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ANALYTIC INEQUALITIES INVOLVING WEIGHTED EXPONENTIAL ψ–BETA FUNCTIONS AND APPLICATIONS

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Abstract. Integral inequalities are the proficient aspect of mathematical analysis. Various techniques have been deployed to acquire to fresh inequalities which are beneficial in various area problems. The aim of this paper is to derive some new analytic inequalities involving generalized weighted exponential beta functions. To attain our primary objectives, we introduce the generalized exponential function $X_{\rho,\flat,\delta}(\varpi)$ and weighted form of exponential beta functions $\mathscr{F}(\rho,\flat,\delta)$. Furthermore, we briefly discuss their properties. we derive several inequalities in association with $X_{\rho,\flat,\delta}(\varpi)$ and $\mathscr{F}(\rho,\flat,\delta)$. As the applications of these new developments ,we conclude some error estimates of Ostrwoski's type inequalities, which show the significance of the obtained results.

1. Introduction and preliminaries

Special functions are particular mathematical functions that have significant role in different fields of pure and applied sciences. Particularly they play vital role in differential equations. The history of special functions is as old as the history of calculus. Today we are familiar with variety of special functions. Gamma and beta functions are one of the most studied basic special functions which were introduced and studied by Euler. Euler also investigated zeta functions, but it was studied extensively by Riemann. Bernoulli defined another special function that is called Bessel functions. Legendre functions were found in late 1700. Gauss unified several special functions with the introduction of Gauss hypergeometric functions. For more details regarding special functions, their properties and applications, see [2].

Now we recall the notions of convex functions.

Let $f:[a,b]\to\mathbb{R}$ is said to be a convex mapping if

$$X((1-\tau)a_1+\tau a_2)\leqslant(1-\tau)X(a_1)+\tau X(a_2),\quad \forall a_1,a_2\in[a,b] \tag{1}$$

where $\tau\in[0,1]$. Next, we present the notion of logarithmic convex functions and which is demonstrated as: Let $f:[a,b]\to\mathbb{R}$ is said to be a logarithmic convex mapping if

$$X((1-\tau)a_1+\tau a_2)\leqslant[X(a_1)]^{(1-\tau)}[X(a_2)]^{\tau},\quad \forall a_1,a_2\in[a,b] \tag{2}$$

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where $\tau \in [0,1]$.

The main motivation of this paper is to introduce and discuss the properties of a four parameter family of functions. Using this function, we define weighted exponential ψ-beta function and discussed its properties. We also present applications of the main results.

We now recall the three parameter family of functions which was introduced and studied by Dragomir and Khosrowshahi in [11].

$$X_{\rho,\flat,\delta}(\varpi) := \exp[\delta\varpi^{\rho}(1-\varpi)^{\flat}], \quad \varpi \in [0,1], \ \ \rho,\flat,\delta \geqslant 0,$$

and they then defined the weighted exponential beta functions by the integral

$$\mathscr{F}(\rho,\flat,\delta) := \int_0^1 \exp[\delta\varpi^{\rho}(1-\varpi)^{\flat}]\mathrm{d}\varpi, \quad \rho,\flat,\delta \geqslant 0.$$

We have the following representation for the generating function $X_{\rho,\flat,\delta}$:

$$X_{\rho,\flat,\delta}(\varpi) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\delta^{n}\varpi^{n\rho}(1-\varpi)^{n\flat},$$

with uniform convergence on $[0,1]$.

As usual, the standard beta function is defined by

$$B(x,y) := \int_0^1 \tau^{x-1}(1-\tau)^{y-1}\mathrm{d}\tau, \ \ x>0, \ y>0,$$

and the k-beta function is defined by [21]

$$B_k(x,y) = \frac{1}{k}\int_0^1 \tau^{\frac{x}{k}-1}(1-\tau)^{\frac{y}{k}-1}, \ \ k>0, \ x>0, \ y>0. \tag{3}$$

The following auxiliary results will play significant role in the development of some of our main results.

LEMMA 1. ([14]) *Let $X : [a_1,a_2] \to \mathbb{R}$ be a continuous and differentiable mapping on (a_1,a_2), whose derivative is bounded on (a_1,a_2) and*

$$\|X'\|_{\infty,(a_1,a_2)} := \sup_{\tau\in(a_1,a_2)} |X'(\tau)| < \infty,$$

then

$$\left|X(\varpi) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} X(\tau)\mathrm{d}\tau\right| \leqslant \left[\frac{1}{4} + \frac{\left(\varpi - \frac{a_1+a_2}{2}\right)^2}{(a_2-a_1)^2}\right](a_2-a_1)\|X'\|_{\infty,(a_1,a_2)}$$

for all $\varpi \in [a_1,a_2]$. Further, the constant $\frac{1}{4}$ is sharp.

LEMMA 2. ([9, 6]) *Let* $X : [a_1,a_2] \to \mathbb{R}$ *be an absolutely continuous mapping on* $[a_1,a_2]$*, then*

$$\left|X(\varpi) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} X(\tau)\mathrm{d}\tau\right| \leqslant \left[\frac{1}{2} + \frac{\left|\varpi - \frac{a_1+a_2}{2}\right|}{a_2-a_1}\right] \|X'\|_{[a_1,a_2],1}$$

for all $\varpi \in [a_1,a_2]$*, where* $\|.\|_1$ *is the Lebesgue norm on* $Ł_1[a_1,a_2]$ *defined as* $\|X'\|_{[a_1,a_2],1} := \int_{a_1}^{a_2} |X(\tau)|\mathrm{d}\tau$.

LEMMA 3. ([7]) *Let* $X : [a_1,a_2] \to \mathbb{R}$ *be an absolutely continuous mapping on* $[a_1,a_2]$*. If* $|X'|^q \in Ł_p[a_1,a_2]$*, then*

$$\begin{aligned}&\left|X(\varpi) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} X(\tau)\mathrm{d}\tau\right| \\ &\leqslant \frac{1}{(1+q)^{\frac{1}{q}}}\left[\left(\frac{\varpi-a_1}{a_2-a_1}\right)^{q+1} + \left(\frac{a_2-\varpi}{a_2-a_1}\right)^{q+1}\right]^{\frac{1}{q}} (a_2-a_1)^{\frac{1}{q}} \|X'\|_{[a_1,a_2],p}\end{aligned}$$

for all $\varpi \in [a_1,a_2]$*, where* $p,q > 1$ *are such that* $\frac{1}{p}+\frac{1}{q} = 1$ *and* $\|.\|_{[a_1,a_2],p}$ *is the* p*-Lebesgue norm on* $Ł_p[a_1,a_2]$ *defined by* $\|X'\|_{[a_1,a_2],p} := \left(\int_{a_1}^{a_2} |X(\tau)|^p \mathrm{d}\tau\right)^{\frac{1}{p}}$.

For more details about the previous lemmas, see [6, 7, 11, 13, 14, 16].

Dragomir and Khosroshahi [12] introduced the concepts of exponential form of beta functions and investigated its key properties. In [3] authors studied the applications of beta functions in probability theory. De sole and Kac [5] derived the integral representations of q analogs of gamma and beta functions and provided the proof of Jacobi's identities for triple product and Ramanujan formula for bilateral hypergeometric series. In [17] Miller formulated the integral representations of generalized beta functions in terms of Whittaker functions. Mohammed [20] utilized the generalized beta functions and preinvex mappings to establish some crucial integral inequalities. Abubakar and Patel [1] introduced the new generalized beta functions based on Wright functions and demonstrated its applications as well. In [4] authors provided a new version of probability density functions involving new generalized beta functions.

2. Results and discussions

In this section, we will discuss our main results. We preserve the same notations as in the previous section. We divide this section into two subsections.

2.1. Some preliminary notions and results

We first introduce some new notions and their related results. We begin by stating the following central definition.

DEFINITION 1. Let $\varpi \in [0,1]$, $\rho, \flat, \delta \geqslant 0$ and $\psi \geqslant 1$. We define:
(i) The four parameters family of functions by

$$X_{\rho,\flat,\delta,\psi}(\varpi) = \exp\Big[\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\Big].$$

(ii) The weighted exponential ψ-beta function $\mathscr{F}_\psi : (0,\infty)^2 \times (0,\infty) \to (0,\infty)$ as

$$\mathscr{F}_\psi(\rho,\flat;\delta) = \int_0^1 X_{\rho,\flat,\delta,\psi}(\varpi)\mathrm{d}\varpi = \int_0^1 \exp\Big[\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\rho}{\psi}}\Big]\mathrm{d}\varpi. \tag{4}$$

With the above, the following lemma may be stated.

