Journal of Mathematical Inequalities
Volume 18, Number 1 (2024), 69–77
doi:10.7153/jmi-2024-18-05

LOWER BOUNDS FOR THE BLOW–UP TIME IN A HIGHER–ORDER NONLINEAR KIRCHHOFF–TYPE EQUATION

YAOJUN YE

(*Communicated by M. Krnić*)

Abstract. This paper is concerned with a nonlinear higher-order Kirchhoff-type equation with dissipation in a bounded domain. By establishing a first order differential inequality technique, a lower bound for the blow-up time is obtained when the blow-up of solution occurs.

1. Introduction

In this paper, we consider the following initial-boundary value problem

$$u_{tt}+\left(\int_\Omega |D^m u|^2 dx\right)^q (-\Delta u)^m + au_t|u_t|^p = bu|u|^r,\ \ x\in\Omega,\ \ t>0, \tag{1.1}$$

$$u=\frac{\partial^i u}{\partial \nu^i}=0,\ \ i=1,2,\cdots,m-1,\ \ x\in\partial\Omega,\ \ t\geqslant 0, \tag{1.2}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in\Omega, \tag{1.3}$$

where $m\geqslant 1$ is a positive integer, $a,b,r>0$ and $p,q\geqslant 0$ are positive constants. $\Omega\subset \mathbf{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$, and $\frac{\partial^i u}{\partial \nu^i}$ denotes the i-order normal derivation of u. $Du=\nabla u=(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\cdots,\frac{\partial u}{\partial x_n})$, $D^m u=\Delta^j u$ for $m=2j$ and $D^m u = D\Delta^j u$ for $m=2j+1$.

In the case of $m=1$, the equation (1.1) becomes a nonlinear Kirchhoff-type equation

$$u_{tt}-\left(\int_\Omega |\nabla u|^2 dx\right)^q \Delta u + au_t|u_t|^p = bu|u|^r \tag{1.4}$$

for $x\in\Omega$, $t>0$. Many authors studied the global existence and asymptotic stability, and the local existence and blow-up of the solution to equation (1.4) with initial-boundary value conditions (see [12, 15, 16, 17, 22, 26]). Ono [17] obtained that the solution to this problem blows up if $r>\max\{p,2q\}$ and the initial energy is negative.

Mathematics subject classification (2020): 35L05, 35L25, 35L75.
Keywords and phrases: Nonlinear higher-order Kirchhoff-type equation, initial-boundary value problem, nonlinear dissipation, blow-up, lower bound for the blow-up time.
This research was supported by the Natural Science Foundation of Zhejiang Province (No. LY17A010009).

Paper JMI-18-05

By using the concavity argument, Zeng et al. [26] proved that the solution for this problem occurs blow-up in finite time with arbitrarily high energy.

When $m > 1$, equation (1.1) described vibrating beams of the Woinowsky-Krieger type with nonlinear dissipation effective in Ω, but without internal material dissipative term of the Kelvin-Voigt type [4, 8, 18]. Gao et al. [7] proved the local existence of solution to the problem (1.1)–(1.3), and gave the upper bounds for the blow-up time (see also [24]). Autuori et al. [2, 4] studied the asymptotic stability of solutions for equation (1.1) with homogeneous Dirichlet boundary condition (1.2). While, for the dual problem of non-continuation of local solutions, they present the global nonexistence of solutions and a priori estimates for the lifespan of maximal solutions ([3, 5]). In the absence of dissipation (i.e., $a = 0$), Galaktionov and Pohozave [6] obtained global existence and nonexistence of solutions for equation (1.1) with initial condition (1.3) in R^n. Under the assumptions of negative initial energy and $r \leqslant p$, Li [11] presented the global existence of solution for the problem (1.1)–(1.3). Meanwhile, he proved the solution blows up at finite time in L^{r+2} norms as $r > \max\{p, 2q\}$. Later, as the initial energy is positive, Messaoudi [13] gave the same result as the one in [11]. Moreover, the lifespan estimates on blow-up time were also established in [11, 13].

