L^p BOUNDS FOR SINGULAR INTEGRAL OPERATORS ALONG TWISTED SURFACES

BADRIYA AL-AZRI AND AHMAD AL-SALMAN

(*Communicated by L. Liu*)

Abstract. This paper concerns the study singular integrals along twisted surfaces of the form

$$\{(\Phi(|v|)u, \Psi(|u|)v) : (u,v) \in \mathbb{R}^n \times \mathbb{R}^m\}.$$

We prove L^p bounds for the corresponding operators when the surfaces are defined by mappings more general than polynomials and convex functions, provided that the kernels are in $L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$.

1. Introduction and statement of results

For $d \geqslant 2$, let $\mathbb{R}^d$ ($d = n$ or $d = m$) be the d-dimensional Euclidean space and $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ equipped with normalized Lebsgue measure $d\sigma$. For non-zero point $y \in \mathbb{R}^n$ ($y \neq 0$), we let $y' = \dfrac{y}{|y|} \in \mathbb{S}^{n-1}$. Let $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ be such that

$$\int_{\mathbb{S}^{n-1}} \Omega(u', .) d\sigma(u') = \int_{\mathbb{S}^{m-1}} \Omega(., v') d\sigma(v') = 0, \qquad (1.1)$$

and

$$\Omega(tx, sy) = \Omega(x, y), \qquad \forall t, s > 0. \qquad (1.2)$$

The classical singular integral operator on product domains associated to the function Ω is defined by

$$\mathscr{S}_{\Omega}(f)(x,y) = \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x-u, y-v) \frac{\Omega(u,v)}{|u|^n |v|^m} du dv. \qquad (1.3)$$

The study of the operator $\mathscr{S}_{\Omega}$ began by Fefferman-Stein in [11] and Fefferman [10]. In [11], Fefferman and Stein showed that $\mathscr{S}_{\Omega}$ is bounded on $L^p(\mathbb{R}^{n+m})$ for $(1 < p < \infty)$ if Ω satisfies certain Lipschitz conditions. Subsequently, several authors have studied the L^p boundedness of the operator $\mathscr{S}_{\Omega}$ under various conditions on Ω. For further results and background information, we refer the readers to consult [2], [8], [9], [11], among others. In particular, in [8], Duoandikoetxea proved that $\mathscr{S}_{\Omega}$ is bounded on L^p

when $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $q > 1$. Subsequently, Fan-Guo-Pan in [9] obtained the same L^p boundedness result but under the condition that Ω lies in certain Block spaces $B_q^{0,0}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ introduced by Jiang and Lu in [12]. For $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, a function Ω lies in the space $B_q^{0,0}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ if $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $\{c_\mu\}$ is a sequence of complex numbers, b_μ is a measurable function satisfying the properties that $supp(b_\mu) = I_\mu$, $\|b_\mu\|_{L^q} \leqslant |I_\mu|^{-1/q'}$, and

$$M_q^{0,0}(\{c_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu|(1 + \log^+(|I_\mu|^{-1})) < \infty.$$

Here, I_μ is an interval on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$. It is well known that Block spaces enjoy the following properties:

$$L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subseteq B_q^{0,0}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}), \qquad B_q^{0,0}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \nsubseteq \bigcup_{q>1} L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}),$$

and

$$B_{q_2}^{0,0}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subset B_{q_1}^{0,0}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \qquad \text{whenever} \qquad 1 < q_1 < q_2.$$

For detailed information about Block spaces, we refer the readers to [12].

In [7], Al-Salman, Al-Qassem, and Pan investigated the L^p boundedness of the operator $\mathscr{S}_\Omega$ under the natural condition $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, i.e.,

$$\int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} |\Omega(u,v)| \log^2(2 + |\Omega(u,v)|) d\sigma(u) d\sigma(v) < \infty. \tag{1.4}$$

They proved that $\mathscr{S}_\Omega$ is bounded on L^p $(1 < p < \infty)$ provided that $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Moreover, they showed that the L^p boundedness of $\mathscr{S}_\Omega$ may fail if the condition $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is replaced by $\Omega \in L(\log L)^{2-\alpha}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for any given $\alpha > 0$. It should be remarked here that the following inclusions hold:

$$L(\log^+ L)^s(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subset L(\log^+ L)^r(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \qquad \text{whenever} \quad r < s$$

and

$$L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subsetneqq L(\log^+ L)^r(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subsetneqq L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \tag{1.5}$$

whenever $q > 1$ and $r \geqslant 1$.

In his investigation of the L^p mapping properties of Marcinkiewicz functions, Al-Salman [3] introduced the following class of mappings that are more general than polynomials and convex functions:

DEFINITION 1.1. ([3]). A function $\psi : [0, \infty) \to \mathbb{R}$ is said to belong to the class $\mathscr{PC}_\lambda(d)$ if there exist a polynomial P belongs to the class $\mathscr{P}_d$ of all real valued polynomials with degree at most d and a mapping $\varphi \in C^{d+1}[0, \infty)$ such that

(i) $\psi(t) = P(t) + \lambda \varphi(t)$

(ii) $P(0) = 0$ and $\varphi^{(j)}(0) = 0$ for $0 \leqslant j \leqslant d$

(iii) $\varphi^{(j)}$ is positive non-decreasing on $(0, \infty)$ for $0 \leqslant j \leqslant d+1$.

For convenience, the polynomial P satisfying the conditions (i) and (ii) above will be denoted by P_ψ. It was pointed out in [3] that the class $\cup_{d \geqslant 0}(\mathscr{PC}_\lambda(d))$ contains the class of polynomials $\mathscr{P}_d$ as well as the class of convex increasing functions. Recently, Al-Azriyah and Al-Salman studied singular integrals on product domains along surfaces determined by mapping that lie in the class $\mathscr{PC}_\lambda(d)$. In fact, Al-Azriyah and Al-Salman proved the following result:

THEOREM 1.1. ([1]). *Let* $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *satisfying* (1.1)–(1.2) *and* (1.4). *If* $\Phi \in \mathscr{PC}_\lambda(d), \Psi \in \mathscr{PC}_\alpha(b)$ *for some* $d, b > 0$ *and* $\lambda, \alpha \in \mathbb{R}$, *then the operator*

$$T_{\Phi,\Psi,\Omega}(f)(x,y) = p.v. \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x - \Phi(|u|)u', y - \Psi(|v|)v') \frac{\Omega(u',v')}{|u|^n |v|^m} \, du dv \qquad (1.6)$$

is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ *for* $1 < p < \infty$ *with* L^p *bounds independent of* $\lambda, \alpha \in \mathbb{R}$ *and the coefficients of the particular polynomials involved in the standard representations of* Φ *and* Ψ *given in Definition* 1.1.

