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BOUNDS FOR THE α–ADJACENCY ENERGY OF A GRAPH

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Abstract. For the adjacency matrix $A(G)$ and diagonal matrix of the vertex degrees $D(G)$ of a simple graph G, the $A_\alpha(G)$ matrix is the convex combinations of $D(G)$ and $A(G)$, and is defined as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, for $0 \leqslant \alpha \leqslant 1$. Let $\rho_1 \geqslant \rho_2 \geqslant \ldots \geqslant \rho_n$ be the eigenvalues of $A_\alpha(G)$ (which we call α-adjacency eigenvalues of the graph G). The generalized adjacency energy also called α-adjacency energy of the graph G is defined as $E^{A_\alpha}(G) = \sum_{i=1}^{n} |\rho_i - \alpha\overline{d}|$, where $\overline{d} = \frac{2m}{n}$ is the average vertex degree, m is the size and n is the order of G. The α-adjacency energy of a graph G merges the theory of energy (adjacency energy) and the signless Laplacian energy, as $E^{A_0}(G) = \mathscr{E}(G)$ and $2E^{A_{\frac{1}{2}}}(G) = QE(G)$, where $\mathscr{E}(G)$ is the energy and $QE(G)$ is the signless Laplacian energy of G. In this paper, we obtain some new upper and lower bounds for the generalized adjacency energy of a graph, in terms of different graph parameters like the vertex covering number, the Zagreb index, the number of edges, the number of vertices, etc. We characterize the extremal graphs attained these bounds.

1. Introduction

In this paper, we consider undirected, simple and finite graphs. A graph is denoted by $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and $E(G)$ is its edge set. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is called *regular* if each of its vertices has the same degree. The adjacency matrix $A(G) = (a_{ij})$ of G is a $(0,1)$-square matrix of order n having (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_G(v_i)$, $i = 1, 2, \ldots, n$, of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively, the Laplacian and the signless Laplacian matrices and their spectrum are respectively, the Laplacian spectrum and signless Laplacian spectrum of G. These matrices are real symmetric and positive semi-definite. We take $0 = \mu_n \leqslant \mu_{n-1} \leqslant \ldots \leqslant \mu_1$ and $0 \leqslant q_n \leqslant q_{n-1} \leqslant \ldots \leqslant q_1$ to be the Laplacian spectrum and the signless Laplacian spectrum of G, respectively. For other undefined notations and terminology from spectral graph theory, the readers are referred to [3, 12, 15, 22].

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Nikiforov in [19], introduced the generalized adjacency matrix $A_\alpha(G)$ (called A_α-matrix) of a graph G as the convex combinations of $D(G)$ and $A(G)$, that is, $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, for $0 \leqslant \alpha \leqslant 1$. Since $A_0(G) = A(G)$, $2A_{\frac{1}{2}}(G) = Q(G)$, $A_1(G) = D(G)$ and $A_\alpha(G) - A_\beta(G) = (\alpha - \beta)L(G)$, any result regarding the spectral properties of A_α-matrix, has its counterpart for each of these particular graph matrices, and these counterparts follow immediately from a single proof. In fact, this matrix merges and generalizes the adjacency and the signless Laplacian spectral theories of a graph to a more general setting. Since the matrix $A_\alpha(G)$ is real symmetric, all its eigenvalues are real. Therefore, we can arrange them as $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_n$. The largest eigenvalue ρ_1 of $A_\alpha(G)$ is called the *generalized adjacency spectral radius or α-spectral radius or A_α-spectral radius* of G. For $\alpha \neq 1$, the A_α-matrix of a connected graph G is nonnegative and irreducible, so by the Perron-Frobenius theorem, the spectral radius $\rho_1(G)$ is the unique eigenvalue and there is a unique positive unit eigenvector X corresponding to $\rho_1(G)$, which is called the *Perron vector* of $A_\alpha(G)$. Note that a vector $X \in \mathbb{R}^n$ is said to be positive if each of its components are positive and nonnegative if all its components are nonnegative. Further results on the spectral properties of the matrix $A_\alpha(G)$ can be found in [8, 16, 17] and the references therein.

The notion of energy of a graph [9] was introduced in 1978 by Ivan Gutman and has its origin in theoretical chemistry. Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ be the adjacency eigenvalues of a graph G. The *energy* of a graph G, denoted by $\mathscr{E}(G)$, is defined as $\mathscr{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. For details on graph energy we refer to the book [15]. This spectrum-based graph invariant has been much studied in both chemical and mathematical literature. Gutman [9] further proposes the study of energy in graphs with an analogue of the energy defined with respect to other (than adjacency) matrices assigned to the graphs. This proposal has been put into effect and extended: the energy of a graph with respect to Laplacian matrix [10], the signless Laplacian matrix [1, 6, 23], the Randić matrix [11] as well as the energy of a graph with respect to the distance matrix [13] has been studied. The concept of energy was extended to digraphs and various energies of digraphs like the energy [21] and the skew energy [2] were put forward and extensively studied. This concept was generalized by Nikiforov by defining the energy of any matrix, see [18, 20].