In [23], the author proves the global existence and nonexistence of the problem (1.1)–(1.3) and gives the decay of energy by applying the lemma of Komornik [10]. Later, under the condition of the positive initial energy, the blow-up solution in the finite time is studied and the lifespan estimate of solution is established (see [24]). Piskin [19] considers a class of system of nonlinear higher-order Kirchhoff-type equations, he proves the blow-up of solution with positive initial energy by using the technique of [21] with a modification in the energy functional due to the different nature of problems.

Yuksekkaya et al [25] deal with the higher-order Kirchhoff-type equation with delay term. They prove the global existence result of solution and discuss the decay of solution by using Nakao's technique [14]. Furthermore, the blow-up result is obtained for negative initial energy under appropriate conditions. Hesameddini [9] considers the higher-order Kirchhoff-type equation with a memory term. Under suitable conditions on relaxation function and the initial data, he proved that the solution blows up in the finite time and the lifespan estimates of solutions are also given.

The lower bound of blow-up time for nonlinear Kirchhoff equations is more difficult to find because the approach for dealing with parabolic equations can not be applied to the problem (1.1)–(1.3). As far as we know, there is no research on the lower bounds for the blow-up time of the problem (1.1)–(1.3). Motivated by the above researches, in this paper, we will focus on studying this question by combining the interpolation inequality with nonlinear estimates.

We denote the space $L^s(\Omega)$ norm by $\|\cdot\|_s$ and $\|\cdot\|$ denotes $L^2(\Omega)$ norm, the Sobolev space $H_0^m(\Omega)$ norm $\|\cdot\|_{H_0^m(\Omega)}$ is replaced by the equivalent norm $\|D^m\cdot\|$. Moreover, $C_i > 0$ $(i = 1, 2, 3, \cdots)$ denote some constants.

This work is organized as follows: In Section 2, some important Lemmas and known conclusions are given. Section 3 is devoted to the proof of the main results.

2. Preliminaries

Firstly, we define the energy function associated with the problem (1.1)–(1.3) by

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2(q+1)}\|D^m u\|^{2(q+1)}-\frac{b}{r+2}\|u\|_{r+2}^{r+2} \tag{2.1}$$

for $u\in H_0^m(\Omega)$, $t\geqslant 0$, and

$$E(0)=\frac{1}{2}\|u_1\|^2+\frac{1}{2(q+1)}\|D^m u_0\|^{2(q+1)}-\frac{b}{r+2}\|u_0\|_{r+2}^{r+2}$$

is the initial energy.

Secondly, we list up two useful lemmas as follows.

LEMMA 2.1. (Sobolev-Poincare inequality (see [1, 20])) *Let s be a number with $2\leqslant s<+\infty$, $n\leqslant 2m$ and $2\leqslant s\leqslant \frac{2n}{n-2m}$, $n>2m$. Then one has the inequality $\|u\|_s\leqslant B\|D^m u\|$ for $\forall u\in H_0^m(\Omega)$, where B is the best constant of Sobolev embedding $H_0^m(\Omega)\hookrightarrow L^s(\Omega)$, which depends on n,s and Ω.*

Finally, we present a local existence and blow-up result of solution to the problem (1.1)–(1.3) (see [7, 11, 13, 24]).

THEOREM 2.1. (Local existence) *Suppose that*

$$0<p<r<+\infty,\ n\leqslant 2m;\ 0<p<r\leqslant\frac{4m}{n-2m},\ n>2m. \tag{2.2}$$

If $(u_0,u_1)\in(H_0^m(\Omega)\cap H^{2m}(\Omega))\times H_0^m(\Omega)$ and $u_0\neq 0$, then the problem (1.1)–(1.3) *has a unique local solution $u(t)$ such that*

$$u\in C([0,T);H_0^m(\Omega)\cap H^{2m}(\Omega)),\ u_t\in C([0,T);L^2(\Omega))\cap L^{p+2}(\Omega\times[0,T)),$$

for $T>0$.

