using the fixed point method, we investigated the Hyers-Ulam stability and also the superstability of cubic bi-derivations on Banach algebras.

Declarations

Availablity of data and materials. Not applicable.

Human and animal rights. We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

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