LEMMA 4. *The function* $\varpi \longmapsto X_{\rho,\flat,\delta,\psi}(\varpi)$ *is monotonically increasing on* $\left[0, \frac{\rho}{\rho+\flat}\right]$ *and decreasing on* $\left[\frac{\rho}{\rho+\flat}, 1\right]$, *and*

$$\max_{\varpi\in[0,1]} X_{\rho,\flat,\delta,\psi}(\varpi) = X_{\rho,\flat,\delta,\psi}\left(\frac{\rho}{\rho+\flat}\right) = \exp\left[\frac{\delta}{\psi}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right]. \tag{5}$$

Proof. From $X_{\rho,\flat,\delta,\psi}(\varpi) = \exp[r_{\rho,\flat}(\varpi)]$, with $r_{\rho,\flat}(\varpi) = \frac{\delta}{\psi}[\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}]$, $\forall \varpi \in [0,1]$, we have

$$X'_{\rho,\flat,\delta,\psi}(\varpi) = r'_{\rho,\flat}(\varpi) X_{\rho,\flat,\delta,\psi}(\varpi),\ \forall \varpi \in [0,1]. \tag{6}$$

Thus $r'_{\rho,\flat}(\varpi)$ and $X'_{\rho,\flat,\delta,\psi}(\varpi)$ have the same sign on $[0,1]$. Moreover we have

$$r'_{\rho,\flat}(\varpi) = \frac{\delta}{\psi}\left[\frac{\rho}{\psi}\varpi^{\frac{\rho}{\psi}-1}(1-\varpi)^{\frac{\flat}{\psi}} - \frac{\flat}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}-1}\right],$$

or, equivalently,

$$r'_{\rho,\flat}(\varpi) = \frac{\delta}{\psi^2}\varpi^{\frac{\rho}{\psi}-1}(1-\varpi)^{\frac{\flat}{\psi}-1}[\rho - (\rho+\flat)\varpi],\ \forall \varpi \in [0,1], \tag{7}$$

This implies that $r'_{\rho,\flat}(\varpi) \geqslant 0$ for $\varpi \in \left[0, \frac{\rho}{\rho+\flat}\right]$ and $r'_{\rho,\flat}(\varpi) \leqslant 0$ for $\varpi \in \left[\frac{\rho}{\rho+\flat}, 1\right]$, which proves our result. □

LEMMA 5. *Let* $\rho, \flat > 1$, $\delta \geqslant 0$ *and* $\psi \geqslant 1$. *Assume that* $\psi < \min\{\rho,\flat\}$, *then we have*

$$\sup_{\varpi\in(0,1)} |r'_{\rho,\flat}(\varpi)| \leqslant \frac{\delta}{\psi^2}\max\{\rho,\flat\}\frac{\left(\frac{\rho}{\psi}-1\right)^{\frac{\rho}{\psi}-1}\left(\frac{\flat}{\psi}-1\right)^{\frac{\flat}{\psi}-1}}{\left(\frac{\rho+\flat}{\psi}-2\right)^{\frac{\rho+\flat}{\psi}-2}}. \tag{8}$$

Proof. Following (7) we can write

$$\left|r'_{\rho,\flat}(\varpi)\right| \leqslant \frac{\delta}{\psi^2} \sup_{\varpi\in(0,1)} \left(\varpi^{\frac{\rho}{\psi}-1}(1-\varpi)^{\frac{\flat}{\psi}-1}\right) \max_{\varpi\in[0,1]} |\rho-(\rho+\flat)\varpi|,$$

and hence

$$\left|r'_{\rho,\flat}(\varpi)\right| \leqslant \frac{\delta}{\psi^2} \max\{\rho,\flat\} \sup_{\varpi\in(0,1)} \left(\varpi^{\frac{\rho}{\psi}-1}(1-\varpi)^{\frac{\flat}{\psi}-1}\right). \tag{9}$$

If for $a,b \geqslant 0$ we set $f(\varpi)=\varpi^a(1-\varpi)^b$, $\varpi\in(0,1)$, and we study the variations of f, it is easy to check that

$$\sup_{\varpi\in(0,1)} f(\varpi) = f\left(\frac{a}{a+b}\right) = \frac{a^a b^b}{(a+b)^{a+b}}.$$

This, when combined with (9), gives (8) after simple algebraic operations, so completes the proof. □

We now give Taylor's type representation for $X_{\rho,\flat,\delta,\psi}(\varpi)$.

PROPOSITION 1. *Let $\rho,\flat,\delta \geqslant 0$, $\psi\geqslant 1$ then, for all $\varpi\in[0,1]$ and $n\geqslant 1$, we have*

$$X_{\rho,\flat,\delta,\psi}(\varpi) = 1+\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}} + \frac{1}{n!}\left[\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right)\right]^{n+1}\int_0^1 \exp\left[s\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right](1-s)^n \mathrm{d}s. \tag{10}$$

Proof. Let $I\subset\mathbb{R}$ be a nonempty closed interval, $c\in I$ and let n be a positive integer. If $X: I\to\mathbb{C}$ is such that the n-th derivative $X^{(n)}$ is absolutely continuous on I, then for each $y\in[0,1]$, $X(y)=T_n(X;c,y)+R_n(X;c,y)$, where

$$T_n(X;c,y)=\sum_{m=0}^{n}\frac{(y-c)^m}{m!}X^{(m)}(c),$$

is the Taylor's polynomial in y, with the remainder given by:

$$R_n(X;c,y)=\frac{1}{n!}\int_c^y (y-\tau)^n X^{(n+1)}(\tau)\mathrm{d}\tau.$$

Now

$$X(y)=\sum_{m=0}^{n}\frac{(y-c)^m}{m!}X^{(m)}(c)+\frac{1}{n!}\int_c^y (y-\tau)^n X^{(n+1)}(\tau)\mathrm{d}\tau.$$

Making the change of variable $\tau=(1-s)c+sy, s\in[0,1]$, we get

$$X(y)=\sum_{m=0}^{n}\frac{(y-c)^m}{m!}X^{(m)}(c)+\frac{(y-c)^{n+1}}{n!}\int_0^1(1-s)^nX^{(n+1)}\big((1-s)c+sy\big)\mathrm{d}s.$$

Applying this to $X(y)=\exp y$ at $c=0$, with $y=\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}$, we obtain

$$\begin{aligned}&\exp\left[\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right]\\&=\sum_{m=0}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m\varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}}\\&\quad+\frac{1}{n!}\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right)^{n+1}\int_0^1\exp\left[s(\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}})\right](1-s)^n\mathrm{d}s,\end{aligned}$$

which is the desired result, so completing the proof. □

COROLLARY 1. *Let* $\rho,\flat,\delta\geqslant 0$ *and* $\psi\geqslant 1$. *For all* $\varpi\in[0,1]$ *we have*

$$X_{\rho,\flat,\delta,\psi}(\varpi)=1+\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m\varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}},\tag{11}$$

with uniform convergence on $[0,1]$.

Proof. It follows from (10) with some elementary techniques of Real Analysis. The details are simple and therefore omitted here. □

The following result concerns a Taylor's type expansion for $\mathscr{F}_{\psi}(\rho,\flat;\delta)$.

PROPOSITION 2. *For* $n\geqslant 1$ *and for any* $\rho,\flat,\delta\geqslant 0$, $\psi\geqslant 1$, *we have*

$$\begin{aligned}&\mathscr{F}_{\psi}(\rho,\flat;\delta)\\&=1+\sum_{m=1}^{n}\frac{1}{m!}\frac{\delta^m}{\psi^{m-1}}B_{\psi}\big(m\rho+\psi,m\flat+\psi\big)\\&\quad+\frac{1}{n!}\int_0^1\left(\int_0^1\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right)^{n+1}\exp[s\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}]\mathrm{d}\varpi\right)(1-s)^n\mathrm{d}s.\end{aligned}$$

Proof. Integrating (10) with respect to $\varpi\in[0,1]$, we get the announced result. The proof is straightforward. □

As for Corollary 1, the following result is immediate from Proposition 2.

COROLLARY 2. *For all* $\rho,\flat,\delta\geqslant 0$, *we have the* ψ*-beta Taylor series expansion*

$$\mathscr{F}_{\psi}(\rho,\flat;\delta)=1+\sum_{m=1}^{\infty}\frac{1}{m!}\frac{\delta^m}{\psi^{m-1}}B_{\psi}\big(m\rho+\psi,m\flat+\psi\big),\tag{12}$$

with uniform convergence on $[0,1]$.

2.2. The main results

In this section we will derive some analytic inequalities for $X_{\rho,\flat,\delta,\psi}$. Some inequalities about $\mathscr{F}_{\psi}(\rho,\flat;\delta)$ will be stated in a parallel manner.

Our first main result reads as follows.