THEOREM 2.2. (Blow-up) *Assume that $r>2q$ and* (2.2) *are valid, if $u_0\in H_0^m(\Omega)$ and $u_1\in L^2(\Omega)$ satisfy (i) $E(0)<0$ or (ii) $0<E(0)<E_1$ and $\|D^m u_0\|>B^{-\frac{r+2}{r-2q}}$, then the solution $u(t)$ in Theorem 2.1 blows up in finite time T_* in the sense of L^{r+2} norm. In other words, that is $\lim\limits_{t\to T_*^-}\|u(t)\|_{r+2}=\infty$.*

3. Main results and their proof

This section will study the lower bounds estimates for the blow-up time of the problem (1.1)–(1.3).

THEOREM 3.1. *Under the assumptions of Theorem 2.1 and Theorem 2.2, if $u(t)$ is a solution of problem* (1.1)–(1.3)*, which blows up in a finite time T_*, then*

$$(i)\quad T_*>\int_{F(0)}^{\infty}\frac{dy}{C_2y^{\alpha}+by+(r+2)|E(0)|}\quad \textit{for}\quad 0<r\leqslant\frac{2m}{n-m},$$

$$(ii)\quad T_* > \int_{F(0)}^{\infty} \frac{dy}{C_3 y^{\beta} + C_4 y^{\frac{\beta}{q+1}} + by + C_5} \quad for \quad \frac{2m}{n-m} < r \leqslant \frac{2m}{n-2m},$$

where $F(0) = \|u_0\|_{r+2}^{r+2}$, *the positive constants* C_i $(i = 2,3,4,5)$ *and the exponents* α, β *will be determined in* (3.8), (3.11), (3.13) *and* (3.14).

Proof. Multiplying equation (1.1) by u_t and integrating over Ω, one has

$$\frac{d}{dt}E(u(t)) = -a\|u_t(t)\|_{p+2}^{p+2} \leqslant 0, \tag{3.1}$$

then $E(t)$ is a nonincreasing function on $t > 0$.

By (3.1) and (2.1), we get that

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2(q+1)}\|D^m u\|^{2(q+1)} - \frac{b}{r+2}\|u\|_{r+2}^{r+2} = E(t) \leqslant E(0),$$

which implies that

$$\|u_t\|^2 + \frac{1}{q+1}\|D^m u\|^{2(q+1)} \leqslant 2\left(|E(0)| + \frac{b}{r+2}\|u\|_{r+2}^{r+2}\right). \tag{3.2}$$

Let $F(t) = \int_\Omega |u|^{r+2}dx$, then

$$F'(t) = (r+2)\int_\Omega |u|^r u u_t dx \leqslant \frac{r+2}{2}\left(\int_\Omega |u|^{2(r+1)}dx + \int_\Omega |u_t|^2 dx\right). \tag{3.3}$$

(i) For $0 < r \leqslant \frac{2m}{n-m}$, we see that $\frac{nr}{m} < 2(r+1) < \frac{2n}{n-2m}$. By applying interpolation inequality, one has

$$\|u\|_{2(r+1)} \leqslant \|u\|_{\frac{nr}{m}}^{\theta}\|u\|_{\frac{2n}{n-2m}}^{1-\theta}, \tag{3.4}$$

where $\frac{1}{2(r+1)} = \frac{\theta}{\frac{nr}{m}} + \frac{1-\theta}{\frac{2n}{n-2m}}$. A direct calculation yields

$$\theta = \frac{\frac{1}{2(r+1)} - \frac{n-2m}{2n}}{\frac{m}{nr} - \frac{n-2m}{2n}} = \frac{r}{r+1}. \tag{3.5}$$