The aim of this paper is to investigate the L^p boundedness of a class of singular integral operators on product domains along twisted surfaces determined by mappings that lie in certain $\mathscr{PC}_\lambda(d)$. Let $h : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a measurable function. For suitable mapping $\Lambda : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ of the form

$$\Lambda(u,v) = (\Phi(|v|)u, \Psi(|u|)v), \qquad (1.7)$$

where $\Phi : (0, \infty) \to \mathbb{R}$ and $\Psi : (0, \infty) \to \mathbb{R}$, we consider the singular integral operator on product domains defined by

$$\mathscr{S}_{\Omega,h,\Lambda}(f)(x,y) = \int_{\mathbb{R}^n \times \mathbb{R}^m} f((x,y) - \Lambda(u,v)) \frac{\Omega(u',v')}{|u|^n |v|^m} h(|u|, |v|) \, du dv. \qquad (1.8)$$

It is clear that, when $\Phi(t) = \Psi(t) = c$-constant and $h \equiv 1$, then $\mathscr{S}_{\Omega,h,\Lambda}$ is the classical operator $\mathscr{S}_\Omega$. In [6], Al-Salman proved the L^p boundedness of $\mathscr{S}_{\Omega,h,\Lambda}$ for $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and $h \in L^\infty([0, \infty) \times [0, \infty))$ provided that the functions Φ and Ψ belong to the class $\mathscr{F}$ of smooth functions $\varphi : (0, \infty) \to \mathbb{R}$ which satisfy $\varphi(0) = 0$ and the following growth conditions:

$$|\varphi(t)| \leqslant C_1 t^d \quad \text{and} \quad \left| \varphi''(t) \right| \geqslant C_2 t^{d-2} \qquad (1.9)$$

for some $d \neq 0$ and $t \in (0, \infty)$, where C_1 and C_2 are positive constants independent of t. It was pointed out in [6] that the operator $\mathscr{S}_{\Omega,h,\Lambda}$ may fail to be bounded on L^p ($1 < p < \infty$) when $\Phi(t) = t$ or $\Psi(t) = t$. In fact, it is shown in [6] that $|\mathscr{S}_{\Omega,\Lambda}(f)(x,y)| = \infty$ if $\Phi(t) = \Psi(t) = t$ for certain choice of f. Furthermore, as a consequence of

the L^p boundedness of certain maximal functions, Al-Salman in [5] deduced the L^p boundedness of $\mathscr{S}_{\Omega,h,\Lambda}$ when the functions Φ and Ψ satisfy some growth conditions similar to (1.9), the function Ω is in $L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, and that h is a measurable function that satisfies

$$\|h\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, r^{-1}s^{-1}drds)} = \left(\int_0^\infty \int_0^\infty |h(r,s)|^2 r^{-1} s^{-1} drds \right)^{\frac{1}{2}} \leqslant 1.$$

In the same paper, Al-Salman obtained the same result for $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ $(q > 1)$.

We remark here that the classes $\mathscr{F}$ and $\cup_{d \geqslant 0}(\mathscr{PC}_\lambda(d))$ are different. In particular, the function $\varphi(t) = t^2 e^{-\frac{1}{t}}$ for $t > 0$ and $\varphi(t) = 0$ for $t \leqslant 0$ lies in $\cup_{d \geqslant 0}(\mathscr{PC}_\lambda(d)) \backslash \mathscr{F}$. On the other hand, the power function $\varphi(t) = \sqrt{t}$ lies in $\mathscr{F} \backslash \cup_{d \geqslant 0}(\mathscr{PC}_\lambda(d))$. Therefore, it is natural to ask if the operators $\mathscr{S}_{\Omega,h,\Lambda}$ in (1.8) are bounded on some L^p if the functions Φ and Ψ are in $\cup_{d \geqslant 0}(\mathscr{PC}_\lambda(d))$. Motivate by the work in [4], we introduce the following class of functions:

DEFINITION 1.2. For $b, d \geqslant 0$ and $\lambda, \alpha \in \mathbb{R}$, we let $\mathscr{H}(d, b, \lambda, \alpha)$ be the class of all pairs (Φ, Φ) of functions Φ and Ψ with $\Phi \in \mathscr{PC}_\lambda(d)$ and $\Psi \in \mathscr{PC}_\alpha(b)$ such that the corresponding polynomials P_Φ and P_Ψ satisfy one of the following conditions:

(i) $P_\Phi(0) = P_\Psi(0) = 0$ and

$$\lim_{t \to 0} \frac{P_\Phi(t)}{t} = \lim_{t \to 0} \frac{P_\Psi(t)}{t} = 0;$$

(ii) $\deg(P_\Phi) + \deg(P_\Psi) = 1$;

(iii) $\deg(P_\Phi)\deg(P_\Psi) = 0$ and $\deg(P_\Phi) + \deg(P_\Psi) > 1$;

(iv) $\deg(P_\Phi) = \deg(P_\Psi) = 1$, $P_\Phi(0) \neq 0$ and $P_\Psi(0) \neq 0$;

(v) $P_\Phi(t) = t$ and $\deg(P_\Psi) > 1$ with $\lim_{t \to 0}(P_\Psi(t) - P_\Psi(0))/t = 0$ or $P_\Psi(t) = t$ and $\deg(P_\Phi) > 1$ with $\lim_{t \to 0}(P_\Phi(t) - P_\Phi(0))/t = 0$.

Our main result is the following:

THEOREM 1.2. *Suppose that* $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *and satisfying* (1.1)– (1.2). *If* $\Lambda(u,v) = (\Phi(|v|)u, \Psi(|u|)v)$, *with* $(\Phi, \Psi) \in \mathscr{H}(d, b, \lambda, \alpha)$ *for* $d, b > 0$, *then the operator* $\mathscr{S}_{\Omega,h,\Lambda}$ *in* (1.8) *is bounded on* $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ *for* $1 < p < \infty$ *with* L^p *bounds independent of* $\lambda, \alpha \in \mathbb{R}$ *and the coefficients of the particular polynomials involved in the standard representations of* Φ *and* Ψ.

In order to prove the above theorem, we consider a special family of maximal functions along twisted surfaces. For $(z_1, z_2) \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ and suitable functions $\Phi, \Psi : [0, \infty) \to \mathbb{R}$, we let

$$v_{\Phi,\Psi}^{(z_1,z_2)}(f)(x,y) = \sup_{k,j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} |f(x - \Phi(\zeta)r z_1, y - \Psi(r) \zeta z_2)| \frac{dr\, d\zeta}{r\zeta}. \qquad (1.10)$$

It is worth mentioning that dealing with maximal functions involving twisted surfaces is more complex than dealing with the classical maximal functions. As far as we know, very little is known about the boundedness of $v_{\Phi,\Psi}^{(z_1,z_2)}$ in (1.10). Recently, Al-Salman [4] proved the L^p boundedness of $v_{\Phi,\Psi}^{(z_1,z_2)}$ when Φ and Ψ are polynomials that satisfy one of the conditions $(i) - (v)$ in the Definition 1.2. Besides that, the same author in [6] obtained that $v_{\Phi,\Psi}^{(z_1,z_2)}$ is bounded on L^p for all $p \in (1,\infty)$, provided that $\Phi, \Psi \in \mathscr{F}$. In this paper, we shall prove the following result:

THEOREM 1.3. *Suppose that* $\Lambda(u,v) = (\Phi(|v|)u, \Psi(|u|)v)$, *where* $\Phi, \Psi \in \mathscr{H}(d,b,\lambda,\alpha)$ *for* $d, b > 0$ *and* $\lambda, \alpha \in \mathbb{R}$. *Let* $v_{\Phi,\Psi}^{(z_1,z_2)}$ *be given by* (1.10). *Then* $v_{\Phi,\Psi}^{(z_1,z_2)}$ *is bounded on* L^p *for all* $1 < p < \infty$ *with* L^p *bounds independent of* $\lambda, \alpha \in \mathbb{R}$ *and the coefficients of the particular polynomials involved in the standard representations of* Φ *and* Ψ.