THEOREM 1. *Let $\rho,\flat,\delta \geqslant 0$ and $\psi \geqslant 1$. For any $p,q>1$ with $\frac{1}{p}+\frac{1}{q}$, we have*

$$0 \leqslant X_{\rho,\flat,\delta,\psi}(\varpi)-1 \leqslant \left(\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho p}{\psi}}\right)-1\right)^{\frac{1}{p}}\left(\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{q\flat}{\psi}}-1\right)\right)^{\frac{1}{q}} \quad (13)$$

for all $\varpi \in [0,1]$. Particularly, we have

$$0 \leqslant \left(X_{\rho,\flat,\delta,\psi}(\varpi)-1\right)^2 \leqslant \left(\exp\left(\frac{\delta}{\psi}\varpi^{\frac{2p}{\psi}}\right)-1\right)\left(\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{2q}{\psi}}-1\right)\right) \quad (14)$$

for all $\varpi \in [0,1]$.

Proof. The Hölder's discrete inequality tells us that we have

$$0 \leqslant \sum_{m=1}^{n} r_m a_{1m} a_{2m} \leqslant \left(\sum_{m=1}^{n} r_m a_{1m}^{p}\right)^{\frac{1}{p}}\left(\sum_{m=1}^{n} r_m a_{2m}^{q}\right)^{\frac{1}{q}}, \quad (15)$$

whenever $r_m, a_{1m}, a_{2m} > 0$, $m \geqslant 1$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Taking $r_m = \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m$, $a_{1m} = \varpi^{\frac{\rho m}{\psi}}$ and $a_{2m} = (1-\varpi)^{\frac{\flat m}{\psi}}$, we have

$$\begin{aligned} 0 &\leqslant \sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}} \\ &\leqslant \left(\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m p}{\psi}}\right)^{\frac{1}{p}}\left(\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m (1-\varpi)^{\frac{\flat m q}{\psi}}\right)^{\frac{1}{q}}, \end{aligned} \quad (16)$$

for all $m \geqslant 1$ and $\varpi \in [0,1]$. The series

$$\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}}, \quad \sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho p m}{\psi}},$$

and

$$\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m (1-\varpi)^{\frac{\flat q m}{\psi}}$$

are convergent, with

$$\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}} = \exp\left[\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right]-1 = X_{\rho,\flat,\delta,\psi}(\varpi)-1, \quad (17)$$

$$\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^{m}\varpi^{\frac{\rho pm}{\psi}}=\exp\left[\frac{\delta}{\psi}\varpi^{\frac{\rho p}{\psi}}\right]-1, \tag{18}$$

and

$$\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^{m}(1-\varpi)^{\frac{\flat qm}{\psi}}=\exp\left[\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat q}{\psi}}\right]-1. \tag{19}$$

Now letting $n\to\infty$ in (16) and substituting (17), (18) and (19) therein we get (13), so completing the proof. □

COROLLARY 3. *Let* $\rho,\flat,\delta\geqslant 0$ *with* $\psi\geqslant 1$*. For any* $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, we have*

$$0\leqslant[\mathscr{F}_{\psi}(\rho,\flat;\delta)-1]\leqslant[\mathscr{F}_{\psi}(p\rho,0;\delta)-1]^{\frac{1}{p}}[\mathscr{F}_{\psi}(0,q\flat;\delta)-1]^{\frac{1}{q}}.$$

Particularly, we have

$$\left(\mathscr{F}_{\psi}(\rho,\flat;\delta,)-1\right)^{2}\leqslant[\mathscr{F}_{\psi}(2\rho,0;\delta)-1][\mathscr{F}_{\psi}(0,2\flat;\delta)-1].$$

Proof. Integrating (13) with respect to $\varpi\in[0,1]$ we get

$$\begin{aligned}&0\leqslant\mathscr{F}_{\psi}(\rho,\flat;\delta)(\varpi)-1\\&\leqslant\int_{0}^{1}\left(\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho p}{\psi}}\right)-1\right)^{\frac{1}{p}}\left(\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{q\flat}{\psi}}-1\right)\right)^{\frac{1}{q}}d\varpi,\end{aligned}$$

which, with the standard Hölder's integral inequality, implies that

$$\mathscr{F}_{\psi}(\rho,\flat;\delta)-1\leqslant\left[\int_{0}^{1}\left(\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho p}{\psi}}\right)-1\right)d\varpi\right]^{\frac{1}{p}}\times\left[\int_{0}^{1}\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{q\flat}{\psi}}-1\right)d\varpi\right]^{\frac{1}{q}},$$

hence the desired result. □

Our second main result is recited in the following.

THEOREM 2. *Let* $\rho,\flat,\delta\geqslant 0$ *and* $\psi\geqslant 1$*. Then for all* $\varpi\in[0,1]$*, we have*

$$\begin{aligned}&\frac{1}{\exp(\frac{\delta}{\psi})-1}\left[\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}\right)-1\right]\left[\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}}\right)-1\right]\\&\leqslant X_{\rho,\flat,\delta,\psi}-1\\&\leqslant\frac{1}{\exp(\frac{\delta}{\psi})-1}\left[\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}\right)-1\right]\left[\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}}\right)-1\right]\\&\quad+\frac{1}{4}\left[\exp\left(\frac{\delta}{\psi}\right)-1\right]\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}.\end{aligned} \tag{20}$$

Proof. Let r_m, a_{1m}, a_{2m} be as in the proof of the previous theorem.

• It is obvious that the sequences $(a_{1m})_m$ and $(a_{2m})_m$ are both monotonic decreasing. According to the Chebychev discrete inequality with the weight $r_m \geqslant 0$ [15] we have

$$\sum_{m=1}^{n} r_m a_{1m} \sum_{m=1}^{n} r_m a_{2m} \leqslant \sum_{m=1}^{n} r_m \sum_{m=1}^{n} r_m a_{1m} a_{2m},$$

which, when replacing r_m, a_{1m} and a_{2m} by their expressions, yields

$$\begin{aligned} &\sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}} \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m (1-\varpi)^{\frac{\flat m}{\psi}} \\ &\leqslant \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}} (1-\varpi)^{\frac{\flat m}{\psi}}. \end{aligned} \tag{21}$$

Using (17), (18), (19) and

$$\sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m = \exp\left(\frac{\delta}{\psi}\right) - 1,$$

and then letting $n \to \infty$ in (21), we get the left inequality in (20).

• Now the Grüss discrete inequality [8] tells us that we have

$$\begin{aligned} &\left| \sum_{m=1}^{n} r_m \sum_{m=1}^{n} r_m a_{1m} a_{2m} - \sum_{m=1}^{n} r_m a_{1m} \sum_{m=1}^{n} r_m a_{2m} \right| \\ &\leqslant \frac{1}{4}\left(\sum_{m=1}^{n} r_m\right)^2 (A-a_1)(B-a_2), \end{aligned} \tag{22}$$

provided that the sequences $(a_{1m})_m$ and $(a_{2m})_m$ are bounded, with $a_1 \leqslant a_{1m} \leqslant A$ and $a_2 \leqslant a_{2m} \leqslant B$. It is easy to see that $0 \leqslant a_{1m} \leqslant \varpi^{\frac{\rho}{\psi}}$ and $0 \leqslant a_{2m} \leqslant (1-\varpi)^{\frac{\flat}{\psi}}$ for all $m \geqslant 1$, which when substituted in (22), imply that

$$\begin{aligned} &\left| \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}} (1-\varpi)^{\frac{\flat m}{\psi}} \right. \\ &\quad \left. - \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}} \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m (1-\varpi)^{\frac{\flat m}{\psi}} \right| \\ &\leqslant \frac{1}{4}\left(\sum_{m=1}^{n}\left(\frac{\delta}{\psi}\right)^m\right)^2 \varpi^{\frac{\rho}{\psi}} (1-\varpi)^{\frac{\flat}{\psi}}. \end{aligned} \tag{23}$$

As previous, letting $n \to \infty$ in (23) we get the right inequality of (20), so completing the proof. □

We have the following main result as well.