Therefore, it follows from (3.4), (3.5), Lemma 2.1 and Hölder inequality that

$$\begin{aligned}\int_\Omega |u|^{2(r+1)}dx = \|u\|_{2(r+1)}^{2(r+1)} &\leqslant \|u\|_{\frac{nr}{m}}^{2r}\|u\|_{\frac{2n}{n-2m}}^{2} \\ &\leqslant B^2|\Omega|^{\frac{2(m-n)(r+2)+4n}{n(r+2)}}\|u\|_{r+2}^{2r}\cdot\|D^m u\|^2 \\ &\leqslant C_1\|u\|_{r+2}^{\frac{2(q+1)r}{q}} + \frac{1}{q+1}\|D^m u\|^{2(q+1)},\end{aligned} \tag{3.6}$$

where $C_1 = \frac{qB^{\frac{2(q+1)}{q}}}{q+1}|\Omega|^{\frac{2(q+1)[(m-n)(r+2)+2n]}{nq(r+2)}}$, and we have used Young's inequality $XY \leqslant \frac{1}{\mu}X^\mu + \frac{1}{\nu}Y^\nu$ with $\mu = \frac{q+1}{q}$, $\nu = q+1$.

We conclude from (3.2), (3.3) and (3.6) that

$$\begin{aligned} F'(t) &\leqslant \frac{r+2}{2}\left[C_1\|u\|_{r+2}^{\frac{2(q+1)r}{q}} + \frac{1}{q+1}\|D^m u\|^{2(q+1)} + \|u_t\|^2\right] \\ &\leqslant \frac{r+2}{2}\left[C_1 F^\alpha(t) + \frac{2b}{r+2}F(t) + 2|E(0)|\right] \\ &= C_2F^\alpha(t) + bF(t) + (r+2)|E(0)|, \end{aligned} \tag{3.7}$$

where

$$C_2 = \frac{(r+2)C_1}{2}, \quad \alpha = \frac{2(q+1)r}{q(r+2)}. \tag{3.8}$$

(3.7) implies that

$$\frac{F'(t)}{C_1F^\alpha(t) + bF(t) + (r+2)|E(0)|} \leqslant 1. \tag{3.9}$$

Integrating both sides of (3.9) over $[0, T_*]$ on t, we yield that

$$T_* \geqslant \int_0^{T_*} \frac{1}{C_1F^\alpha(t) + bF(t) + (r+2)|E(0)|} d(F(t)). \tag{3.10}$$

By (3.10) and $\lim\limits_{t\to T_*^-} \|u(t)\|_{r+2}^{r+2} = \infty$ in Theorem 2.2, we obtain that

$$T_* > \int_{F(0)}^{\infty} \frac{dy}{C_2y^\alpha + by + (r+2)|E(0)|}.$$

(ii) When $\frac{2m}{n-m} < r \leqslant \frac{2m}{n-2m}$, we have $r+2 < 2(r+1) < \frac{2n}{n-2m}$. From interpolation inequality, we derive that

$$\|u\|_{2(r+1)} \leqslant \|u\|_{r+2}^{1-\vartheta}\|u\|_{\frac{2n}{n-2m}}^{\vartheta}, \tag{3.11}$$

where $\frac{1}{2(r+1)} = \frac{1-\vartheta}{r+2} + \frac{\vartheta}{\frac{2n}{n-2m}}$. Thus, we gain that

$$\vartheta = \frac{\frac{1}{r+2} - \frac{1}{2(r+1)}}{\frac{1}{r+2} - \frac{n-2m}{2n}} = \frac{nr}{(r+1)[2n-(n-2m)(r+2)]}.$$