The Laplacian energy $LE(G)$ of a graph G was put forward by Gutman et al. in [10] and is defined as $LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$. Likewise, the signless Laplacian energy $QE(G)$ of G was put forward by Abreu et al. in [1] and is defined as $QE(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right|$, where $\frac{2m}{n}$ is the average degree of the graph G.

Motivated by the above works, Guo and Zhou in [8] introduced an energy like quantity based on the matrix $A_\alpha(G)$. Let us define the auxiliary eigenvalues s_i, corresponding to the eigenvalues of $A_\alpha(G)$ as

$$s_i = \rho_i - \frac{2\alpha m}{n}.$$

The α-adjacency energy(also called generalized adjacency energy), denoted by $E^{A_\alpha}(G)$ of a graph G is defined as the mean deviation of the eigenvalues of $A_\alpha(G)$, that is,

$$E^{A_\alpha}(G)=\sum_{i=1}^{n}\left|\rho_i-\frac{2\alpha m}{n}\right|=\sum_{i=1}^{n}|s_i|. \tag{1.1}$$

It can be easily verified that $\sum_{i=1}^{n}s_i=0$. It is clear from the definition that $E^{A_0}(G)=\mathscr{E}(G)$, the (adjacency) energy of graph G and $2E^{A_{\frac{1}{2}}}(G)=QE(G)$, the signless Laplacian energy of G. From this it follows that the concept of α-adjacency energy of a graph G merges the theories of (adjacency) energy and the signless Laplacian energy of a graph G. Therefore, it will be interesting to study the quantity $E^{A_\alpha}(G)$ and explore some properties like the bounds, the dependence on the structure of G and the dependence on the parameter α. For some basic properties and bounds for α-adjacency energy, we refer to [24].

The paper is organized as follows. Section 2 provides an overview of some known results from the literature. In Section 3, we present some new upper and lower bounds for the generalized adjacency energy of a graph, in terms of different graph parameters like the vertex covering number, the Zagreb index, the number of edges and the number of vertices. We characterize the extremal graphs which achieve these bounds.

2. Preliminary results

In this section, we mention some known results from the literature which will help in obtaining the main results of this paper.

Let $\mathbb{M}_{m\times n}(\mathbb{R})$ be the set of all $m\times n$ matrices with real entries, that is,

$$\mathbb{M}_{m\times n}(\mathbb{R})=\{X : X=(x_{ij})_{m\times n}, x_{ij}\in\mathbb{R}\}.$$

For $M\in\mathbb{M}_{m\times n}(\mathbb{R})$, the *Frobenius norm* is defined as

$$\|M\|_F=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|m_{ij}|^2}=\sqrt{trace(M^tM)},$$

where the *trace* of a square matrix is defined as sum of the diagonal entries. Further, if $MM^t=M^tM$, then $\|M\|_F^2=\sum_{i=1}^{n}|\lambda_i(M)|^2$, where λ_i is the i^{th} eigenvalue of the matrix M.

The *Zagreb index* $Zg(G)$ of a graph G is defined as the sum of squares of vertex degrees, that is, $Zg(G)=\sum_{u\in V(G)}d_G^2(u)$.

The following Lemma was obtained in [24] and gives some basic properties of the α-adjacency matrix of the graph G.

LEMMA 2.1. *Let G be a connected graph of order n with m edges having vertex degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$. Then*

(1) $\sum_{i=1}^{n} \rho_i = 2\alpha m$; **(2)** $\sum_{i=1}^{n} \rho_i^2 = \alpha^2 Zg(G) + (1-\alpha)^2 \|A(G)\|_F^2$;

(3) $\sum_{i=1}^{n} s_i^2 = \alpha^2 Zg(G) + (1-\alpha)^2 \|A(G)\|_F^2 - \dfrac{4\alpha^2 m^2}{n}$;

(4) $\rho(G) \geqslant \dfrac{2m}{n}$, *equality holds if and only if G is a regular graph; and*

(5) $\rho(G) \geqslant \sqrt{\dfrac{Zg(G)}{n}}$, *equality holds if and only if G is a regular graph.*

From part 3 of Lemma 2.1, we have $\sum_{i=1}^{n} s_i^2 = (1-\alpha)^2 \|A(G)\|_F^2 + \sum_{i=1}^{n} \left(\alpha d_i - \dfrac{2\alpha m}{n}\right)^2$.

Let

$$2S(G) := (1-\alpha)^2 \|A(G)\|_F^2 + \sum_{i=1}^{n} \left(\alpha d_i - \frac{2\alpha m}{n}\right)^2. \tag{2.2}$$

We observe that $2S(G) = (1-\alpha)^2 \|A(G)\|_F^2$ if and only if G is a regular graph, otherwise $2S(G) > (1-\alpha)^2 \|A(G)\|_F^2$. Further $2S(G) = \|A(G) - \frac{2\alpha m}{n}\mathbb{I}_n\|_F^2 = \sum_{i=1}^{n} s_i^2$, where $\mathbb{I}_n$ is the identity matrix of order n.