As a consequence of Theorem 1.3, we have the following result:

THEOREM 1.4. *Suppose that* $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ *satisfying* (1.1)–(1.2) *and* Φ, Ψ
$\in \mathscr{H}(d,b,\lambda,\alpha)$, *for* $d, b \geqslant 0$. *Let* $v_{\Omega,\Lambda_{\Phi,\Psi}}$ *be the maximal function*

$$v_{\Omega,\Lambda_{\Phi,\Psi}}(f)(x,y) = \sup_{k,j \in \mathbb{Z}} \left| \iint_{\substack{2^k < |v| < 2^{k+1} \\ 2^j < |u| < 2^{j+1}}} f((x,y) - \Lambda(u,v)) \frac{\Omega(u',v')}{|u|^n |v|^m} du\, dv \right|. \qquad (1.11)$$

Then, there exists a constant $C_p > 0$ *such that*

$$\|v_{\Omega,\Lambda_{\Phi,\Psi}}(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leqslant C_p \|\Omega\|_{L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}$$

for all $1 < p < \infty$ *with* L^p *bounds independent of* $\lambda, \alpha \in \mathbb{R}$ *and the coefficients of the particular polynomials involved in the standard representations of* Φ *and* Ψ.

Historically, Al-Salman in [5] obtained the L^p boundedness of $v_{\Omega,\Lambda_{\Phi,\Psi}}$ where Φ and Ψ satisfy (1.9). In [4], the same author proved that $v_{\Omega,\Lambda_{\Phi,\Psi}}$ is bounded on L^p $(1 < p < \infty)$ provided that Φ and Ψ are polynomials satisfying one of the conditions $(i)-(v)$ in Definition 1.2. We remark here that, in light of the relations (1.5) and the

definition of the class $\mathscr{H}(d,b,\lambda,\alpha)$, Theorem 1.2, and Theorem 1.4 are fundamental generalizations of Theorem 4.3 and Corollary 4.4 in [5] respectively. Furthermore, Theorem 1.2, Theorem 1.3, and Theorem 1.4 generalize the corresponding results in [4], [6], and [5].

This paper is organized as follows. In Section 2, we present some preliminary lemmas. In Section 3, we will develop and prove some maximal function results. The proofs of Theorem 1.3 and Theorem 1.4 will be presented in Section 4. Finally, the proof of Theorem 1.2 will be presented in Section 5.

Throughout this paper, the letter C will stand for a constant that may vary at each occurrence but it is independent of the essential variables.

2. Preliminary tools

This section is devoted to recall some known lemmas. We start by recalling the following lemma in [1] (see also [3]):

LEMMA 2.1. ([1]). *If* $\varphi \in C^{d+1}[0,\infty)$ *and satisfies the conditions* (i)–(iii) *in Definition* 1.1, *then*

(i) $\varphi(\alpha r) \leqslant \alpha \varphi(r)$ *for* $0 \leqslant \alpha \leqslant 1$ *and* $r > 0$

(ii) $\varphi(\alpha r) \geqslant \alpha \varphi(r)$ *for* $\alpha \geqslant 1$ *and* $r > 0$.

(iii) $\varphi^{(d+1)}(r) \geqslant r^{-d-1}\varphi(r)$ *for* $r > 0$.

The following Proposition will play a key role in this paper:

PROPOSITION 2.2. ([6]). *Let* $L : \mathbb{R}^n \to \mathbb{R}^n$ *and* $H : \mathbb{R}^m \to \mathbb{R}^m$ *be linear transformations. Let* $\{\sigma_{k,j} : k, j \in \mathbb{Z}\}$ *be a sequence of Borel measures on* $\mathbb{R}^n \times \mathbb{R}^m$. *Let* $\rho_1, \rho_2 \in \mathbb{R}$ *and let* ε_1 *and* ε_2 *be defined by*

$$\varepsilon_i = \begin{cases} 1, & \rho_i \geqslant -1 \\ -1, & \rho_i < -1, \end{cases} \qquad i = 1,2.$$

Suppose that for some $a > 1$, $\alpha, \beta, C > 0$, *and* $B > 1$, *the following hold for* $k, j \in \mathbb{Z}$, $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$:

(i) $|\widehat{\sigma}_{j,k}(\xi,\eta)| \leqslant CB^2 (a^{\varepsilon_2 jB} a^{\varepsilon_1 \rho_1 kB} |L(\xi)|)^{\pm\frac{\alpha}{B}} (a^{\varepsilon_1 kB} a^{\varepsilon_2 \rho_2 jB} |H(\eta)|)^{\pm\frac{\beta}{B}}$,

(ii) $\| \sup_{j,k} (||\sigma_{j,k}| * f|) \|_q \leqslant CB^2 \|f\|_q$, $\forall q \in (1,\infty)$.

Then for $1 < p < \infty$, *there exists a positive constant* C_p *such that*

$$\left\| \sum_{k,j\in\mathbb{Z}} \sigma_{j,k} * f \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leqslant C_p B^2 \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \tag{2.1}$$

and

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} |\sigma_{j,k} * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leqslant C_p B^2 \|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \tag{2.2}$$

hold for all f in $L^p(\mathbb{R}^n\times\mathbb{R}^m)$. The constant C_p is independent of B and the linear transformations L and H.

Finally, the following lemma was proved by Al-Salman in [4]:

LEMMA 2.3. ([4]). *Let $L:\mathbb{R}^n\to\mathbb{R}^n$ and $H:\mathbb{R}^m\to\mathbb{R}^m$ be linear transformations. Let $\rho_1,\rho_2 > -1$ be such that $\rho_1\rho_2 \neq 1$. For $(i,j)\in\{(1,1),(1,0),(0,1),(0,0)\}$, let $\sigma^{(i,j)} = \{\sigma_{t,s}^{(i,j)} : t,s\in\mathbb{R}\}$ be a family of measures on $\mathbb{R}^n\times\mathbb{R}^m$. Suppose that for some α, β and $C > 0$, the following hold for $t,s\in\mathbb{R}$, $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^m$:*

(i) $|\widehat{\sigma}_{t,s}^{(i,j)}(\xi,\eta)| \leqslant 1$

(ii) $|\widehat{\sigma}_{t,s}^{(1,1)}(\xi,\eta)| \leqslant C\,(2^t\,2^{\rho_1 s}|L(\xi)|)^{-\alpha}\,(2^s\,2^{\rho_2 t}|H(\eta)|)^{-\beta}$

(iii) $\left|\left(\widehat{\sigma}_{t,s}^{(1,1)} - \widehat{\sigma}_{t,s}^{(1,0)}\right)(\xi,\eta)\right| \leqslant C\,(2^t\,2^{\rho_1 s}|L(\xi)|)^{-\alpha}\,(2^s\,2^{\rho_2 t}|H(\eta)|)^{\beta}$

(iv) $\left|\left(\widehat{\sigma}_{t,s}^{(1,1)} - \widehat{\sigma}_{t,s}^{(0,1)}\right)(\xi,\eta)\right| \leqslant C\,(2^t\,2^{\rho_1 s}|L(\xi)|)^{\alpha}\,(2^s\,2^{\rho_2 t}|H(\eta)|)^{-\beta}$

(v) $\left|\left(\widehat{\sigma}_{t,s}^{(1,1)} - \widehat{\sigma}_{t,s}^{(0,1)} - \widehat{\sigma}_{t,s}^{(1,0)} + \widehat{\sigma}_{t,s}^{(0,0)}\right)(\xi,\eta)\right| \leqslant C\,(2^t\,2^{\rho_1 s}|L(\xi)|)^{\alpha}\,(2^s\,2^{\rho_2 t}|H(\eta)|)^{\beta}$