By (3.11), we have

$$\begin{aligned} \int_\Omega |u|^{2(r+1)}dx &\leqslant B^{2\vartheta(r+1)}\|u\|_{r+2}^{2(r+1)(1-\vartheta)}\|D^m u\|^{2\vartheta(r+1)} \\ &\leqslant B^{2\vartheta(r+1)}(\|u\|_{r+2}^{r+2} + \|D^m u\|^2)^\beta, \end{aligned} \tag{3.12}$$

where

$$\beta = \frac{(r+1)(2+\vartheta r)}{r+2}. \tag{3.13}$$

From (3.2), (3.3) and (3.12), We conclude that

$$\begin{aligned}
F'(t) &\leqslant \frac{r+2}{2}\left[B^{2\vartheta(r+1)}(F(t)+\|D^m u\|^2)^\beta + \|u_t\|^2\right]\\
&\leqslant \frac{r+2}{2}\left[2^{\beta-1}B^{2\vartheta(r+1)}F^\beta(t) + 2^{\beta-1}B^{2\vartheta(r+1)}\|D^m u\|^{2\beta} + \|u_t\|^2\right]\\
&\leqslant \frac{r+2}{2}\left[2^{\beta-1}B^{2\vartheta(r+1)}F^\beta(t) + 2^{\beta-1}B^{2\vartheta(r+1)}\left(2(q+1)|E(0)| + \frac{2b(q+1)}{r+2}F(t)\right)^{\frac{\beta}{q+1}}\right.\\
&\quad \left. +2|E(0)| + \frac{2b}{r+2}F(t)\right]\\
&\leqslant \frac{r+2}{2}\left[2^{\beta-1}B^{2\vartheta(r+1)}F^\beta(t) + (q+1)^{\frac{\beta}{q+1}} 2^{\frac{(q+1)(\beta-2)+2\beta}{q+1}} B^{2\vartheta(r+1)}|E(0)|^{\frac{\beta}{q+1}}\right.\\
&\quad \left. +(q+1)^{\frac{\beta}{q+1}} 2^{\frac{(q+1)(\beta-2)+2\beta}{q+1}} b^{\frac{\beta}{q+1}}(r+2)^{-\frac{\beta}{q+1}} B^{2\vartheta(r+1)} F^{\frac{\beta}{q+1}}(t) + 2|E(0)| + \frac{2b}{r+2}F(t)\right]\\
&= C_3 F^\beta(t) + C_4 F^{\frac{\beta}{q+1}}(t) + bF(t) + C_5,
\end{aligned} \tag{3.14}$$

where

$$\begin{gathered}
C_3 = (r+2)2^{\beta-2}B^{2\vartheta(r+1)},\\
C_4 = (q+1)^{\frac{\beta}{q+1}} 2^{\frac{(q+1)(\beta-2)+2\beta-(q+1)}{q+1}} b^{\frac{\beta}{q+1}} (r+2)^{1-\frac{\beta}{q+1}} B^{2\vartheta(r+1)},
\end{gathered} \tag{3.15}$$

and

$$C_5 = (r+2)(q+1)^{\frac{\beta}{q+1}} 2^{\frac{(q+1)(\beta-2)+2\beta-(q+1)}{q+1}} B^{2\vartheta(r+1)} |E(0)|^{\frac{\beta}{q+1}} + (r+2)|E(0)|. \tag{3.16}$$

By $\lim_{t\to T_*^-} \|u(t)\|_{r+2}^{r+2} = \infty$ in Theorem 2.2, one has

$$T_* > \int_{F(0)}^{\infty} \frac{dy}{C_3 y^\beta + C_4 y^{\frac{\beta}{q+1}} + by + C_5}.$$

This completes the proof of Theorem 3.1. □

THEOREM 3.2. *Under the assumptions of Theorem 3.1, if r satisfies*

$$\frac{2m}{n-2m} < r < \frac{4m(n-m)}{n(n-2m)} \ for\ n > 2m, \tag{3.17}$$

then the blow-up time T_* has the following estimate

$$T_* > \int_{H(0)}^{\infty} \frac{dy}{C_9 y^\alpha + C_{10} y^\beta + C_{11}},$$

where $H(0) = \int_\Omega |u_0|^{\frac{2n-2m}{n-2m}} dx$, $\gamma = \frac{(r+2)(q+1)(1-\lambda)(n-2m)}{[2(q+1)-\lambda(r+2)](n-m)}$ *and* $\delta = \frac{n\gamma}{(q+1)(n-2m)}$. *The constants* $C_i > 0$ $(i = 9,10,11)$ *are defined by* (3.28) *and* (3.29).