It is well known that a graph G has two distinct eigenvalues if and only if $G \cong K_n$, where K_n represents the complete graph on n vertices. Using this fact it can be easily proved that a graph G has two distinct α-adjacency eigenvalues, for $\alpha \neq 1$, if and only if $G \cong K_n$. The α-adjacency spectrum of K_n is given in next lemma, which can be found in [19].

LEMMA 2.2. *The spectrum of $A_\alpha(K_n)$ is $\{(n-1), (n\alpha-1)^{[n-1]}\}$, where $\rho^{[j]}$ means that the algebraic multiplicity of the eigenvalue ρ is j.*

The following Lemma can be found in [24].

LEMMA 2.3. *Let G be graph of order n with m edges and let σ be the number of generalized adjacency eigenvalues of G greater than or equal to $\dfrac{2\alpha m}{n}$. Then*

$$E^{A_\alpha}(G) = 2\Big(\sum_{i=1}^{\sigma} \rho_i - \frac{2\alpha\sigma m}{n}\Big) = 2\max_{1\leqslant i\leqslant k}\left\{\sum_{i=1}^{k} \rho_i - \frac{2\alpha k m}{n}\right\} = 2\max_{1\leqslant i\leqslant k}\left\{S_{k,\alpha} - \frac{2\alpha k m}{n}\right\},$$

where $S_{k,\alpha} = \sum_{i=1}^{k} \rho_i$ is the sum of the k largest generalized adjacency eigenvalues of G.

A very interesting and useful Lemma due to Fulton [5] is as follows.

LEMMA 2.4. *Let A and B be two real symmetric matrices, both of order n. If k, $1 \leqslant k \leqslant n$, is a positive integer, then*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leqslant \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B),$$

where $\lambda_i(X)$ is the i^{th} eigenvalue of X.

The following Arithmetic-Geometric mean inequality can be found in [14].

LEMMA 2.5. *If $a_1, a_2, \ldots, a_n$ are non-negative numbers, then*

$$\begin{aligned} n\left[\frac{1}{n}\sum_{j=1}^{n} a_j - \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}\right] &\leqslant n\sum_{j=1}^{n} a_j - \left(\sum_{j=1}^{n}\sqrt{a_j}\right)^2 \\ &\leqslant n(n-1)\left[\frac{1}{n}\sum_{j=1}^{n} a_j - \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}\right]. \end{aligned}$$

Moreover, equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

A tree of order n with a vertex of degree $n-1$ is denoted by $K_{1,n-1}$ and is called star. The following Lemma gives the α-adjacency spectrum of $K_{1,n-1}$.

LEMMA 2.6. [19] *The A_α-spectrum of $K_{1,n-1}$ is*

$$\left\{\frac{1}{2}\left(\alpha n \pm \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha}\right), \alpha^{[n-2]}\right\}.$$

3. Bounds for the α-adjacency energy of a graph

In this section, we obtain some bounds for the α-adjacency energy, in terms of the order n, the size m, the *Zagreb index*, the vertex covering number and the parameter α, associated with structure of the graph G. We characterize the extremal graphs attaining these bounds.

We first obtain two upper bounds for the smallest generalized adjacency eigenvalues ρ_n. Using the Rayleigh quotient theorem for $0 \neq X = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$, we have

$$\rho_n \leqslant \frac{X^t A_\alpha(G) X}{X^t X}.$$

Let v_n be the vertex with minimum degree $\delta = d_n$ in G. Taking $x_n = 1$ and $x_i = 0$, for $i \leqslant n-1$, we get $\rho_n \leqslant d_n$. Since G is a connected graph, it is easy to see that $X = (0,0,\ldots,0,1)$ is not an eigenvector of $A_\alpha(G)$ corresponding to eigenvalue ρ_n. So, we have

$$\rho_n < \delta. \tag{3.3}$$

Let v_i and v_j be two adjacent vertices in G. Taking $x_i = 1$, $x_j = -1$ and $x_k = 0$, for $k \neq i, j$ and proceeding similarly as above we get

$$\rho_n \leqslant \frac{\alpha(d_i + d_j) - 2(1-\alpha)}{2}. \tag{3.4}$$

Equality occurs in (3.4) if and only if $X = (0, \ldots, 1, 0, \ldots, -1, 0, \ldots, 0)^t$ is an eigenvector of $A_\alpha(G)$ corresponding to the eigenvalue ρ_n. Suppose that such an X is an eigenvector corresponding to the eigenvalue ρ_n. Then using the eigenequation $A_\alpha(G)X = \rho_n X$, we get

(i) $\alpha d_i - (1-\alpha) = \rho_n$,

(ii) $\alpha d_j - (1-\alpha) = -\rho_n$ and

(iii) $(1-\alpha)\Big(a_{ik} - a_{jk}\Big) = 0$, for all $k \neq i, j$.