(vi) $\left|\left(\widehat{\sigma}_{t,s}^{(1,0)} - \widehat{\sigma}_{t,s}^{(0,0)}\right)(\xi,\eta)\right| \leqslant C\,(2^t\,2^{\rho_1 s}|L(\xi)|)^{\alpha}$

(vii) $\left|\left(\widehat{\sigma}_{t,s}^{(0,1)} - \widehat{\sigma}_{t,s}^{(0,0)}\right)(\xi,\eta)\right| \leqslant C\,(2^s\,2^{\rho_2 t}|H(\eta)|)^{\beta}$

(viii) For $(i,j)\in\{(1,0),(0,1),(0,0)\}$, the maximal function

$$(\sigma^{(i,j)})^*(f)(x,y) = \sup_{t,s}(||\sigma_{t,s}^{(i,j)}| * f|(x,y))$$

satisfies

$$\|(\sigma^{(i,j)})^*(f)\|_q \leqslant C\|f\|_q \tag{2.3}$$

for any $1 < q < \infty$. Then for $1 < p < \infty$ there exists positive constant C_p such that the maximal function

$$(\sigma^{(1,1)})^*(f)(x,y) = \sup_{t,s}(||\sigma_{t,s}^{(1,1)}| * f|(x,y)) \tag{2.4}$$

satisfies

$$\|(\sigma^{(1,1)})^*(f)\|_p \leqslant C_p\|f\|_p \tag{2.5}$$

for all $f\in L^p(\mathbb{R}^n\times\mathbb{R}^n)$. The constant C_p is independent of the linear transformations L and H.

Here, we remark that 2^t in the above lemma can be replaced by $\varphi(2^t)$ where φ is convex increasing. Similarly for 2^s.

3. Introductory maximal inequalities

This section is devoted to establish some necessary maximal inequalities. We start by establishing the following lemma:

LEMMA 3.1. *Let $z_1 \in \mathbb{R}^n$ and $z_2 \in \mathbb{R}^m$. For $N, M \geqslant 0$, suppose that $(\Phi, P_\Psi) \in \mathscr{H}(N, M, \lambda, 0)$ are such that $\Phi(t) = P_\Phi(t) + \lambda \varphi(t)$ and $P_\Psi(t) = \sum_{i=2}^{M} b_i t^i$ where $M = \deg(P_\Psi) \geqslant 2$. Then the maximal function*

$$
\upsilon_{\Phi, P_\Psi}^{(z_1, z_2)}(f)(x, y) = \sup_{t, s \in \mathbb{R}} \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} |f(x - \Phi(\zeta) r z_1, y - P_\Psi(r) \zeta z_2)| \frac{dr \, d\zeta}{r \zeta}, \tag{3.1}
$$

is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $1 < p < \infty$. The L^p bounds of $\upsilon_{\Phi, P_\Psi}^{(z_1, z_2)}$ may depend on the degrees of the polynomials P_Φ and P_Ψ, but they are independent of $\lambda \in \mathbb{R}$, the coefficients of the polynomials P_Φ, P_Ψ and the points z_1 and z_2.

Proof of Lemma 3.1. The proof of above lemma is based on an induction argument on the $\deg(P_\Psi) = M \geqslant 2$. First, for $M = 2$, we argue in three cases as follow:

Case 1. Assume that P_Φ and P_Ψ satisfy the assumption (i) in Definition 1.2. Let

$$
P_\Phi(t) = \sum_{i=2}^{N} a_i t^i \qquad \text{and} \qquad P_\Psi(t) = b_2 t^2.
$$

Thus, by using the Riesz representation theorem, we define the family of measures $\{\upsilon_{t,s}^{(\Phi, 2)} : t, s \in \mathbb{R}\}$ by setting, for any $f \in \mathscr{C}_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$,

$$
\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x, y) d\upsilon_{t,s}^{(\Phi, 2)}(x, y) = \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} f(\Phi(\zeta) r z_1, (b_2 r^2) \zeta z_2) \frac{dr \, d\zeta}{r \zeta}. \tag{3.2}
$$

Then

$$
\upsilon_{\Phi, P_\Psi}^{(z_1, z_2)}(f)(x, y) = \sup_{t, s} \left| (\upsilon_{t,s}^{(\Phi, 2)} * f)(x, y) \right|. \tag{3.3}
$$

Notice that

$$
\begin{aligned}
\widehat{\upsilon}_{t,s}^{(\Phi, 2)}(\xi, \eta) &= \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} e^{i(\xi, \eta) \cdot (\Phi(\zeta) r z_1, (b_2 r^2) \zeta z_2)} \frac{dr \, d\zeta}{r \zeta} \\
&= \int_{1}^{2} \int_{1}^{2} e^{i[(\xi \cdot z_1) \Phi(2^s \zeta) 2^t r + (\eta \cdot z_2) b_2 (2^t r)^2 2^s \zeta]} \frac{dr \, d\zeta}{r \zeta}.
\end{aligned} \tag{3.4}
$$

Also, we define the measures $\{v_{t,s}^{(\Phi,1)} : t,s \in \mathbb{R}\}$, $\{v_{t,s}^{(P_\Phi,1)} : t,s \in \mathbb{R}\}$, and $\{v_{t,s}^{(P_\Phi,2)} : t,s \in \mathbb{R}\}$ via the Fourier transform by

$$\widehat{v}_{j,k}^{(P_\Phi,2)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1)P_\Phi(2^s \zeta) 2^t r + (\eta \cdot z_2) b_2 (2^t r)^2 2^s \zeta]} \frac{dr \, d\zeta}{r\zeta}, \tag{3.5}$$

$$\widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta) = \widehat{v}_{t,s}^{(\Phi,2)}(\xi,0), \tag{3.6}$$

and

$$\widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta) = \widehat{v}_{t,s}^{(P_\Phi,2)}(\xi,0). \tag{3.7}$$

In addition, corresponding to the measures $\{v_{t,s}^{(\Phi,1)}\}$, $\{v_{t,s}^{(P_\Phi,1)}\}$, and $\{v_{t,s}^{(P_\Phi,2)}\}$, we define the maximal functions $(v^{(\Phi,1)})^*$, $(v^{(P_\Phi,1)})^*$, and $(v^{(P_\Phi,2)})^*$ by

$$(v^{(\Phi,1)})^*(f)(x,y) = \sup_{t,s} \left| (v_{t,s}^{(\Phi,1)} * f)(x,y) \right|, \tag{3.8}$$

$$(v^{(P_\Phi,1)})^*(f)(x,y) = \sup_{t,s} \left| (v_{t,s}^{(P_\Phi,1)} * f)(x,y) \right|, \tag{3.9}$$

and

$$(v^{(P_\Phi,2)})^*(f)(x,y) = \sup_{t,s} \left| (v_{t,s}^{(P_\Phi,2)} * f)(x,y) \right|. \tag{3.10}$$

Now, notice that

$$\begin{aligned}
(v^{(\Phi,1)})^*(f)(x,y) &= \sup_{t,s} \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} |f(x - \Phi(\zeta) r z_1, y)| \frac{d\zeta \, dr}{\zeta \, r} \\
&\leqslant \sup_{t,s} \int_{2^s}^{2^{s+1}} \left(\int_{\Phi(\zeta) 2^t}^{\Phi(\zeta) 2^{t+1}} |f(x - u z_1, y)| \frac{du}{u} \right) \frac{d\zeta}{\zeta} \\
&\leqslant C \left(M_{z_1} + M_{-z_1} \right) f(x,y),
\end{aligned} \tag{3.11}$$

where M_{z_1} and M_{-z_1} are the directional Hardy-Littlewood maximal functions in the direction of z_1 and $-z_1$ respectively (acting on the x-variable) and C is a positive constant. Therefore,

$$\|(v^{(\Phi,1)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.12}$$

for all $1 < p < \infty$. Similarly, we get

$$\|(v^{(P_\Phi,1)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.13}$$

for all $1 < p < \infty$. On the other hand, by Theorem 1.2 in [4], we have

$$\|(v^{(P_\Phi,2)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.14}$$

for all $1 < p < \infty$ with constant C_p independent of z_1, z_2 and the coefficients of the polynomials P_Φ and P_Ψ.