Proof. According to (3.17), the interpolation inequality and Lemma 2.1, we receive that

$$\|u\|_{r+2}^{r+2} \leqslant \|u\|_{\frac{2n-2m}{n-2m}}^{(1-\lambda)(r+2)} \cdot \|u\|_{\frac{2n}{n-2m}}^{\lambda(r+2)} \leqslant B^{\lambda(r+2)} \|u\|_{\frac{2n-2m}{n-2m}}^{(1-\lambda)(r+2)} \cdot \|D^m u\|^{\lambda(r+2)}, \tag{3.18}$$

where $\frac{1}{r+2} = \frac{(n-2m)(1-\lambda)}{2n-2m} + \frac{\lambda(n-2m)}{2n}$. Thus, we get that $0 < \lambda = \frac{n[(n-2m)(r+2)-(2n-2m)]}{m(n-2m)(r+2)} < 1$. By $r < \frac{4m(n-m)}{n(n-2m)}$ in (3.18), we have $0 < \frac{\lambda(r+2)}{2} < 1$.

From (3.18) and Young's inequality, we obtain

$$\begin{aligned} b\|u\|_{r+2}^{r+2} &\leqslant \frac{2(q+1)-\lambda(r+2)}{2(q+1)} \left(bB^{\lambda(r+2)}\right)^{\frac{2(q+1)}{2(q+1)-\lambda(r+2)}} \|u\|_{\frac{2n-2m}{n-2m}}^{\frac{2(r+2)(q+1)(1-\lambda)}{2(q+1)-\lambda(r+2)}} + \frac{\lambda(r+2)}{2(q+1)} \|D^m u\|^{2(q+1)} \\ &\leqslant \frac{2(q+1)-\lambda(r+2)}{2(q+1)} \left(bB^{\lambda(r+2)}\right)^{\frac{2(q+1)}{2(q+1)-\lambda(r+2)}} \|u\|_{\frac{2n-2m}{n-2m}}^{\frac{2(r+2)(q+1)(1-\lambda)}{2(q+1)-\lambda(r+2)}} + \frac{1}{q+1} \|D^m u\|^{2(q+1)}. \end{aligned} \tag{3.19}$$

Combining (3.2) with (3.19) yields

$$\begin{aligned} b\|u\|_{r+2}^{r+2} \leqslant{}& \frac{2(q+1)-\lambda(r+2)}{2(q+1)} \left(bB^{\lambda(r+2)}\right)^{\frac{2(q+1)}{2(q+1)-\lambda(r+2)}} \left(\int_\Omega |u|^{\frac{2n-2m}{n-2m}} dx\right)^\gamma \\ &+ \frac{2b}{r+2} \|u\|_{r+2}^{r+2} + 2|E(0)|, \end{aligned} \tag{3.20}$$

where $\gamma = \frac{(r+2)(q+1)(1-\lambda)(n-2m)}{[2(q+1)-\lambda(r+2)](n-m)}$. According to (3.20), we can derive that

$$\frac{b}{r+2} \|u\|_{r+2}^{r+2} \leqslant C_6 \left(\int_\Omega |u|^{\frac{2n-2m}{n-2m}} dx\right)^\gamma + \frac{2}{r} |E(0)| \tag{3.21}$$

with $C_6 = \frac{2(q+1)-\lambda(r+2)}{2r(q+1)} \left(bB^{\lambda(r+2)}\right)^{\frac{2(q+1)}{2(q+1)-\lambda(r+2)}}$.