The equations (i) and (ii) imply that $d_i = d_j$ and (iii) implies that $a_{ik} = a_{jk}$, for all $k \neq i, j$. This gives that for a connected graph G equality occurs in (3.4) if and only if there exist adjacent vertices v_i and v_j in G, such that $d_i = d_j$ and v_i and v_j are adjacent to all other vertices of G. That is, if and only if G is of the form $K_2 \vee H$, where H is a graph on $n-2$ vertices. Note that join of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = G_1 \vee G_2$ having vertex set $V(G) = V_1 \cup V_2$ and edge set $E(G)$ consisting of all the edges in G_1 and G_2 together with edges joining each vertex of G_1 with every vertex of G_2.

The following theorem gives a lower bound for the α-adjacency of a graph, in terms of the number of edges, the number of vertices and the minimum degree.

THEOREM 3.1. *Let G be a connected graph of order n with m edges and let $\alpha \in (0,1)$. Let d_i and d_j be respectively the degrees of the adjacent vertices v_i and v_j in G. If $A_\alpha(G)$ is non-singular, then $E^{A_\alpha}(G) \geqslant \frac{4\alpha m}{n} - \alpha\Big(d_i + d_j - 2(1-\alpha)\Big)$, with equality if and only if $\sigma = n-1$ and G is of the form $K_2 \vee H$, where H is a graph on $n-2$ vertices. If $A_\alpha(G)$ is singular, then $E^{A_\alpha}(G) \geqslant \frac{4\alpha m}{n}$, with equality if and only if $\sigma = n-1$.*

Proof. From the Lemma 2.3 and the fact that $\rho_1 + \rho_2 + \cdots + \rho_n = 2\alpha m$, we have

$$\begin{aligned}
E^{A_\alpha}(G) &= 2 \max_{1 \leqslant i \leqslant k} \left\{ \sum_{i=1}^{k} \rho_i(G) - \frac{2\alpha i m}{n} \right\} \\
&= 2 \max_{1 \leqslant i \leqslant k} \left\{ 2\alpha m - \sum_{i=k+1}^{n} \rho_i(G) - \frac{2\alpha m k}{n} \right\} \\
&= 2 \max_{1 \leqslant i \leqslant k} \left\{ \frac{2\alpha m}{n}(n-k) - \sum_{i=k+1}^{n} \rho_i(G) \right\} \\
&\geqslant \frac{4\alpha m}{n}(n-(n-1)) - 2\rho_n(G).
\end{aligned}$$

If $A_\alpha(G)$ is singular, then $\rho_n = 0$ and the result follows in this case. If $A_\alpha(G)$ is non-singular, then using the inequality (3.4) the results follows. It is clear that equality occurs in this case if and only if $\sigma = n-1$ and $2\rho_n = \alpha(d_i + d_j) - 2(1-\alpha)$. Now, using the discussion before this theorem, it follows that equality occurs in this case if and only if $\sigma = n-1$ and G is of the form $K_2 \vee H$, where H is a graph on $n-2$ vertices. This completes the proof. $\square$

A subset S of the vertex set $V(G)$ is said to be a *covering set* of G if every edge of G is incident to at least one vertex in S. A covering set with minimum cardinality among all covering sets is called the *minimum covering set* of G and its cardinality, denoted by $\tau = \tau(G)$, is called *vertex covering number* of the graph G. The following result gives an upper bound for the generalized adjacency energy of a graph, in terms of the vertex covering number and the number of edges.

THEOREM 3.2. *Let G be a connected graph of order $n \geqslant 2$ and m edges having vertex covering number τ. Let σ be the number of generalized adjacency eigenvalues which are greater than or equal to $\frac{2\alpha m}{n}$.*

(i) *If $\alpha \leqslant 0.5$, then*

$$E^{A_\alpha}(G) \leqslant \alpha(m-\tau) + 2\gamma(\alpha) + 2\alpha\sigma\left(\tau - \frac{2m}{n}\right),$$

where

$$\gamma(\alpha) = \frac{1}{2}\sum_{i=1}^{\tau}\sqrt{\alpha^2(m_i+1)^2 + 4m_i(1-2\alpha)}.$$

(ii) *If $\alpha > 0.5$, then*

$$E^{A_\alpha}(G) \leqslant 2\alpha(m-\tau) + 2\alpha\sigma\left(\tau - \frac{2m}{n}\right).$$

Further, if $G \cong K_{1,n-1}$, then equality occurs in both (i) and (ii).

Proof. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. Let τ be the vertex covering number and C be a minimum vertex covering set of G. Without loss of generality let $C = \{v_1, v_2, \dots, v_\tau\}$.