Now, it is easy to show that

$$|\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta)| + |\widehat{\upsilon}_{t,s}^{(\Phi,1)}(\xi,\eta)| + |\widehat{\upsilon}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| + |\widehat{\upsilon}_{t,s}^{(P_\Phi,2)}(\xi,\eta)| \leqslant C. \tag{3.15}$$

Furthermore, we have

$$\begin{aligned}
&\left| \frac{d^{N+1}}{d\zeta^{N+1}} [(\xi \cdot z_1)\Phi(2^s\zeta)2^t r + (\eta \cdot z_2)(b_2(2^t r)^2)2^s\zeta] \right| \\
&= |(2^t r)\lambda\, 2^{(N+1)s}\,\varphi^{(N+1)}(2^s\zeta)(\xi \cdot z_1)| \\
&\geqslant |\lambda\, 2^t\, 2^{(N+1)s}(2^s\zeta)^{-N-1}\varphi(2^s)(\xi \cdot z_1)| \\
&\geqslant |\lambda\, 2^t\, \varphi(2^s)(\xi \cdot z_1)|.
\end{aligned} \tag{3.16}$$

Thus, by Van der Corput lemma in [13], we get

$$\begin{aligned}
|\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta)| &\leqslant \int_1^2 \left| \int_1^2 e^{i[(\xi \cdot z_1)\Phi(2^s\zeta)2^t r + (\eta \cdot z_2)(b_2(2^t r)^2)2^s\zeta]} \frac{d\zeta}{\zeta} \right| \frac{dr}{r} \\
&\leqslant C|\lambda\, 2^t\, \varphi(2^s)(\xi \cdot z_1)|^{-\frac{1}{(N+1)}}.
\end{aligned} \tag{3.17}$$

On the other hand, we have

$$\left| \frac{d^2}{dr^2}[(\xi \cdot z_1)\Phi(2^s\zeta)2^t r + (\eta \cdot z_2)(b_2(2^t r)^2)2^s\zeta] \right| = |2b_2(2^s\zeta)2^{2t}(\eta \cdot z_2)| \\
\geqslant |b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)|. \tag{3.18}$$

Similarly, by Van der Corput lemma in [13], we obtain

$$\begin{aligned}
|\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta)| &\leqslant \int_1^2 \left| \int_1^2 e^{i[(\xi \cdot z_1)\Phi(2^s\zeta)2^t r + (\eta \cdot z_2)b_2(2^t r)^2 2^s\zeta]} \frac{dr}{r} \right| \frac{d\zeta}{\zeta} \\
&\leqslant C|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)|^{-\frac{1}{2}}.
\end{aligned} \tag{3.19}$$

Also, we have

$$|\widehat{\upsilon}_{t,s}^{(\Phi,1)}(\xi,\eta)| \leqslant C|\lambda\, 2^t\, \varphi(2^s)(\xi \cdot z_1)|^{-\frac{1}{(N+1)}}, \tag{3.20}$$

$$|\widehat{\upsilon}_{t,s}^{(P_\Phi,2)}(\xi,\eta)| \leqslant C|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)|^{-\frac{1}{2}}. \tag{3.21}$$

By interpolation between the estimates (3.17) and (3.19), we get

$$|\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta)| \leqslant C|\lambda\, 2^t\, \varphi(2^s)(\xi \cdot z_1)|^{-\frac{1}{2(N+1)}}|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)|^{-\frac{1}{4}}. \tag{3.22}$$

Then, by (3.17), (3.19), (3.20), and (3.21), we get

$$\begin{aligned}
|\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(\Phi,1)}(\xi,\eta)| &\leqslant |\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta)| + |\widehat{\upsilon}_{t,s}^{(\Phi,1)}(\xi,\eta)| \\
&\leqslant C|\lambda\, 2^t\, \varphi(2^s)(\xi \cdot z_1)|^{-\frac{1}{(N+1)}}.
\end{aligned} \tag{3.23}$$

Similarly

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P\Phi,2)}(\xi,\eta)| \leqslant C \left|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)\right|^{-\frac{1}{2}}. \tag{3.24}$$

In contrast, we can obtain that

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta)| = \left| \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} e^{i(\xi \cdot z_1)\Phi(\zeta)r} \left(e^{[i(\eta \cdot z_2)(b_2 r^2)\,\zeta]} - 1\right) \frac{dr}{r}\frac{d\zeta}{\zeta} \right|$$
$$\leqslant C \left|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)\right|. \tag{3.25}$$

By combining (3.25) and the trivial estimate

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta)| \leqslant C,$$

we get

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta)| \leqslant C \left|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)\right|^{\frac{1}{2}}. \tag{3.26}$$

By the same procedure as in (3.25)–(3.26), we get

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P\Phi,2)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})\,(\xi \cdot z_1)\right|^{\frac{1}{2}}. \tag{3.27}$$

Thus, by (3.23)–(3.24) and (3.26)–(3.27), we obtain

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^s)(\xi \cdot z_1)\right|^{-\frac{1}{2(N+1)}} \left|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)\right|^{\frac{1}{4}}, \tag{3.28}$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P\Phi,2)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi \cdot z_1)\right|^{\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)\right|^{-\frac{1}{4}}. \tag{3.29}$$

Also, we can show that

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta) + \widehat{v}_{t,s}^{(P\Phi,1)}(\xi,\eta)|$$
$$= \left| \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} \left(e^{i(\xi \cdot z_1)\Phi(\zeta)r} - e^{i(\xi \cdot z_1)P\Phi(\zeta)r}\right) \left(e^{i(\eta \cdot z_2)(b_2 r^2)\,\zeta} - 1\right) \frac{dr}{r}\frac{d\zeta}{\zeta} \right|$$
$$\leqslant \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} \left|\left(e^{i(\xi \cdot z_1)\lambda \varphi(\zeta)r} - 1\right)\right| \left|\left(e^{i(\eta \cdot z_2)(b_2 r^2)\,\zeta} - 1\right)\right| \frac{dr}{r}\frac{d\zeta}{\zeta}$$
$$\leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi \cdot z_1)\right| \left|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)\right|. \tag{3.30}$$

By combining the estimate (3.30) and the trivial estimate

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta) + \widehat{v}_{t,s}^{(P\Phi,1)}(\xi,\eta)| \leqslant C,$$

we get

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta) + \widehat{v}_{t,s}^{(P\Phi,1)}(\xi,\eta)|$$
$$\leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi \cdot z_1)\right|^{\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s(\eta \cdot z_2)\right|^{\frac{1}{4}}. \tag{3.31}$$

The following two estimates are straight-forward

$$|\widehat{\upsilon}_{t,s}^{(\Phi,1)} - \widehat{\upsilon}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})\,(\xi\cdot z_1)\right|^{\frac{1}{4}}, \tag{3.32}$$

$$|\widehat{\upsilon}_{t,s}^{(P_\Phi,2)} - \widehat{\upsilon}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| \leqslant C \left|b_2\, 2^{2t}\, 2^s\,(\eta\cdot z_2)\right|^{\frac{1}{4}}. \tag{3.33}$$

Thus, by (3.12)–(3.14), (3.22), (3.28), (3.29), (3.31)–(3.33), and Lemma 2.3 and the remark that follows its statement, we obtain the L^p boundedness of the inequality (3.3) with L^p bounds independent of $\lambda \in \mathbb{R}$, the coefficients of the polynomials P_Φ, P_Ψ and the points z_1 and z_2. The proof is complete under assumption (i). Now, we move to the proof under condition (iii).