Let $H(t) = \int_\Omega |u|^{\frac{2n-2m}{n-2m}} dx$, then, from Young's inequality and Lemma 2.1, we have

$$\begin{aligned} H'(t) &= \frac{2n-2m}{n-2m} \int_\Omega |u|^{\frac{2m}{n-2m}} u u_t dx \leqslant \frac{n-m}{n-2m} \left(\int_\Omega |u|^{\frac{2n}{n-2m}} dx + \|u_t\|^2\right) \\ &\leqslant \frac{n-m}{n-2m} \left(B^{\frac{2n}{n-2m}} \|D^m u\|^{\frac{2n}{n-2m}} + \|u_t\|^2\right). \end{aligned} \tag{3.22}$$

By (3.2) and (3.21), one has

$$\begin{aligned} \|D^m u\|^{\frac{2n}{n-2m}} &\leqslant \left(2(q+1)C_6 H^\gamma(t) + \frac{4(q+1)}{r} |E(0)| + 4(q+1)|E(0)|\right)^{\frac{n}{(q+1)(n-2m)}} \\ &\leqslant C_7 H^\delta(t) + C_8 \end{aligned} \tag{3.23}$$

and

$$\|u_t\|^2 \leqslant 2C_6 H^\gamma(t) + 4r^{-1}|E(0)| + 2|E(0)|, \tag{3.24}$$

where $\delta = \frac{n\gamma}{(q+1)(n-2m)}$,

$$C_7 = 2^{\frac{2m(q+1)-n(q-1)}{(q+1)(n-2m)}} \left[(q+1)C_6\right]^{\frac{n}{(q+1)(n-2m)}}, \tag{3.25}$$

$$C_8 = 2^{\frac{3n-(q+1)(n-2m)}{(q+1)(n-2m)}} \left[(q+1)r^{-1}|E(0)| + (q+1)|E(0)|\right]^{\frac{n}{(q+1)(n-2m)}}. \tag{3.26}$$

We conclude from (3.22)–(3.24) that

$$H'(t) \leqslant C_9 H^\gamma(t) + C_{10} H^\delta(t) + C_{11}, \tag{3.27}$$

where

$$C_9 = \frac{2(n-m)}{(n-2m)} C_6, \quad C_{10} = \frac{n-m}{n-2m} B^{\frac{2n}{n-2m}} \tag{3.28}$$

and

$$C_{11} = \frac{n-m}{n-2m}\left[B^{\frac{2n}{n-2m}} C_8 + \frac{4}{r}|E(0)| + 2|E(0)|\right]. \tag{3.29}$$

Then, it follows from (3.21) and Theorem 2.2 that

$$\lim_{t \to T_*} \int_\Omega |u|^{\frac{2n-2m}{n-2m}} dx = +\infty. \tag{3.30}$$

By (3.27) and (3.30), we obtain that

$$T_* > \int_{H(0)}^{\infty} \frac{dy}{C_9 y^\gamma + C_{10} y^\delta + C_{11}}.$$

The proof of Theorem 3.2 is finished. □

REFERENCES

[1] R. A. ADAMS AND J. J. F. FOURNIER, *Sobolev Spaces*, Academic Press, 2003.

[2] G. AUTUORI AND P. PUCCI, *Local asymptotic stability for polyharmonic Kirchhoff systems*, Appl. Anal., Special Volume dedicated to Prof. P. L. Butzer, **90** (2011), 493–514.

[3] G. AUTUORI, F. COLASUONNO AND P. PUCCI, *Lifespan estimates for solutions of polyharmonic Kirchhoff systems*, Math. Models Methods Appl. Sci., **22** (2012), 1–36.

[4] G. AUTUORI, P. PUCCI AND M. C. SALVATORI, *Asymptotic stability for nonlinear Kirchhoff systems*, Nonlinear Anal., RWA., **10** (2009), 889–909.

[5] G. AUTUORI, P. PUCCI AND M. C. SALVATORI, *Global Nonexistence for Nonlinear Kirchhoff Systems*, Archive Rat. Mech. Anal., **196** (2010), 489–516.