THEOREM 3. *Let* $\rho,\flat,\delta \geqslant 0$ *and* $\psi \geqslant 1$. *Then for all* $\varpi \in (0,1)$ *we have*

$$\left(\exp\left[\frac{\delta}{\psi}\right]-1\right)\left[\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right]^{\frac{\delta\exp[\frac{\delta}{\psi}]}{\psi[\exp(\frac{\delta}{\psi})-1]}} \leqslant X_{\rho,\flat,\delta,\psi}(\varpi)-1. \tag{24}$$

Proof. The Jensen's discrete inequality [19], applied to the concave function $t \longmapsto \ln t$, $t \in (0,\infty)$, gives

$$\ln\left(\frac{\sum_{m=1}^{n} r_m u_m}{\sum_{m=1}^{n} r_m}\right) \geqslant \frac{\sum_{m=1}^{n} r_m \ln(u_m)}{\sum_{m=1}^{n} r_m},$$

where $u_m := \varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}}$ and $r_m = \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m$, $m \geqslant 1$. Therefore we have

$$\ln\left(\frac{\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}}}{\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m}\right) \geqslant \frac{\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \ln[\varpi^{\frac{\rho n}{\psi}}(1-\varpi)^{\frac{\flat m}{\psi}}]}{\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m}$$
$$= \frac{\sum_{m=1}^{n}\frac{1}{(m-1)!}\left(\frac{\delta}{\psi}\right)^m \ln[\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}]}{\sum_{m=1}^{n}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m}, \tag{25}$$

for all $\varpi \in (0,1)$ and $n \geqslant 1$. It is easy to see that

$$\sum_{m=1}^{\infty}\frac{1}{(m-1)!}\left(\frac{\delta}{\psi}\right)^m = \frac{\delta}{\psi}\exp\left(\frac{\delta}{\psi}\right)$$

and

$$\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m = \exp\left(\frac{\delta}{\psi}\right)-1.$$

Taking $n \to \infty$ in (23) and using the representation (11), we get

$$\ln\left(\frac{X_{\rho,\flat,\delta,\psi}(\varpi)-1}{\exp(\frac{\delta}{\psi})-1}\right) \geqslant \frac{\delta\exp(\frac{\delta}{\psi})}{\psi[\exp(\frac{\delta}{\psi})-1]}\ln[\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}]$$
$$= \ln\left[\left(\varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}}\right)^{\frac{\delta\exp(\frac{\delta}{\psi})}{\psi[\exp(\frac{\delta}{\psi})-1]}}\right], \tag{26}$$

hence (24), then completes the proof. □

We may also state the following result.

THEOREM 4. *Let* $\rho,\flat,\delta \geqslant 0$ *and* $\psi \geqslant 1$. *Then for all* $\varpi \in (0,1)$, *we have*

$$0 \leqslant X_{\rho,\flat,\delta,\psi}(\varpi)-1$$
$$\leqslant \left(\exp\left(\frac{\delta}{\psi}\right)-1\right)\frac{[\exp(\frac{\delta}{\psi}\varpi^{\frac{2\rho}{\psi}})-1][\exp(\frac{\delta}{\psi}(1-\varpi)^{\frac{2\flat}{\psi}})-1]}{[\exp(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}})-1][\exp(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}})-1]}. \tag{27}$$

Proof. The Cauchy-Bunyakovsky-Schwarz weighted inequality, namely

$$\left(\sum_{m=1}^{n} r_m a_{1m}^2\right)\left(\sum_{m=1}^{n} r_m a_{2m}^2\right) \geqslant \frac{\left(\sum_{m=1}^{n} r_m a_{1m}\right)\left(\sum_{m=1}^{n} r_m a_{2m}\right)\left(\sum_{m=1}^{n} r_m a_{1m} a_{2m}\right)}{\sum_{m=1}^{n} r_m},$$

with $a_{1m} = \varpi^{\frac{\rho m}{\psi}}$, $a_{2m} = (1-\varpi)^{\frac{\flat m}{\psi}}$ and $r_m = \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m$, yields

$$\begin{aligned} &\sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{2\rho m}{\psi}} \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m (1-\varpi)^{\frac{2\flat m}{\psi}} \\ &\geqslant \frac{\sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}} \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m (1-\varpi)^{\frac{\flat m}{\psi}} \sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m \varpi^{\frac{\rho m}{\psi}} (1-\varpi)^{\frac{\flat m}{\psi}}}{\sum_{m=1}^{n} \frac{1}{m!}\left(\frac{\delta}{\psi}\right)^m}. \end{aligned} \tag{28}$$

As previously, since all the series involved in (28) are convergent, then taking limit $n \to \infty$ and using (11), we get the required result. □

THEOREM 5. *For any* $(\rho_1, \flat_2, \delta_1), (\rho_2, \flat_2, \delta_2) \in [0,\infty) \times [0,\infty) \times [0,\infty)$, $\psi \geqslant 1$ *and* $\tau \in [0,1]$*, the inequality*

$$\begin{aligned} &X_{(1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2,(1-\tau)\delta_1+\tau\delta_2,\psi}(\varpi) \\ &\leqslant (1-\tau)^2 X_{\rho_1,\flat_1,\delta_1,\psi}(\varpi) + (1-\tau)\tau X_{\rho_2,\flat_2,\delta_1,\psi}(\varpi) \\ &\quad + \tau(1-\tau) X_{\rho_1,\flat_1,\delta_2,\psi}(\varpi) + \tau^2 X_{\rho_2,\flat_2,\delta_2,\psi}(\varpi), \end{aligned} \tag{29}$$

holds true for all $\varpi \in (0,1)$.

Proof. Fix $\varpi \in (0,1)$. We have

$$\begin{aligned} &(1-\tau)(\rho_1, \flat_1, \delta_1) + \tau(\rho_2, \flat_2, \delta_2) \\ &= ((1-\tau)\rho_1 + \tau\rho_2, (1-\tau)\flat_1 + \tau\flat_2, (1-\tau)\delta_1 + \tau\delta_2) \in (0,\infty) \times (0,\infty) \times (0,\infty) \end{aligned}$$

and so

$$\begin{aligned} &X_{(1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2,(1-\tau)\delta_1+\tau\delta_2,\psi}(\varpi) - 1 \\ &= \sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{(1-\tau)\delta_1+\tau\delta_2}{\psi}\right)^m \varpi^{\frac{[(1-\tau)\rho_1+\tau\rho_2]m}{\psi}} (1-\varpi)^{\frac{[(1-\tau)\flat_1+\tau\flat_2]m}{\psi}} \\ &= \sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{(1-\tau)\delta_1+\tau\delta_2}{\psi}\right)^m \varpi^{\frac{(1-\tau)\rho_1 m}{\psi}} (1-\varpi)^{\frac{(1-\tau)\flat_1 m}{\psi}} \varpi^{\frac{\tau\rho_2 m}{\psi}} (1-\varpi)^{\frac{\tau\flat_2 m}{\psi}} \\ &= \sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{(1-\tau)\delta_1+\tau\delta_2}{\psi}\right)^m [\varpi^{\frac{\rho_1 m}{\psi}} (1-\varpi)^{\frac{\flat_1 m}{\psi}}]^{(1-\tau)} [\varpi^{\frac{\rho_2 m}{\psi}} (1-\varpi)^{\frac{\flat_2 m}{\psi}}]^{\tau} := A. \end{aligned}$$

Using the convexity of the power function $t \longmapsto t^m$, $m \geqslant 1$, we have

$$\left(\frac{(1-\tau)\delta_1+\tau\delta_2}{\psi}\right)^m \leqslant (1-\tau)\left(\frac{\delta_1}{\psi}\right)^m+\tau\left(\frac{\delta_2}{\psi}\right)^m,$$

for all $\delta_1, \delta_2 > 0$, $\psi \geqslant 1$ and $\tau \in [0,1]$. It follows that

$$\begin{aligned}
A &\leqslant \sum_{m=1}^{\infty}\frac{1}{m!}\left[(1-\tau)\left(\frac{\delta_1}{\psi}\right)^m+\tau\left(\frac{\delta_2}{\psi}\right)^m\right][\varpi^{\frac{\rho_1 m}{\psi}}(1-\varpi)^{\frac{\flat_1 m}{\psi}}]^{(1-\tau)}[\varpi^{\frac{\rho_2 m}{\psi}}(1-\varpi)^{\frac{\flat_2 m}{\psi}}]^{\tau}\\
&= \sum_{m=1}^{\infty}\frac{1}{m!}\left[(1-\tau)\left(\frac{\delta_1}{\psi}\right)^m\right][\varpi^{\frac{\rho_1 m}{\psi}}(1-\varpi)^{\frac{\flat_1 m}{\psi}}]^{(1-\tau)}[\varpi^{\frac{\rho_2 m}{\psi}}(1-\varpi)^{\frac{\flat_2 m}{\psi}}]^{\tau}\\
&\quad+\sum_{m=1}^{\infty}\frac{1}{m!}\left[\tau\left(\frac{\delta_2}{\psi}\right)^m\right][\varpi^{\frac{\rho_1 m}{\psi}}(1-\varpi)^{\frac{\flat_1 m}{\psi}}]^{(1-\tau)}[\varpi^{\frac{\rho_2 m}{\psi}}(1-\varpi)^{\frac{\flat_2 m}{\psi}}]^{\tau}.
\end{aligned}$$

By Young's inequality we have

$$\begin{aligned}
&[\varpi^{\frac{\rho_1 m}{\psi}}(1-\varpi)^{\frac{\flat_1 m}{\psi}}]^{(1-\tau)}[\varpi^{\frac{\rho_2 m}{\psi}}(1-\varpi)^{\frac{\flat_2 m}{\psi}}]^{\tau}\\
&\leqslant [(1-\tau)\varpi^{\frac{\rho_1 m}{\psi}}(1-\varpi)^{\frac{\flat_1 m}{\psi}}]+[\tau x^{\frac{\rho_2 m}{\psi}}(1-\varpi)^{\frac{\flat_2 m}{\psi}}].
\end{aligned}$$