Case 2. Assume that P_Φ and P_Ψ satisfy the Condition (iii) in Definition 1.2. Let $P_\Phi(t) = c$-*constant* and $P_\Psi(t) = b_2 t^2$. We define the same families of measures as in (3.2)–(3.10) in the Case 1. But here we replace $P_\Phi(t) = \sum\limits_{i=2}^{N} b_i t^i$ by $P_\Phi(t) = c$-*constant*. Thus, as in (3.11), we can prove that

$$\|(\upsilon^{(\Phi,1)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.34}$$

for all $1 < p < \infty$. Similarly, we get

$$\|(\upsilon^{(P_\Phi,1)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.35}$$

for all $1 < p < \infty$. Also, we have

$$
\begin{aligned}
(\upsilon^{(P_\Phi,2)})^*(f)(x,y) &= \sup_{t,s} \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} |f(x - crz_1, y - (b_2 r)^2\, \zeta z_2)| \frac{d\zeta\, dr}{\zeta r} \\
&\leqslant \sup_{t,s} \int_{2^t}^{2^{t+1}} \left(\int_{(b_2 r)^2\, 2^s}^{(b_2 r)^2\, 2^{s+1}} |f(x - crz_1, y - uz_2)| \frac{du}{u} \right) \frac{dr}{r} \\
&\leqslant M_{z_1}^{(1)} \circ (M_{z_2}^{(2)} + M_{-z_2}^{(2)}) f(x,y),
\end{aligned}
\tag{3.36}
$$

where $M_{z_1}^{(1)}$ is the directional Hardy-Littlewood maximal functions in the direction of z_1 (acting on the x-variable) and M_{z_1} and M_{-z_1} are as in (3.11). Hence,

$$\|(\upsilon^{(P_\Phi,2)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.37}$$

for all $1 < p < \infty$ with constant C_p independent of z_1, z_2 and the coefficients of the polynomials P_Φ and P_Ψ.

On the other hand, we can show that

$$|\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta)| + |\widehat{\upsilon}_{t,s}^{(\Phi,1)}(\xi,\eta)| + |\widehat{\upsilon}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| + |\widehat{\upsilon}_{t,s}^{(P_\Phi,2)}(\xi,\eta)| \leqslant C, \tag{3.38}$$

$$|\widehat{\upsilon}_{t,s}^{(\Phi,2)}(\xi,\eta)| \leqslant C\, |\lambda\, 2^t\, \varphi(2^s)(\xi\cdot z_1)|^{-\frac{1}{4}}\, |b_2\, 2^{2t}\, 2^s(\eta\cdot z_2)|^{-\frac{1}{4}}, \tag{3.39}$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^s)(\xi\cdot z_1)\right|^{-\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{\frac{1}{4}}, \quad (3.40)$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P_\Phi,2)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1)\right|^{\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{-\frac{1}{4}}, \quad (3.41)$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P_\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta) + \widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta)|$$
$$\leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1)\right|^{\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{\frac{1}{4}}, \quad (3.42)$$

$$|\widehat{v}_{t,s}^{(\Phi,1)} - \widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1)\right|^{\frac{1}{4}}, \quad (3.43)$$

$$|\widehat{v}_{t,s}^{(P_\Phi,2)} - \widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| \leqslant C \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{\frac{1}{4}}. \quad (3.44)$$

Thus, by (3.34)–(3.37), (3.38)–(3.44), and Lemma 2.3, we obtain the L^p boundedness of the inequality (3.3) with L^p bounds independent of $\lambda \in \mathbb{R}$, the coefficients of the polynomials P_Φ, P_Ψ and the points z_1 and z_2. This completes the proof of Case 2.

Case 3. Assume that P_Φ and P_Ψ satisfy the Condition (v) in Definition 1.2. Let $P_\Phi(t) = t$ and $P_\Psi(t) = b_2 t^2$. We define the similar families of measures as in (3.2)–(3.10) in the Case 1, with $P_\Phi(t) = t$. As in (3.11), we have

$$\|(v^{(\Phi,1)})^*(f)\|_p \leqslant C_p \|f\|_p \quad (3.45)$$

for all $1 < p < \infty$. Similarly, we get

$$\|(v^{(P_\Phi,1)})^*(f)\|_p \leqslant C_p \|f\|_p \quad (3.46)$$

for all $1 < p < \infty$. Now, by Lemma 3.2 in [4], we get

$$\|(v^{(P_\Phi,2)})^*(f)\|_p \leqslant C_p \|f\|_p \quad (3.47)$$

for all $1 < p < \infty$ with constant C_p independent of z_1, z_2 and the coefficients of the polynomials P_Φ and P_Ψ. Also, we can prove that

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta)| + |\widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta)| + |\widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| + |\widehat{v}_{t,s}^{(P_\Phi,2)}(\xi,\eta)| \leqslant C, \quad (3.48)$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^s)(\xi\cdot z_1)\right|^{-\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s(\eta\cdot z_2)\right|^{-\frac{1}{4}}, \quad (3.49)$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^s)(\xi\cdot z_1)\right|^{-\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{\frac{1}{4}}, \quad (3.50)$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P_\Phi,2)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1)\right|^{\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{-\frac{1}{4}}, \quad (3.51)$$

$$|\widehat{v}_{t,s}^{(\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(P_\Phi,2)}(\xi,\eta) - \widehat{v}_{t,s}^{(\Phi,1)}(\xi,\eta) + \widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta)|$$
$$\leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1)\right|^{\frac{1}{4}} \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{\frac{1}{4}}, \quad (3.52)$$

$$|\widehat{v}_{t,s}^{(\Phi,1)} - \widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| \leqslant C \left|\lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1)\right|^{\frac{1}{4}}, \quad (3.53)$$

$$|\widehat{v}_{t,s}^{(P_\Phi,2)} - \widehat{v}_{t,s}^{(P_\Phi,1)}(\xi,\eta)| \leqslant C \left|b_2\, 2^{2t}\, 2^s\, (\eta\cdot z_2)\right|^{\frac{1}{4}}. \quad (3.54)$$

Then, we can follow the same procedure as in previous cases. We omit details. This concludes the third case.