Let $G_1, G_2, \dots, G_\tau$ be the spanning subgraphs of G corresponding to the vertices $v_1, v_2, \dots, v_\tau$ of C, having vertex set same as G and edge sets defined as follows.

$$E(G_i) = \{v_iv_t : v_t \in N(v_i) \setminus \{v_1, v_2, \dots, v_{i-1}\}\}, \ \ i = 1, 2, \dots, \tau.$$

For $i = 1, 2, \dots, \tau$, let $m_i = |E(G_i)|$. It is clear that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_\tau)$ and $G_i = K_{1,m_i} \cup (n - m_i - 1)K_1$, for all $i = 1, 2, \dots, \tau$. Further, it is easy to see that the generalized adjacency matrix $A_\alpha(G)$ of G can be decomposed as

$$A_\alpha(G) = A_\alpha(G_1) + A_\alpha(G_2) + \cdots + A_\alpha(G_\tau). \tag{3.5}$$

By Lemma 2.6, the generalized adjacency spectrum of $G_i = K_{1,m_i} \cup (n-m_i-1)K_1$ is

$$\left\{ \frac{1}{2}\left(\alpha(m_i+1) \pm \sqrt{\alpha^2(m_i+1)^2+4m_i(1-2\alpha)}\right), \alpha^{[m_i-1]}, 0^{[n-m_i-1]} \right\}.$$

It is easy to see that $\alpha(m_i+1) \pm \sqrt{\alpha^2(m_i+1)^2+4m_i(1-2\alpha)} < 0$ for $\alpha > \frac{1}{2}$ and $\alpha(m_i+1) \pm \sqrt{\alpha^2(m_i+1)^2+4m_i(1-2\alpha)} \geqslant 0$ for $\alpha \leqslant \frac{1}{2}$. Using Lemma 2.4, we have

$$S_{k,\alpha}(G_i) \leqslant \frac{1}{2}\left(\alpha(m_i+1) + \sqrt{\alpha^2(m_i+1)^2+4m_i(1-2\alpha)}\right) + (k-1)\alpha$$

for $\alpha \leqslant \frac{1}{2}$ and

$$S_{k,\alpha}(G_i) \leqslant \alpha(m_i+1) + (k-2)\alpha$$

for $\alpha > \frac{1}{2}$, where $i = 1,2,\ldots,\tau$. Now, for $\alpha \leqslant \frac{1}{2}$, applying Lemma 2.4 to equation (3.5), we get

$$\begin{aligned} S_{k,\alpha}(G) &\leqslant S_{k,\alpha}(G_1) + S_{k,\alpha}(G_2) + \cdots + S_{k,\alpha}(G_\tau) \\ &\leqslant \sum_{i=1}^{\tau}\left(\frac{\alpha(m_i+1) + \sqrt{\alpha^2(m_i+1)^2+4m_i(1-2\alpha)}}{2} + (k-1)\alpha\right) \\ &= \frac{\alpha m}{2} + \gamma(\alpha) + \tau\alpha\left(k - \frac{1}{2}\right) \end{aligned} \tag{3.6}$$

where $\gamma(\alpha) = \frac{1}{2}\sum_{i=1}^{\tau}\sqrt{\alpha^2(m_i+1)^2+4m_i(1-2\alpha)}$. Similarly, for $\alpha > \frac{1}{2}$, we get

$$S_{k,\alpha}(G) \leqslant \alpha\Big(m + \tau(k-1)\Big). \tag{3.7}$$

Lastly, using the Lemma 2.3 together with inequalities (3.6) and (3.7), we get

$$\begin{aligned} E^{A_\alpha}(G) &= 2\left(S_{\sigma,\alpha}(G) - \frac{2\sigma\alpha m}{n}\right) \\ &\leqslant \alpha m + 2\gamma(\alpha) + 2\tau\alpha\left(\sigma - \frac{1}{2}\right) - \frac{4\sigma\alpha m}{n} \\ &= \alpha(m-\tau) + 2\gamma(\alpha) + 2\alpha\sigma\left(\tau - \frac{2m}{n}\right) \end{aligned}$$

for $\alpha \leqslant 0.5$ and

$$\begin{aligned} E^{A_\alpha}(G) &= 2\left(S_{\sigma,\alpha}(G) - \frac{2\sigma\alpha m}{n}\right) \\ &\leqslant 2\alpha m + 2\alpha\tau(\sigma-1) - \frac{4\sigma\alpha m}{n} \\ &= 2\alpha(m-\tau) + 2\alpha\sigma\left(\tau - \frac{2m}{n}\right) \end{aligned}$$

for $\alpha > 0.5$. From the equation (3.5) and the Lemma 2.3, it is clear that if $G \cong K_{1,n-1}$, then equality occurs. This completes the proof. $\square$

Using Cauchy-Schwarz's inequality to $\gamma(\alpha)$, we get

$$\begin{aligned}\gamma(\alpha) &= \frac{1}{2}\sum_{i=1}^{\tau}\sqrt{\alpha^2(m_i+1)^2+4m_i(1-2\alpha)}\\ &\leqslant \frac{1}{2}\sqrt{\tau\sum_{i=1}^{\tau}\left(\alpha^2(m_i+1)^2+4m_i(1-2\alpha)\right)}\\ &= \frac{1}{2}\sqrt{\alpha^2\tau\sum_{i=1}^{\tau}(m_i+1)^2+4m\tau(1-2\alpha)}.\end{aligned}$$

This inequality together with the part (i) of Theorem 3.2 gives the following upper bound for the generalized adjacency energy for $\alpha \leqslant 0.5$.