[6] V. A. GALAKTIONOV AND S. I. POHOZAEV, *Blow-up and critical exponents for nonlinear hyperbolic equations*, Nonlinear Anal., **53** (2003), 453–466.

[7] Q. GAO, F. LI AND Y. WANG, *Blow up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation*, Cent. Eur. J. Math., **9** (2011), 686–698.

[8] G. C. GORAIN, *Exponential energy decay estimates for the solutions of n-dimensional Kirchhoff type wave equation*, Appl. Math. Comput., **177** (2006), 235–242.

[9] E. HESAMEDDINI, *Blow-up of the Solution for Higher-Order Integro-Differential Equation with Nonlinear Dissipation*, Applied Mathematical Sciences, **5** (2011), 3575–3583.
[10] V. KOMORNIK, *Exact Controllability and Stabilization*, The Multiplier Method, Masson, Paris, 1994.
[11] F. C. LI, *Global existence and blow-up of solutions for a higher-order kirchhoff-type equation with nonlinear dissipation*, Appl. Math. Letters, **17** (2004), 1409–1414.
[12] T. MATSUYAMA AND R. IKEHATA, *On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms*, J. Math. Anal. Appl., **204** (1996), 729–753.
[13] S. A. MESSAOUDI AND B. SAID HOUARI, *A blow-up result for a higher-order nonlinear Kirchhoff-type hyperbolic equation*, Appl. Math. Lett., **20** (2007), 866–871.
[14] N. NAKAO, *A difference inequality and its application to nonlinear evolution equations*, J. Math. Soc. Japan, **30** (1978), 747–762.
[15] K. ONO, *On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type*, Math. Meth. Appl. Sci., **20** (1997), 151–177.
[16] K. ONO, *On global existence, decay and blow-up of solutions for some mildly degenerate Kirchhoff strings*, J. Differential Equations, **137** (1997), 273–301.
[17] K. ONO, *On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation*, J. Math. Anal. Appl., **216** (1997), 321–342.
[18] V. PATA AND S. ZELIK, *Smooth attractors for strongly damped wave equations*, Nonlinearity, **19** (2006), 1495–1506.
[19] E. PISKIN, *Blow up of Solutions for a System of Nonlinear Higher-order Kirchhoff-type Equations*, Mathematics and Statistics, **6** (2014), 219–229.
[20] E. PISKIN, *Sobolev Spaces*, Sekin Publishing, 2017.
[21] E. VITILLARO, *Global nonexistence theorems for a class of evolution equations with dissipation*, Arch. Ration. Mech. Anal., **149** (1999), 155–182.
[22] S. T. WU AND L. Y. TSAI, *Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with some dissipation*, Nonlinear Anal., TMA., **65** (2006), 243–264.
[23] Y. J. YE, *Global existence and energy decay estimate of solutions for a higher-order Kirchhoff type equation with damping and source term*, Nonlinear Analysis: RWA **14** (2013) 2059–2067.
[24] Y. J. YE AND X. X. TAO, *Initial Boundary Value Problem for Higher-order Nonlinear Kirchhoff-type Equation*, ACTA Mathematica Sinica (Chinese Series), **62** (2019), 923–938.
[25] H. YUKSEKKAYA, E. PISKIN, S. M. BOULAARAS AND B. B. CHERIF, *Existence, Decay, and Blow-Up of Solutions for a Higher-Order Kirchhoff-Type Equation with Delay Term*, Journal of Function Spaces, **2021** (2021), 11 pages, 866–871, https://doi.org/10.1155/2021/4414545.
[26] R. ZENG, C. L. MU AND S. M. ZHOU, *A blow-up result for Kirchhoff-type equations with high energy*, Math. Methods Appl. Sci., **34** (2011), 479–486.

(Received March 29, 2022)

Yaojun Ye
Department of Mathematics and Statistics
Zhejiang University of Science and Technology
Hangzhou 310023, P. R. China
e-mail: yjye2013@163.com