Next, assume that the L^p boundedness of $v_{\Phi,P_\Psi}^{(z_1,z_2)}$ in (3.1) holds for all polynomials P_Ψ with degree less than or equal $M - 1 \geqslant 1$. Let

$$P_\Psi(t) = \sum_{i=2}^{M} b_i t^i \qquad \text{and} \qquad P_\Psi^{(M)}(t) = \sum_{i=2}^{M-1} b_i t^i.$$

In light of the conditions on P_Ψ, we shall prove the L^p boundedness of $v_{\Phi,P_\Psi}^{(z_1,z_2)}$ in (3.1) under the assumption (i), (iii), and (v) in Definition 1.2. We start by assuming that P_Φ and P_Ψ satisfy the Condition (i) in Definition 1.2. Let

$$P_\Phi(t) = \sum_{i=2}^{N} a_i t^i.$$

We define the family of measures $\{v_{t,s}^{(\Phi,M)}, v_{t,s}^{(\Phi,M-1)}, v_{t,s}^{(P_\Phi,M)}, v_{t,s}^{(P_\Phi,M-1)} : t,s \in \mathbb{R}\}$ via the Fourier transform by

$$\widehat{v}_{t,s}^{(\Phi,M)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1)\Phi(2^s \zeta)2^t r + (\eta \cdot z_2) P_\Psi(2^t r) 2^s \zeta]} \frac{dr\,d\zeta}{r\zeta}, \tag{3.55}$$

$$\widehat{v}_{t,s}^{(\Phi,M-1)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1)\Phi(2^s \zeta)2^t r + (\eta \cdot z_2) P_\Psi^{(M)}(2^t r) 2^s \zeta]} \frac{dr\,d\zeta}{r\zeta}, \tag{3.56}$$

$$\widehat{v}_{t,s}^{(P_\Phi,M)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1) P_\Phi(2^s \zeta)2^t r + (\eta \cdot z_2) P_\Psi(2^t r) 2^s \zeta]} \frac{dr\,d\zeta}{r\zeta}, \tag{3.57}$$

and

$$\widehat{v}_{t,s}^{(P_\Phi,M-1)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1) P_\Phi(2^s \zeta)2^t r + (\eta \cdot z_2) P_\Psi^{(M)}(2^t r) 2^s \zeta]} \frac{dr\,d\zeta}{r\zeta}. \tag{3.58}$$

Then, the maximal function $v_{\Phi,P_\Psi}^{(z_1,z_2)}$ is given by

$$v_{\Phi,P_\Psi}^{(z_1,z_2)}(f)(x,y) = \sup_{t,s} \left| (v_{t,s}^{(\Phi,M)} * f)(x,y) \right|. \tag{3.59}$$

Now, let $(v^{(\Phi,M-1)})^*$, $(v^{(P_\Phi,M-1)})^*$, and $(v^{(P_\Phi,M)})^*$ be the maximal functions given by

$$(v^{(\Phi,M-1)})^*(f)(x,y) = \sup_{t,s} \left| (v_{t,s}^{(\Phi,M-1)} * f)(x,y) \right|, \tag{3.60}$$

$$(v^{(P_\Phi,M)})^*(f)(x,y) = \sup_{t,s} \left| (v_{t,s}^{(P_\Phi,M)} * f)(x,y) \right|, \tag{3.61}$$

and

$$(v^{(P_\Phi,M-1)})^*(f)(x,y) = \sup_{t,s} \left|(v_{t,s}^{(P_\Phi,M-1)} * f)(x,y)\right|. \tag{3.62}$$

By induction assumption, we observe that

$$\|(v^{(\Phi,M-1)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.63}$$

for all $1 < p < \infty$.

Furthermore, by Theorem 1.2 in [4], we obtain that

$$\|(v^{(P_\Phi,M)})^*(f)\|_p \leqslant C_p \|f\|_p. \tag{3.64}$$

Similarly, we get

$$\|(v^{(P_\Phi,M-1)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{3.65}$$

for all $1 < p < \infty$ with constant C_p independent of the points z_1 and z_2 and the coefficients of the polynomials P_Φ and P_Ψ.

Also, it is easy to see that for all $(l,r) \in \{(\Phi,M),(\Phi,M-1),(P_\Phi,M),(P_\Phi,M-1)\}$, we have

$$|\widehat{v}_{t,s}^{(l,r)}(\xi,\eta)| \leqslant C. \tag{3.66}$$

Now, by the properties of φ in Lemma 2.1, we can obtain that

$$\left|\frac{d^{N+1}}{d\zeta^{N+1}}[(\xi \cdot z_1)\Phi(2^s \zeta) 2^t r + (\eta \cdot z_2) P_\Psi(2^t r) 2^s \zeta]\right| \geqslant C|\lambda 2^t \varphi(2^s)(\xi \cdot z_1)|. \tag{3.67}$$

On the other hand, we have

$$\left|\frac{d^M}{dr^M}[(\xi \cdot z_1)\Phi(2^s \zeta) 2^t r + (\eta \cdot z_2) P_\Psi(2^t r) 2^s \zeta]\right| \geqslant C|b_M M! 2^s 2^{Mt} (\eta \cdot z_2)|. \tag{3.68}$$

By (3.67), (3.68), and Van der Corput lemma in [13], we get

$$|\widehat{v}_{t,s}^{(\Phi,M)}(\xi,\eta)| \leqslant C|\lambda 2^t \varphi(2^s)(\xi \cdot z_1)|^{-\frac{1}{N+1}}, \tag{3.69}$$

$$|\widehat{v}_{t,s}^{(\Phi,M)}(\xi,\eta)| \leqslant C|b_M M! 2^s 2^{Mt} (\eta \cdot z_2)|^{-\frac{1}{M}}, \tag{3.70}$$

and

$$|\widehat{v}_{t,s}^{(\Phi,M-1)}(\xi,\eta)| \leqslant C|\lambda 2^t \varphi(2^s)(\xi \cdot z_1)|^{-\frac{1}{N+1}}, \tag{3.71}$$

$$|\widehat{v}_{t,s}^{(P_\Phi,M)}(\xi,\eta)| \leqslant C|b_M M! 2^s 2^{Mt} (\eta \cdot z_2)|^{-\frac{1}{M}}. \tag{3.72}$$

By (3.66), (3.69), and (3.70), we have

$$|\widehat{v}_{t,s}^{(\Phi,M)}(\xi,\eta)| \leqslant C|\lambda 2^t \varphi(2^s)(\xi \cdot z_1)|^{-\frac{1}{2(N+1)}} |b_M M! 2^s 2^{Mt} (\eta \cdot z_2)|^{-\frac{1}{2M}}. \tag{3.73}$$

Also, we have

$$
\begin{aligned}
&|\widehat{\upsilon}_{t,s}^{(\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(\Phi,M-1)}(\xi,\eta)| \\
&= \left| \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} e^{i(\xi\cdot z_1)\Phi(\zeta)\,r}\, e^{i(\eta\cdot z_2)\,P_{\Psi_M}(r)\,\zeta} \left(e^{[i(\eta\cdot z_2)\,b_M\, r^M\,\zeta]} - 1 \right) \frac{dr}{r}\frac{d\zeta}{\zeta} \right| \\
&\leqslant C \left| b_M\, 2^{Mt}\, 2^s\,(\eta\cdot z_2) \right|.
\end{aligned}
\tag{3.74}
$$

By interpolation between (3.74) and the trivial estimate

$$
|\widehat{\upsilon}_{t,s}^{(\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(\Phi,M-1)}(\xi,\eta)| \leqslant C,
$$

we obtain that

$$
|\widehat{\upsilon}_{t,s}^{(\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(\Phi,M-1)}(\xi,\eta)| \leqslant C \left| b_M\, 2^{Mt}\, 2^s\,(\eta\cdot z_2) \right|^{\frac{1}{M}}.
\tag{3.75}
$$

Similarly, we can get that

$$
|\widehat{\upsilon}_{t,s}^{(\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(P_\Phi,M)}(\xi,\eta)| \leqslant C \left| \lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1) \right|^{\frac{1}{N+1}}.
\tag{3.76}
$$