COROLLARY 3.3. *Let G be a connected graph of order $n \geqslant 2$ and m edges having vertex covering number τ and let $\alpha \leqslant 0.5$. Let σ be the number of generalized adjacency eigenvalues of G which are greater than or equal to $\frac{2\alpha m}{n}$. Then*

$$E^{A_\alpha}(G) \leqslant \alpha(m-\tau)+\sqrt{\alpha^2\tau\sum_{i=1}^{\tau}(m_i+1)^2+4m\tau(1-2\alpha)}+2\alpha\sigma\left(\tau-\frac{2m}{n}\right).$$

Moreover, if $G \cong K_{1,n-1}$, then equality occurs.

Taking $\alpha = 0$ in Corollary 3.3 and using the fact $E^{A_0}(G) = \mathscr{E}(G)$, we obtain the following upper bound for energy of a graph, which was obtained in [7].

COROLLARY 3.4. *Let G be a connected graph of order $n \geqslant 2$ and m edges having vertex covering number τ. Then*

$$E(G) \leqslant 2\sqrt{\tau m}.$$

Moreover, if $G \cong K_{1,n-1}$, then equality occurs.

We note that for a connected graph having maximum degree Δ and vertex covering number τ, we always have $m \leqslant \tau\Delta$. This fact together with Corollary 3.4 gives that the upper bound given by Wang and Ma [25] for the energy follows from the upper bound given by Theorem 2.2 in [7].

The following result gives upper and lower bounds for the generalized adjacency energy in terms of order n, the parameter $\|A(G)\|_F^2$ and the determinant of the generalized adjacency matrix $A_\alpha(G)$.

THEOREM 3.5. *Let G be a connected graph of order $n \geqslant 3$ with m edges having Zagreb index $Zg(G)$ and maximum degree Δ. Then*

$$E^{A_\alpha}(G) \leqslant \Delta - \frac{2\alpha m}{n} + \sqrt{(n-2)\left(\gamma_1 - (1-\alpha)^2\frac{4m^2}{n^2}\right) + (n-1)\left((1-\alpha)\frac{2m}{n}\right)^{\frac{-2}{n-1}}\gamma_2}$$

and

$$E^{A_\alpha}(G) \geqslant (1-\alpha)\frac{2m}{n} + \sqrt{\gamma_1 - \left(\Delta - \frac{2\alpha m}{n}\right)^2 + (n-1)(n-2)\left(\Delta - \frac{2\alpha m}{n}\right)^{\frac{-2}{n-1}}\gamma_2},$$

where $\gamma_1 = \alpha^2 Zg(G) + (1-\alpha)^2\|A(G)\|_F^2 - \dfrac{4\alpha^2 m^2}{n}$ and $\gamma_2 = \left|\det\left(A_\alpha(G) - \dfrac{2m\alpha}{n}I\right)\right|^{\frac{2}{n-1}}$. Equality occurs in both inequalities if and only if $G \cong K_n$ or G is a Δ-regular graph with three distinct generalized adjacency eigenvalues, $\rho_1 = \Delta$ and the other two eigenvalues with absolute value $\sqrt{\dfrac{\alpha^2 Zg(G) + (1-\alpha)^2\|A(G)\|_F^2}{n-1}}$.

Proof. Replacing n by $n-1$ and setting $a_j = |s_j|^2$, for $j = 2,\ldots,n$ in Lemma 2.5, we have

$$\alpha \leqslant (n-1)\sum_{j=2}^{n}|s_j|^2 - \left(\sum_{j=2}^{n}|s_j|\right)^2 \leqslant (n-2)\alpha,$$

that is,

$$\alpha \leqslant (n-1)\sum_{j=2}^{n}|s_j|^2 - \left(E^{A_\alpha}(G) - |s_1|\right)^2 \leqslant (n-2)\alpha, \tag{3.8}$$

where

$$\begin{aligned}\alpha &= (n-1)\left[\frac{1}{n-1}\sum_{j=2}^{n}|s_j|^2 - \left(\prod_{j=2}^{n}|s_j|^2\right)^{\frac{1}{n-1}}\right]\\ &= \sum_{j=2}^{n}|s_j|^2 - (n-1)\left(\prod_{j=2}^{n}|s_j|\right)^{\frac{2}{n-1}}\\ &= \sum_{j=2}^{n}|s_j|^2 - \frac{(n-1)}{|s_1|^{\frac{2}{n-1}}}\left|\det\left(A_\alpha(G) - \frac{2m\alpha}{n}I\right)\right|^{\frac{2}{n-1}}.\end{aligned}$$