Therefore we have

$$\begin{aligned}
A &\leqslant \sum_{m=1}^{\infty}\frac{1}{m!}\left[(1-\tau)\left(\frac{\delta_1}{\psi}\right)^m\right](1-\tau)[\varpi^{\frac{\rho_1}{\psi}}(1-\varpi)^{\frac{\flat_1}{\psi}}]^m+\tau[\varpi^{\frac{\rho_2}{\psi}}(1-\varpi)^{\frac{\flat_2}{\psi}}]^m\\
&\quad+\sum_{m=1}^{\infty}\frac{1}{m!}\left[\tau\left(\frac{\delta_2}{\psi}\right)^m\right](1-\tau)[\varpi^{\frac{\rho_1}{\psi}}(1-\varpi)^{\frac{\flat_1}{\psi}}]^m+\tau[\varpi^{\frac{\rho_2}{\psi}}(1-\varpi)^{\frac{\flat_2}{\psi}}]^m.\\
&= (1-\tau)^2\sum_{m=1}^{\infty}\frac{1}{m!}\left[\left(\frac{\delta_1}{\psi}\right)^m\right][\varpi^{\frac{\rho_1}{\psi}}(1-\varpi)^{\frac{\flat_1}{\psi}}]^m\\
&\quad+(1-\tau)\tau\sum_{m=1}^{\infty}\frac{1}{m!}\left[\left(\frac{\delta_1}{\psi}\right)^m\right][\varpi^{\frac{\rho_2}{\psi}}(1-\varpi)^{\frac{\flat_2}{\psi}}]^m\\
&\quad+\tau(1-\tau)\sum_{m=1}^{\infty}\frac{1}{m!}\left[\left(\frac{\delta_2}{\psi}\right)^m\right][\varpi^{\frac{\rho_1}{\psi}}(1-\varpi)^{\frac{\flat_1}{\psi}}]^m\\
&\quad+\tau^2\sum_{m=1}^{\infty}\frac{1}{m!}\left[\left(\frac{\delta_2}{\psi}\right)^m\right][\varpi^{\frac{\rho_2}{\psi}}(1-\varpi)^{\frac{\flat_2}{\psi}}]^m.\\
&= (1-\tau)^2[X_{\rho_1,\flat_1,\delta_1,\psi}(\varpi)-1]+(1-\tau)\tau[X_{\rho_2,\flat_2,\delta_1,\psi}(\varpi)-1]\\
&\quad+\tau(1-\tau)[X_{\rho_1,\flat_1,\delta_2,\psi}(\varpi)-1]+\tau^2X_{\rho_2,\flat_2,\delta_2,\psi}(\varpi)\\
&= (1-\tau)^2X_{\rho_1,\flat_1,\delta_1,\psi}(\varpi)+(1-\tau)\tau X_{\rho_2,\flat_2,\delta_1,\psi}(\varpi)\\
&\quad+\tau(1-\tau)X_{\rho_1,\flat_1,\delta_2,\psi}(\varpi)+\tau^2X_{\rho_2,\flat_2,\delta_2,\psi}(\varpi)-1.
\end{aligned}$$

This completes the proof. □

COROLLARY 4. *(i) For fixed* $\varpi \in (0,1)$, $\psi \geqslant 1$ *and* $\delta \geqslant 0$, *the function* $(\rho,\flat) \longmapsto X_{\rho,\flat,\delta,\psi}(\varpi)$ *is globally convex on* $[0,\infty)\times[0,\infty)$.

(ii) For fixed $\rho,\flat \geqslant 0$, $\psi \geqslant 1$ *and* $\varpi \in (0,1)$, *the function* $\delta \longmapsto X_{\rho,\flat,\delta,\psi}(\varpi)$ *is convex on* $[0,\infty)$.

Proof. (i) Let $(\rho_1,\flat_1),(\rho_2,\flat_2) \in [0,\infty)\times[0,\infty)$ and $\tau \in [0,1]$. Then by using Theorem 5 for $\delta_1=\delta_2=\delta$, we get

$$\begin{aligned}
&X_{(1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2,\delta,\psi}(\varpi)\\
&\leqslant (1-\tau)^2X_{\rho_1,\flat_1,\delta,\psi}(\varpi)+(1-\tau)\tau X_{\rho_2,\flat_2,\delta,\psi}(\varpi)\\
&\quad +(1-\tau)\tau f_{\rho_1,\flat_1,\delta,\psi}(\varpi)+\tau^2X_{\rho_2,\flat_2,\delta,\psi}(\varpi)\\
&=[(1-\tau)^2+\tau(1-\tau)]X_{\rho_1,\flat_1,\delta,\psi}(\varpi)+[\tau(1-\tau)+\tau^2]X_{\rho_2,\flat_2,\delta,\psi}(\varpi)\\
&=(1-\tau)X_{\rho_1,\flat_1,\delta,\psi}(\varpi)+\tau X_{\rho_2,\flat_2,\delta,\psi}(\varpi).
\end{aligned}$$

Whence the desired result.

(ii) Fix $\varpi \in (0,1)$ and $\rho,\flat \geqslant 0$. Theorem 5 with $\rho_1=\rho_2=\rho$ and $\flat_1=\flat_2=\flat$ implies that

$$\begin{aligned}
&X_{\rho,\flat,(1-\tau)\delta_1+\tau\delta_2,\psi}(\varpi)\\
&\leqslant (1-\tau)^2X_{\rho,\flat,\delta_1,\psi}(\varpi)+(1-\tau)\tau X_{\rho,\flat,\delta_2,\psi}(\varpi)\\
&\quad +\tau(1-\tau)X_{\rho,\flat,\delta_1,\psi}(\varpi)+\tau^2X_{\rho,\flat,\delta_2,\psi}(\varpi)\\
&=[(1-\tau)^2+(1-\tau)\tau]X_{\rho,\flat,\delta_1,\psi}(\varpi)+[\tau(1-\tau)+\tau^2]X_{\rho,\flat,\delta_2,\psi}(\varpi).\\
&=(1-\tau)X_{\rho,\flat,\delta_1,\psi}(\varpi)+\tau X_{\rho,\flat,\delta_2}(\varpi).
\end{aligned}$$

This completes the proof. □

COROLLARY 5. *For any* $(\rho_1,\flat_1,\delta_1),(\rho_2,\flat_2,\delta_2) \in [0,\infty)\times[0,\infty)\times[0,\infty)$, $\psi \geqslant 1$ *and* $\tau \in [0,1]$, *we have*

$$\begin{aligned}
&\mathscr{F}_\psi\Big((1-\tau)(\rho_1,\flat_1,\delta_1)+\tau(\rho_2,\flat_2,\delta_2)\Big)\\
&\leqslant (1-\tau)^2\mathscr{F}_\psi(\rho_1,\flat_1,\delta_1)+(1-\tau)\tau F_\psi(\rho_1,\flat_1,\delta_2)\\
&\quad +(1-\tau)\tau F_\psi(\rho_2,\flat_2,\delta_1)+\tau^2\mathscr{F}_\psi(\rho_2,\flat_2,\delta_2). \qquad (30)
\end{aligned}$$

Proof. Integrating (29) with respect to $\varpi \in [0,1]$ and using (4), we get the desired result. The details are simple and therefore omitted here. □

COROLLARY 6. *For fixed* $\delta \geqslant 0$ *and* $\psi \geqslant 1$, *the function* $(\rho,\flat) \longmapsto \mathscr{F}_\psi(\rho,\flat;\delta)$ *is globally convex on* $[0,\infty)\times[0,\infty)$. *Also* $\delta \longmapsto \mathscr{F}_\psi(\rho,\flat;\delta)$ *is convex on* $[0,\infty)$ *for any* $\rho,\flat \geqslant 0$ *and* $\psi \geqslant 1$.

THEOREM 6. *Let* $\rho,\flat,\delta \geqslant 0$ *and* $\psi \geqslant 1$. *Then we have*

$$0 \leqslant \mathscr{F}_\psi(\rho,\flat;\delta) - 1 \leqslant \frac{1}{\exp(\frac{\delta}{\psi}) - 1}[\mathscr{F}_\psi(\rho,0;\delta) - 1][\mathscr{F}_\psi(0,\flat;\delta) - 1] + \frac{1}{4}\left[\exp\left(\frac{\delta}{\psi}\right) - 1\right]\psi B_\psi(\rho+\psi,\flat+\psi). \tag{31}$$

Proof. Integrating the right inequality of (20) with respect to $\varpi \in [0,1]$, we get

$$\begin{aligned}
0 &\leqslant \mathscr{F}_\psi(\rho,\flat;\delta) - 1 \\
&\leqslant \frac{1}{\exp(\frac{\delta}{\psi}) - 1}\int_0^1 \left[\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}\right) - 1\right]\left[\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}}\right) - 1\right] d\varpi \\
&\quad + \frac{1}{4}\left[\exp\left(\frac{\delta}{\psi}\right) - 1\right]\int_0^1 \varpi^{\frac{\rho}{\psi}}(1-\varpi)^{\frac{\flat}{\psi}} d\varpi \\
&= \frac{1}{\exp(\frac{\delta}{\psi}) - 1}\int_0^1 \left[\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}\right) - 1\right]\left[\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}}\right) - 1\right] d\varpi \\
&\quad + \frac{1}{4}\left[\exp\left(\frac{\delta}{\psi}\right) - 1\right]\psi B_\psi(\rho+\psi+\flat+\psi).
\end{aligned} \tag{32}$$

Now, let

$$X(\varpi) := \exp\left(\frac{\delta\varpi^{\frac{\rho}{\psi}}}{\psi}\right) - 1, \quad Y(\varpi) = \exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}}\right) - 1, \ \ \varpi \in [0,1].$$

It is obvious that $\varpi \longmapsto X(\varpi)$ is an increasing function and $\varpi \longmapsto X(\varpi)$ is a decreasing one. The Chebysev's integral inequality [18] tells us that for the opposite monotonic functions $X,Y : [0,1] \to \mathbb{R}$, we have

$$\int_0^1 X(\varpi)Y(\varpi)d\varpi \leqslant \int_0^1 X(\varpi)d\varpi \int_0^1 Y(\varpi)d\varpi,$$

or, equivalently,

$$\begin{aligned}
&\int_0^1 \left[\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}\right) - 1\right]\left[\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}}\right) - 1\right]d\varpi \\
&\leqslant \int_0^1 \left[\exp\left(\frac{\delta}{\psi}\varpi^{\frac{\rho}{\psi}}\right) - 1\right]d\varpi \int_0^1 \left[\exp\left(\frac{\delta}{\psi}(1-\varpi)^{\frac{\flat}{\psi}}\right) - 1\right]d\varpi \\
&= [\mathscr{F}_\psi(\rho,0;\delta) - 1][\mathscr{F}_\psi(0,\flat;\delta) - 1].
\end{aligned}$$

Substituting this in (32) we get (31). □

Finally, we now discuss the logarithmic convexity property for $\mathscr{F}_\psi(\rho,\flat;\delta)$.

THEOREM 7. *For each* $\delta \geqslant 0$ *and* $\psi \geqslant 1$, *the function* $(\rho,\flat) \longmapsto \mathscr{F}_\psi(\rho,\flat;\delta)$ *is logarithmically convex on* $[0,\infty) \times [0,\infty)$.

Proof. Fix $\delta \geqslant 0$, $\psi \geqslant 1$ and let $(\rho_1,\flat_1),(\rho_2,\flat_2) \in [0,\infty)\times[0,\infty)$. By (12) we have

$$
\begin{aligned}
&\mathscr{F}_\psi\big((1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2;\delta\big)-1\\
&=\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}B_\psi\left([(1-\tau)\rho_1+\tau\rho_2]m+\psi,[(1-\tau)\flat_1+\tau\flat_2]m+\psi\right)\\
&=\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}B_\psi\left((1-\tau)(\rho_1 m+\psi)+\tau(\rho_2 m+\psi),(1-\tau)(\flat_1 m+\psi)+\tau(\flat_2 m+\psi)\right)\\
&=\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}B_\psi\left((1-\tau)\left(\rho_1 m+\psi,\flat_1 m+\psi\right)+\tau\left(\rho_2 m+\psi,\flat_2 m+\psi\right)\right),
\end{aligned}
$$

which, with the fact that the ψ-beta function is logarithmically convex [21], implies that

$$
\begin{aligned}
&\mathscr{F}_\psi\big((1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2;\delta\big)-1\\
&\leqslant\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}[B_\psi\left(\rho_1 m+\psi,\flat_1 m+\psi\right)]^{1-\tau}[B_\psi\left(\rho_2 m+\psi,\flat_2 m+\psi\right)]^{\tau}.
\end{aligned}
$$

Now using the Holder's weighted inequality with $p=\frac{1}{1-\tau}>1$, $q=\frac{1}{\tau}>1$, we get

$$
\begin{aligned}
&\mathscr{F}_\psi\big((1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2;\delta\big)-1\\
&\leqslant\left[\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}\left([B_\psi\left(\rho_1 m+\psi,\flat_1 m+\psi\right)]^{1-\tau}\right)^{\frac{1}{1-\tau}}\right]^{1-\tau}\\
&\quad\times\left[\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}\left([B_\psi\left(\rho_2 m+\psi,\flat_2 m+\psi\right)]^{\tau}\right)^{\frac{1}{\tau}}\right]^{\tau}.\\
&=\left[\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}B_\psi\big(\rho_1 m+\psi,\flat_1 m+\psi\big)\right]^{1-\tau}\\
&\quad\times\left[\sum_{m=1}^{\infty}\frac{\delta^m}{m!\psi^{m-1}}B_\psi\big(\rho_2 m+\psi,\flat_2 m+\psi\big)\right]^{\tau},
\end{aligned}
$$

which, with (12) again, yields

$$
\begin{aligned}
&\mathscr{F}_\psi\big((1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2;\delta\big)-1\\
&\leqslant[\mathscr{F}_\psi(\rho_1,\flat_1;\delta)-1]^{1-\tau}[\mathscr{F}_\psi(\rho_2,\flat_2;\delta)-1]^{\tau}.
\end{aligned}
\tag{33}
$$

If we apply the Hölder's discrete inequality to the right side of (33) we then obtain

$$
\begin{aligned}
&\mathscr{F}_\psi\big((1-\tau)\rho_1+\tau\rho_2,(1-\tau)\flat_1+\tau\flat_2;\delta\big)\\
&\leqslant[\mathscr{F}_\psi(\rho_1,\flat_1;\delta)-1]^{1-\tau}[\mathscr{F}_\psi(\rho_2,\flat_2;\delta)-1]^{\tau}+1\\
&=[\mathscr{F}_\psi(\rho_1,\flat_1;\delta)-1]^{1-\tau}[\mathscr{F}_\psi(\rho_2,\flat_2;\delta)-1]^{\tau}+1^{1-\tau}1^{\tau}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \left[\left(\left([\mathscr{F}_\psi(\rho_1,\flat_1;\delta)-1]^{1-\tau}\right)^{\frac{1}{1-\tau}}+1\right]^{1-\tau} \\
&\quad\times\left[\left(\left([\mathscr{F}_\psi(\rho_2,\flat_2;\delta)-1]^{\tau}\right)^{\frac{1}{\tau}}+1\right]^{\tau} \\
&= [\mathscr{F}_\psi(\rho_1,\flat_1;\delta)]^{1-\tau}[\mathscr{F}_\psi(\rho_2,\flat_2;\delta)]^{\tau}.
\end{aligned}$$

The proof is finished. □

3. Applications

In this section, we will discuss some applications of our results. We give two types of applications. The first concerns error bounds through Ostrowski type inequalities by using the generalized exponential beta function whereas the second application is about Ostrowski and trapezoid type of quadrature schemes.

3.1. Error bounds via Ostrowski type inequalities

Our first main result here is the following.

THEOREM 8. *Let* $\rho,\flat>1$, $\delta\geqslant 0$ *and* $1\leqslant\psi<\min\{\rho,\flat\}$. *Then, for any* $\varpi\in[0,1]$*, we have*

$$\begin{aligned}
&\left|\mathscr{F}_\psi(\rho,\flat;\delta)-X_{\rho,\flat;\delta,\psi}(\varpi)\right| \\
&\leqslant\left|\frac{1}{4}+\left(\varpi-\frac{1}{2}\right)^2\right|\frac{\delta}{\psi^2}\max\{\rho,\flat\}\frac{\left(\frac{\rho}{\psi}-1\right)^{\frac{\rho}{\psi}-1}\left(\frac{\flat}{\psi}-1\right)^{\frac{\flat}{\psi}-1}}{\left(\frac{\rho+\flat}{\psi}-2\right)^{\frac{\rho+\flat}{\psi}-2}} \\
&\quad\times\exp\left[\frac{\delta}{\psi^2}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right].
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&\left|\mathscr{F}_\psi(\rho,\flat;\delta)-\exp\left(\frac{\delta}{\psi 2^{\frac{\rho+\flat}{\psi}}}\right)\right| \\
&\leqslant\frac{1}{4}\frac{\delta}{\psi^2}\max\{\rho,\flat\}\frac{\left(\frac{\rho}{\psi}-1\right)^{\frac{\rho}{\psi}-1}\left(\frac{\flat}{\psi}-1\right)^{\frac{\flat}{\psi}-1}}{\left(\frac{\rho+\flat}{\psi}-2\right)^{\frac{\rho+\flat}{\psi}-2}} \\
&\quad\times\exp\left[\frac{\delta}{\psi^2}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right].
\end{aligned}$$