Using (iii) of Lemma 2.1 and the value of α, it follows from the left inequality of (3.8) that

$$\left(E^{A_\alpha}(G) - |s_1|\right)^2 \leqslant (n-2)\sum_{j=2}^{n}|s_j|^2 + \frac{(n-1)}{|s_1|^{\frac{2}{n-1}}}\left|\det\left(A_\alpha(G) - \frac{2m\alpha}{n}I\right)\right|^{\frac{2}{n-1}},$$

that is,

$$E^{A_\alpha}(G) \leqslant s_1 + \sqrt{(n-2)(\gamma_1 - s_1^2) + (n-1)s_1^{\frac{-2}{n-1}} \left| \det\left(A_\alpha(G) - \frac{2m\alpha}{n} I\right)\right|^{\frac{2}{n-1}}}, \quad (3.9)$$

where $\gamma_1 = \alpha^2 Zg(G) + (1-\alpha)^2 \|A(G)\|_F^2 - \frac{4\alpha^2 m^2}{n}$ and $s_1 \geqslant 0$. Now, by part (iv) of Lemma 2.1, we have $\rho_1 \geqslant \frac{2m}{n}$, giving that $s_1 \geqslant (1-\alpha)\frac{2m}{n}$. Also, using the fact $A_\alpha(G)$ is a non-negative irreducible matrix (as G is connected), it follows that $\rho_1 \leqslant \Delta$, where Δ is the maximum degree. Note that equality occurs in this last inequality if and only if G is a regular graph. This gives that $s_1 = \rho_1 - \frac{2\alpha m}{n} \leqslant \Delta - \frac{2\alpha m}{n}$. Using the inequalities $s_1 \geqslant (1-\alpha)\frac{2m}{n}$ and $s_1 \leqslant \Delta - \frac{2\alpha m}{n}$ in (3.9), we get the first inequality.

Again, using the value of α, it follows from the right inequality of (3.8) that

$$\left(E^{A_\alpha}(G) - |s_1|\right)^2 \geqslant \sum_{j=2}^{n} |s_j|^2 + (n-1)(n-2)|s_1|^{\frac{-2}{n-1}} \left| \det\left(A_\alpha(G) - \frac{2m\alpha}{n} I\right)\right|^{\frac{2}{n-1}},$$

that is,

$$E^{A_\alpha}(G) \geqslant s_1 + \sqrt{\gamma_1 - s_1^2 + (n-1)(n-2)s_1^{\frac{-2}{n-1}} \left| \det\left(A_\alpha(G) - \frac{2m\alpha}{n} I\right)\right|^{\frac{2}{n-1}}}. \quad (3.10)$$

Now, using the inequalities $s_1 \geqslant (1-\alpha)\frac{2m}{n}$ and $s_1 \leqslant \Delta - \frac{2\alpha m}{n}$ in (3.10) we get the second inequality.

Equality occurs in the first inequality if and only if equality occurs in Lemma 2.5, $s_1 = (1-\alpha)\frac{2m}{n}$ and $s_1 = \Delta - \frac{2\alpha m}{n}$. From Lemma 2.1, it is clear that $s_1 = (1-\alpha)\frac{2m}{n}$ and $s_1 = \Delta - \frac{2\alpha m}{n}$ if and only if G is a Δ-regular graph. Also, equality occurs in Lemma 2.5 if and only if $|s_2|^2 = |s_3|^2 = \cdots = |s_n|^2$, that is, if and only if $|s_2| = |s_3| = \cdots = |s_n|$, as ρ_i are real numbers. The following cases arise.

Case 1. If G is a Δ-regular graph with $s_2 = s_3 = \cdots = s_n$, then $\rho_2 = \rho_3 = \cdots = \rho_n$ giving that G is a connected graph with two distinct generalized adjacency eigenvalues. Using Lemma 2.2, it follows that $G \cong K_n$ in this case.

Case 2. On the other hand if G is a Δ-regular graph with at least one s_i different from s_1, then $|s_2| = |s_3| = \cdots = |s_n|$ gives there exist a positive integer k, such that $s_2 = \cdots = s_k$ and $s_{k+1} = \cdots = s_n$. That is, $\rho_2 = \cdots = \rho_k = \theta$ and $\rho_{k+1} = \cdots = \rho_n = -\theta$. Since, $\sum_{j=2}^{n} \rho_j^2 = \alpha^2 Zg(G) + (1-\alpha)^2 \|A(G)\|_F^2$, it follows that equality occurs in this case if G is a connected Δ-regular graph with three distinct generalized adjacency eigenvalues, namely, the eigenvalue $\rho_1 = \Delta_1$ and the other two eigenvalues with absolute value $\sqrt{\dfrac{\alpha^2 Zg(G) + (1-\alpha)^2\|A(G)\|_F^2}{n-1}}$. Similarly, we can discuss the equality case for second inequality.

Conversely, it is easy to see that equality occurs in each of the inequalities for the mentioned cases. This completes the proof. □

Proceeding similar as in Theorem 3.5 and using part (v) of Lemma 2.1 we obtain the following result.