Proof. According to Lemma 1, the following inequality

$$\left|X_{\rho,\flat;\delta,\psi}(\varpi)-\int_0^1 X_{\rho,\flat;\delta,\psi}(\tau)d\tau\right|\leqslant\left[\frac{1}{4}+\left(\varpi-\frac{1}{2}\right)^2\right]\|X'_{\rho,\flat;\delta,\psi}\|_{\infty,(0,1)} \tag{34}$$

holds for all $\varpi\in[0,1]$. Writing again (6) as

$$X'_{\rho,\flat;\delta,\psi}(\varpi)=r'_{\rho,\flat}(\varpi)X_{\rho,\flat;\delta,\psi}(\varpi),\quad \forall\varpi\in[0,1], \tag{35}$$

we then obtain

$$\|X'_{\rho,\flat;\delta,\psi}\|_{\infty,(0,1)}:=\sup_{\varpi\in(0,1)}|X'_{\rho,\flat;\delta,\psi}(\varpi)|\leqslant\sup_{\varpi\in(0,1)}|r'_{\rho,\flat}(\varpi)|\sup_{\varpi\in(0,1)}X_{\rho,\flat;\delta,\psi}(\varpi),$$

which, with (5) and (8), implies the first part of Theorem 8. The second part follows by taking $\varpi=\frac{1}{2}$. □

THEOREM 9. *For any $\varpi\in[0,1]$, we have the following assertions:*
(i) If $\rho,\flat>1$, $\delta\geqslant 0$ and $1\leqslant\psi<\min\{\rho,\flat\}$, then there holds

$$\left|\frac{X_{\rho,\flat;\delta,\psi}(\varpi)}{\mathscr{F}_\psi(\rho,\flat;\delta)}-1\right|\leqslant\left[\frac{1}{2}+\left|\varpi-\frac{1}{2}\right|\right]\frac{\delta}{\psi^2}\max\{\rho,\flat\}\frac{\left(\frac{\rho}{\psi}-1\right)^{\frac{\rho}{\psi}-1}\left(\frac{\flat}{\psi}-1\right)^{\frac{\flat}{\psi}-1}}{\left(\frac{\rho+\flat}{\psi}-2\right)^{\frac{\rho+\flat}{\psi}-2}}.$$

In particular one has

$$\left|\frac{\exp\left[\frac{\delta}{\psi 2^{\frac{\rho+\flat}{\psi}}}\right]}{\mathscr{F}_\psi(\rho,\flat;\delta)}-1\right|\leqslant\frac{1}{2}\frac{\delta}{\psi^2}\max\{\rho,\flat\}\frac{\left(\frac{\rho}{\psi}-1\right)^{\frac{\rho}{\psi}-1}\left(\frac{\flat}{\psi}-1\right)^{\frac{\flat}{\psi}-1}}{\left(\frac{\rho+\flat}{\psi}-2\right)^{\frac{\rho+\flat}{\psi}-2}}.$$

(ii) If $\rho,\flat>0$, $\delta\geqslant 0$ and $\psi\geqslant 1$, then we have

$$\begin{aligned}&\left|\mathscr{F}_\psi(\rho,\flat;\delta)-X_{\rho,\flat;\delta,\psi}(\varpi)\right|\\&\leqslant\left[\frac{1}{2}+\left|\varpi-\frac{1}{2}\right|\right]\frac{\delta}{\psi}\max\{\rho,\flat\}B_\psi(\rho,\flat)\exp\left[\frac{\delta}{\psi}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right],\end{aligned}$$

and particularly,

$$\begin{aligned}&\left|\mathscr{F}_\psi(\rho,\flat;\delta)-\exp\left[\frac{\delta}{\psi 2^{\frac{\rho+\flat}{\psi}}}\right]\right|\\&\leqslant\frac{1}{2}\frac{\delta}{\psi}\max\{\rho,\flat\}B_\psi(\rho,\flat)\exp\left[\frac{\delta}{\psi}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right].\end{aligned}$$

Proof. (i) By Lemma 2, we have

$$\left|\mathscr{F}_\psi\big(\rho,\flat;\delta\big) - X_{\rho,\flat;\delta,\psi}(\varpi)\right| \leqslant \left[\frac{1}{2} + \left|\varpi - \frac{1}{2}\right|\right] \|X'_{\rho,\flat;\delta,\psi}\|_{(0,1),1}, \quad \forall\varpi \in [0,1]. \quad (36)$$

Using (35) we can write

$$\begin{aligned}\|X'_{\rho,\flat;\delta,\psi}\|_{(0,1),1} &= \int_0^1 |r'_{\rho,\flat}(\tau)| X_{\rho,\flat;\delta,\psi}(\tau)\mathrm{d}\tau \\ &\leqslant \sup_{\tau\in(0,1)} |r'_{\rho,\flat}(\tau)| \int_0^1 X_{\rho,\flat;\delta,\psi}(\tau)\mathrm{d}\tau \\ &= \Big(\mathscr{F}_\psi\big(\rho,\flat;\delta\big)\Big) \sup_{\tau\in(0,1)} |r'_{\rho,\flat}(\tau)|,\end{aligned}$$

which, with (8) and (36), yields the first inequality. To obtain the second inequality we take $\varpi = \frac{1}{2}$.

(ii) Now, if we write

$$\begin{aligned}\|X'_{\rho,\flat;\delta,\psi}\|_{(0,1),1} &= \int_0^1 |r'_{\rho,\flat}(\tau)| X_{\rho,\flat;\delta,\psi}(\tau)\mathrm{d}\tau \\ &\leqslant \sup_{\tau\in(0,1)} X_{\rho,\flat;\delta,\psi}(\tau) \int_0^1 |r'_{\rho,\flat}(\tau)|\mathrm{d}\tau,\end{aligned}$$

and we use (7) we get

$$\|X'_{\rho,\flat;\delta,\psi}\|_{(0,1),1} \leqslant \frac{\delta}{\psi^2}\max\{\rho,\flat\} \sup_{\tau\in(0,1)} X_{\rho,\flat;\delta,\psi}(\tau) \int_0^1 \varpi^{\frac{\rho}{\psi}-1}(1-\varpi)^{\frac{\flat}{\psi}-1}\mathrm{d}\varpi. \quad (37)$$

Substituting (37) in (36), and using (3) and (5), we obtain the third inequality of the theorem. The fourth inequality follows when taking $\varpi = \frac{1}{2}$. □

We also have the following result.

THEOREM 10. *Let* $\rho,\flat,\delta \geqslant 0, \psi \geqslant 1$*. For any* $p,q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*, the following inequality*

$$\begin{aligned}&\left|\mathscr{F}_\psi\big(\rho,\flat;\delta\big) - X_{\rho,\flat;\delta,\psi}(\varpi)\right| \\ &\leqslant \frac{1}{(q+1)^{\frac{1}{q}}}\left[\varpi^{q+1} + (1-\varpi)^{q+1}\right]^{\frac{1}{q}} \\ &\quad \times \frac{\delta}{\psi}\max\{\rho,\flat\}\left[B_\psi\big(p\rho + (1-p)\psi, p\flat + (1-p)\psi\big)\right]^{\frac{1}{p}} \\ &\quad \times \exp\left[\frac{\delta}{\psi}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right]\end{aligned}$$

holds for all $\varpi \in [0,1]$ *and particularly, we have*

$$\begin{aligned}&\left|\mathscr{F}_\psi(\rho,\flat;\delta) - \exp\left[\frac{\delta}{\psi 2^{\frac{\rho+\flat}{\psi}}}\right]\right| \\ &\leqslant \frac{1}{2(q+1)^{\frac{1}{q}}}\frac{\delta}{\psi}\max\{\rho,\flat\}\left[B_\psi(p\rho+(1-p)\psi, p\flat+(1-p)\psi)\right]^{\frac{1}{p}} \\ &\quad\times \exp\left[\frac{\delta}{\psi}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right].\end{aligned}$$

Proof. According to Lemma 3 we have, for all $\varpi \in [0,1]$,

$$\left|\mathscr{F}_\psi(\rho,\flat;\delta) - X_{\rho,\flat;\delta,\psi}(\varpi)\right| \leqslant \frac{1}{(q+1)^{\frac{1}{q}}}\left[\varpi^{q+1} + (1-\varpi)^{q+1}\right]^{\frac{1}{q}} \|X'_{\rho,\flat;\delta,\psi}\|_{(0,1),p}. \tag{38}$$