THEOREM 3.6. *Let G be a connected graph of order $n \geqslant 3$ with m edges having Zagreb index $Zg(G)$ and maximum degree Δ. Then*

$$E^{A_\alpha}(G) \leqslant \Delta - \frac{2\alpha m}{n} + \sqrt{(n-2)\left(\gamma_1 - \left(\sqrt{\frac{Zg(G)}{n}} - \frac{2\alpha m}{n}\right)^2\right) + (n-1)\left(\sqrt{\frac{Zg(G)}{n}} - \frac{2m\alpha}{n}\right)^{\frac{-2}{n-1}}\gamma_2}$$

and

$$E^{A_\alpha}(G) \geqslant \sqrt{\frac{Zg(G)}{n}} - \frac{2m\alpha}{n} + \sqrt{\gamma_1 - \left(\Delta - \frac{2\alpha m}{n}\right)^2 + (n-1)(n-2)\left(\Delta - \frac{2\alpha m}{n}\right)^{\frac{-2}{n-1}}\gamma_2},$$

where $\gamma_1 = \alpha^2 Zg(G) + (1-\alpha)^2\|A(G)\|_F^2 - \dfrac{4\alpha^2 m^2}{n}$ and $\gamma_2 = \left|\det\left(A_\alpha(G) - \dfrac{2m\alpha}{n}I\right)\right|^{\frac{2}{n-1}}$. Equality occurs in both the inequalities if and only if $G \cong K_n$ or G is a Δ-regular graph with three distinct generalized adjacency eigenvalues, $\rho_1 = \Delta$ and the other two eigenvalues with absolute value $\sqrt{\dfrac{\alpha^2 Zg(G) + (1-\alpha)^2\|A(G)\|_F^2}{n-1}}$.

The following result gives an upper bound for the generalized adjacency energy in terms of vertex degrees and the energy of a graph.

THEOREM 3.7. *Let G be a connected graph of order $n \geqslant 3$ with m edges and vertex degrees $d_1, d_2, \ldots, d_n$. Let σ be the number of generalized adjacency eigenvalues of G which are greater than or equal to $\dfrac{2\alpha m}{n}$. Then*

$$E^{A_\alpha}(G) \leqslant (1-\alpha)\mathscr{E}(G) + \alpha\sum_{i=1}^{\sigma}\left(d_i - \frac{2m}{n}\right).$$

Equality occurs if G is a regular graph.

Proof. Applying Lemma 2.4 to

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

we get

$$\sum_{i=1}^{k} \rho_i(G) \leqslant \alpha \sum_{i=1}^{k} d_i + (1-\alpha) \sum_{i=1}^{k} \lambda_i, \tag{3.11}$$

where $\lambda_i(G)$ are the adjacency eigenvalues of G. Let σ be the number of generalized adjacency eigenvalues of G which are greater than or equal to $\frac{2\alpha m}{n}$, then $1 \leqslant \sigma \leqslant n$. From the definition of energy, we have

$$\mathscr{E}(G) = 2 \max_{1\leqslant j\leqslant n} \sum_{i=1}^{k} \lambda_i(G) \geqslant 2 \sum_{i=1}^{\sigma} \lambda_i(G)$$

This together with inequality 3.11 gives

$$\begin{aligned} 2\sum_{i=1}^{\sigma} \rho_i(G) &\leqslant 2\alpha \sum_{i=1}^{\sigma} d_i + 2(1-\alpha) \sum_{i=1}^{\sigma} \lambda_i(G) \\ 2\sum_{i=1}^{\sigma} \rho_i - \frac{4\alpha\sigma m}{n} &\leqslant 2\alpha \sum_{i=1}^{\sigma} d_i + (1-\alpha)E(G) - \frac{4\alpha\sigma m}{n}. \end{aligned}$$

Thus, using Lemma 2.3, it follows that

$$E^{A_\alpha}(G) \leqslant (1-\alpha)\mathscr{E}(G) + 2\alpha \sum_{i=1}^{\sigma} \left(d_i - \frac{2m}{n} \right).$$

If G is regular graph then it is clear that equality occurs. $\square$

4. Conclusion

In this paper our aim was to present some new upper and lower bounds for the generalized adjacency energy of a graph, in terms of different graph structural parameters. Formally, we presented bounds connected the generalized adjacency energy with the vertex degrees, the Zagreb index (a well-known topological index), the vertex covering number, the number of vertices and the number of edges of a graph.

The importance of our results is that for $\alpha = 0$, we obtain the corresponding results for the energy (adjacency energy) and for $\alpha = 0.5$, we obtain the corresponding results for the signless Laplacian energy of a graph, giving that the results obtained in this paper present generalizations and extensions of the corresponding results obtained for the energy and the signless Laplacian energy. Further, in the language of Linear Algebra, the generalized adjacency energy of a graph G represents the trace norm of the matrix $A_\alpha(G) - \frac{2\alpha m}{n}\mathbb{I}_n$. Trace norm of a general matrix is an interesting and important concept in Matrix Theory and has been extensively studied for graph matrices. This gives another motivation for the study of the generalized adjacency energy and its importance from Matrix Theory point of view.

Although, we have presented some upper and lower bounds for the generalized adjacency energy connecting it with different graph parameters associated with the structure of the graph, there are many other graph parameters like the clique number, the independence number, the chromatic number, the domination number, etc or the topological indices like, the second Zagreb index, the Randić index, etc, whose connection with the generalized adjacency energy will be an interesting research problem for future research.

Declarations

Availability of data and material. The data used to support the findings of this study are included within the article.

Competing interests. The authors declare that there is no conflict of interests regarding the publication of this paper.

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