Now, using (6) and (7) we get

$$\begin{aligned}\|X'_{\rho,\flat;\delta,\psi}\|^p_{(0,1),p} &= \left(\frac{\delta}{\psi^2}\right)^p \int_0^1 \tau^{(\frac{\rho}{\psi}-1)p}(1-\tau)^{(\frac{\flat}{\psi}-1)p}\left|\rho-(\rho+\flat)\tau\right|^p \left(X_{\rho,\flat;\delta,\psi}(\tau)\right)^p \mathrm{d}\tau \\ &\leqslant \left(\frac{\delta}{\psi^2}\right)^p \left(\max(\rho,\flat)\right)^p \sup_{\tau\in(0,1)}\left(X_{\rho,\flat;\delta,\psi}(\tau)\right)^p \int_0^1 \tau^{(\frac{\rho}{\psi}-1)p}(1-\tau)^{(\frac{\flat}{\psi}-1)p}.\end{aligned}$$

This, after simple manipulations and the use of (3) and (5), substituted in (38) yields the first inequality. The second one follows by taking $\varpi = \frac{1}{2}$. □

3.2. Ostrowski and trapezoid type of quadrature schemes

Let $J_s : a_1 = \varpi_0 < \varpi_1 < \ldots \varpi_{s-1} < \varpi_s = a_2$ be a partition of the interval $[a_1, a_2]$, and a_{1i} $(i = 0,1,\ldots,s+1)$ be the $(s+2)$-points such that $a_{10} = a_1, a_{1i} \in [\varpi_{s-1}, \varpi_s]$ $(i = 1,2,\ldots,s)$ and $a_{1s+1} = a_2$. We set $h_i := \varpi_{i+1} - \varpi_i$, $\forall i \in \{0,1,2,\ldots,s-1\}$, the sub-interval size, and $w(h) := \max\{h_i : i = 0,1,2,\ldots,s-1\}$.

Our aim here is to approximate the following integral by demanding;

$$\int_{a_1}^{a_2} X(\tau)\mathrm{d}\tau = M_s(X, J_s, a_{1s+1}) + N_s(X, J_s, a_{1s+1}),$$

where

$$M_s(X, J_s, a_{1s+1}) = \sum_{i=0}^{s}(a_{1i+1} - a_{1i})X(\varpi_i) \tag{39}$$

is the Ostrowski quadrature rule and $N_s(X, J_s, a_{1s+1})$ is its associated remainder.

If in (39) we take

$$a_{10} = a_1,\ a_{11} = \frac{a_1+\varpi_1}{2}, \ldots,\ a_{1s-1} = \frac{\varpi_{s-2}+\varpi_{s-1}}{2},\ a_{1s} = \frac{\varpi_s+\varpi_{s-1}}{2},\ a_{1s+1} = a_2,$$

then we get

$$M_s(X,J_s,a_1) = \frac{1}{2}\left[(\varpi_1 - a_1)X(a_1)\sum_{i=i}^{s-1}[(\varpi_{i+1}-\varpi_i)X(\varpi_i)+(a_2-\varpi_{s-1})X(a_2)]\right]$$
$$:= T(X,J_s),$$

which is called the trapezoid quadrature rule.

If we choose an equidistance partition of $[a_1,a_2]$, i.e.

$$J_s : \varpi_i = a_1 + (a_2-a_1)\frac{i}{s}, \quad i = 0,1,\ldots,s$$

we then obtain the equidistance trapezoid quadrature rule given by:

$$T(X,J_s) := \frac{X(a_1)+X(a_2)}{2s}(a_2-a_1) + \frac{(a_2-a_1)}{s}\sum_{i=1}^{s-1} X\left(a_1+(a_2-a_1)\frac{i}{s}\right).$$

• In [14] Dragomir and Rassias derived the error bounds for functions X such that $X' \in Ł_\infty[a_1,a_2]$, as

$$|N_s(X,J_s,a_{1s+1})| \leqslant \left[\frac{1}{4}\sum_{i=0}^{s-1} h_i^2 + \sum_{i=0}^{s-1}\left(a_{1i+1} - \frac{\varpi_i+\varpi_{i+1}}{2}\right)^2\right]\|X'\|_{\infty,(a_1,a_2)}. \tag{40}$$

For trapezoid rule error bound we have

$$|N_s(X,J_s)| \leqslant \frac{1}{4}\sum_{i=0}^{s-1} h_i^2 \|X'_{\rho,\flat}\|_{\infty,[a_1,a_2]} \leqslant \frac{1}{4}(a_2-a_1)w(h)\|X'\|_{\infty,(a_1,a_2)}.$$

• In [6] Dragomir established error bounds for 1-norm functions, given by

$$|N_s(X,J_s,a_{1s+1})| \leqslant \left[\frac{1}{2}w(h) + \max_{i=1,2,\ldots,n}\left|a_{1i+1} - \frac{\varpi_i+\varpi_{i+1}}{2}\right|\right]\|X'_{\rho,\flat}\|_{1,(a_1,a_2)}. \tag{41}$$

• In [7] he established a generalized error bound for functions X such that $X' \in L_p(a_1,a_2)$ as

$$|N_s(X,J_s,a_{1s+1})|$$
$$\leqslant \frac{1}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{s-1}(a_{1i+1}-\varpi_i)^{q+1} + (\varpi_{i+1}-a_{1i+1})^{q+1}\right]^{\frac{1}{q}}\|X'\|_{p,(a_1,a_2)}. \tag{42}$$

We now state the following result.

THEOREM 11. *With the previous notations, if we set*

$$\mathscr{F}_\psi(\rho,\flat;\delta) := \int_0^1 X_{\rho,\flat;\delta,\psi}(\varpi)d\varpi = M_s\big(X_{\rho,\flat;\delta,\psi},J_s,a_{1s+1}\big) + N_s(X_{\rho,\flat;\delta,\psi},J_s,a_{1s+1}),$$

then the remainder $N_s(X_{\rho,\flat;\delta,\psi}, J_s, a_{1s+1})$ *satisfies the following assertions:*

(i) Under the hypotheses of Theorem 8, we have

$$\begin{aligned}
&\left|N_s(X_{\rho,\flat;\delta,\psi}, J_s, a_{1s+1})\right| \\
&\leqslant \left[\frac{1}{4}\sum_{i=0}^{s-1} h_i^2 + \sum_{i=0}^{s-1}\left(a_{1i+1} - \frac{\varpi_i + \varpi_{i+1}}{2}\right)^2\right] \\
&\quad \times \frac{\delta}{\psi^2}\max\{\rho,\flat\}\frac{\left(\frac{\rho}{\psi}-1\right)^{\frac{\rho}{\psi}-1}\left(\frac{\flat}{\psi}-1\right)^{\frac{\flat}{\psi}-1}}{\left(\frac{\rho+\flat}{\psi}-2\right)^{\frac{\rho+\flat}{\psi}-2}} \\
&\quad \times \exp\left[\frac{\delta}{\psi^2}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right].
\end{aligned}$$

(ii) Under the assumptions of Theorem 9, we have

$$\begin{aligned}
&\left|N_s(X_{\rho,\flat;\delta,\psi}, J_s, a_{1s+1})\right| \\
&\leqslant \left[\frac{1}{2}w(h) + \max_{i=1,2,\dots,n}\left|a_{1i+1} - \frac{\varpi_i + \varpi_{i+1}}{2}\right|\right] \\
&\quad \times \frac{\delta}{\psi}\max\{\rho,\flat\}B_\psi(\rho,\flat)\exp\left[\frac{\delta}{\psi}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right].
\end{aligned}$$

(iii) With the hypotheses of Theorem 10, there holds

$$\begin{aligned}
&\left|N_s(X_{\rho,\flat;\delta,\psi}, J_s, a_{1s+1})\right| \\
&\leqslant \frac{1}{(q+1)^{\frac{1}{q}}}\left[\sum_{i=0}^{s-1}(a_{1i+1} - \varpi_i)^{q+1} + (\varpi_{i+1} - a_{1i+1})^{q+1}\right]^{\frac{1}{q}} \\
&\quad \times \frac{\delta}{\psi}\max\{\rho,\flat\}\left[B_\psi\left(p\rho + (1-p)\psi, p\flat + (1-p)\psi\right)\right]^{\frac{1}{p}} \\
&\quad \times \exp\left[\frac{\delta}{\psi}\left(\frac{\rho}{\rho+\flat}\right)^{\frac{\rho}{\psi}}\left(\frac{\flat}{\rho+\flat}\right)^{\frac{\flat}{\psi}}\right].
\end{aligned}$$

Proof. The proof follows when using the inequalities (40), (41) and (42), respectively, and the bounds for $\|X'_{\rho,\flat}\|_{1,(a_1,a_2)}$ as in Theorem 8, Theorem 9 and Theorem 10, respectively. The details are straightforward and therefore omitted here. □

4. Conclusion

In the current investigation, we have studied the new generalized family of exponential functions involving four parameters and new generalized exponential beta functions, which are based on generalized exponential functions. We have incorporated with the properties of these newly proposed specials functions such as their taylor's representation, convexity property and some analytical inequalities are established. Also, we have provided applications to Ostrowski's inequalities and quadrature rules. In future, we will try to investigate these functions in probability theory and their q-variants.

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