By (3.69)–(3.72) and (3.75)–(3.76), we obtain

$$
\begin{aligned}
&|\widehat{\upsilon}_{t,s}^{(\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(\Phi,M-1)}(\xi,\eta)| \\
&\leqslant C \left| \lambda\, 2^t\, \varphi(2^s)(\xi\cdot z_1) \right|^{-\frac{1}{2(N+1)}} \left| b_M\, 2^{Mt}\, 2^s\,(\eta\cdot z_2) \right|^{\frac{1}{2M}},
\end{aligned}
\tag{3.77}
$$

and

$$
\begin{aligned}
&|\widehat{\upsilon}_{t,s}^{(\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(P_\Phi,M)}(\xi,\eta)| \\
&\leqslant C \left| \lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1) \right|^{\frac{1}{2(N+1)}} \left| b_M\, M!\, 2^s\, 2^{Mt}\,(\eta\cdot z_2) \right|^{-\frac{1}{2(M)}}.
\end{aligned}
\tag{3.78}
$$

By the same procedure as in (3.31), we get

$$
\begin{aligned}
&|\widehat{\upsilon}_{t,s}^{(\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(P_\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(\Phi,M-1)}(\xi,\eta) + \widehat{\upsilon}_{t,s}^{(P_\Phi,M-1)}(\xi,\eta)| \\
&\leqslant C \left| \lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1) \right|^{\frac{1}{2(N+1)}} \left| b_M\, 2^{Mt}\, 2^s\,(\eta\cdot z_2) \right|^{\frac{1}{2M}}.
\end{aligned}
\tag{3.79}
$$

Also, we have

$$
|\widehat{\upsilon}_{t,s}^{(\Phi,M-1)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(P_\Phi,M-1)}(\xi,\eta)| \leqslant C \left| \lambda\, 2^t\, \varphi(2^{s+1})(\xi\cdot z_1) \right|^{\frac{1}{2(N+1)}},
\tag{3.80}
$$

$$
|\widehat{\upsilon}_{t,s}^{(P_\Phi,M)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(P_\Phi,M-1)}(\xi,\eta)| \leqslant C \left| b_M\, 2^{Mt}\, 2^s\,(\eta\cdot z_2) \right|^{\frac{1}{2M}}.
\tag{3.81}
$$

Finally, by (3.63)–(3.65), (3.73), (3.77)–(3.81), and Lemma 2.3 and the remark that follows its statement, we establish the L^p boundedness for $\upsilon_{\Phi,P_\Psi}^{(z_1,z_2)}$. The proof under Assumption (i) is complete. The proof under the assumptions (iii) and (v) follow by similar argument with minor modification. We omit the details. This completes the proof. $\square$

4. Proofs of Theorems 1.3 and 1.4

In this section, we shall present the proofs of Theorems 1.3 and 1.4. We will carry out the proof of Theorem 1.3 by a double induction argument along with Lemma 2.3.

Proof of Theorem 1.3. To prove Theorem 1.3, we start by using double induction on the degrees $d = \deg(P_\Phi)$ and $b = \deg(P_\Psi)$. First, for $d = 2$ and b arbitrary, the L^p boundedness of $v_{\Phi,\Psi}^{(z_1,z_2)}$ is satisfied by Lemma 3.1. Similarly, for $b = 2$ and d arbitrary. Next, we assume that the L^p bounds $(1 < p < \infty)$ for $v_{\Phi,\Psi}^{(z_1,z_2)}$ holds for all Φ with degree of P_Φ less than $d+1$ and P_Ψ of any degree. Furthermore, we assume the L^p boundedness also holds for all Ψ with degree of P_Ψ less than $b+1$ and Φ of any degree. Thus, we assume that $\deg(\Phi) = d+1$ and $\deg(\Psi) = b+1$. Let

$$\Phi(t) = P_\Phi(t) + \lambda\,\varphi_1(t) \qquad \text{and} \qquad \Psi(t) = P_\Psi(t) + \alpha\varphi_2(t) \qquad (4.1)$$

where

$$P_\Phi(t) = \sum_{i=2}^{d+1} a_i t^i \qquad \text{and} \qquad P_\Psi(t) = \sum_{i=2}^{b+1} c_i t^i. \qquad (4.2)$$

For $t, s \in \mathbb{R}$, we define the family of measures $\{v_{t,s}^{(d+1,b+1)} : t, s \in \mathbb{R}\}$, $\{v_{t,s}^{(d+1,b)} : t, s \in \mathbb{R}\}$, $\{v_{t,s}^{(d,b+1)} : t, s \in \mathbb{R}\}$, and $\{v_{t,s}^{(d,b)} : t, s \in \mathbb{R}\}$ by

$$\int\!\!\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y) dv_{t,s}^{(d+1,b+1)}(x,y) = \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} f(\Phi(\zeta)\,r z_1, \Psi(r)\,\zeta z_2)\frac{dr\,d\zeta}{r\zeta}, \qquad (4.3)$$

$$\int\!\!\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y) dv_{t,s}^{(d+1,b)}(x,y) = \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} f(\Phi(\zeta)\,r z_1, P_\Psi(r)\,\zeta z_2)\frac{dr\,d\zeta}{r\zeta}, \qquad (4.4)$$

$$\int\!\!\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y) dv_{t,s}^{(d,b+1)}(x,y) = \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} f(P_\Phi(\zeta)\,r z_1, \Psi(r)\,\zeta z_2)\frac{dr\,d\zeta}{r\zeta}, \qquad (4.5)$$

and

$$\int\!\!\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x,y) dv_{t,s}^{(d,b)}(x,y) = \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} f(P_\Phi(\zeta)\,r z_1, P_\Psi(r)\,\zeta z_2)\frac{dr\,d\zeta}{r\zeta}. \qquad (4.6)$$

Now, let

$$(v^{(i,r)})^*(f)(x,y) = \sup_{t,s}\left|(v_{t,s}^{(i,r)} * f)(x,y)\right|, \qquad (4.7)$$

where $i = d, d+1, r = b, b+1$. Then

$$v_{\Phi,\Psi}^{(z_1,z_2)}(f)(x,y) = \sup_{t,s}\left|(v_{t,s}^{(d+1,b+1)} * f)(x,y)\right|. \qquad (4.8)$$

For $(i,r) \in \{(d+1,b), (d,b+1), (d,b)\}$, induction assumption implies that

$$\|(\upsilon^{(i,r)})^*(f)\|_p \leqslant C_p \|f\|_p \tag{4.9}$$

for all $1 < p < \infty$. Now, we move to obtain the Fourier estimates of the measures $\{\upsilon_{t,s}^{(d+1,b+1)} : t,s \in \mathbb{Z}\}$, $\{\upsilon_{t,s}^{(d+1,b)} : t,s \in \mathbb{R}\}$, $\{\upsilon_{t,s}^{(d,b+1)} : t,s \in \mathbb{R}\}$, and $\{\upsilon_{t,s}^{(d,b)} : t,s \in \mathbb{R}\}$. Notice that

$$\widehat{\upsilon}_{t,s}^{(d+1,b+1)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1)\Phi(2^s \zeta)2^t r + (\eta \cdot z_2)\Psi(2^t r)2^s \zeta]} \frac{dr d\zeta}{r\zeta}, \tag{4.10}$$

$$\widehat{\upsilon}_{t,s}^{(d+1,b)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1)\Phi(2^s \zeta)2^t r + (\eta \cdot z_2)P_\Psi(2^t r)2^s \zeta]} \frac{dr d\zeta}{r\zeta}, \tag{4.11}$$

$$\widehat{\upsilon}_{t,s}^{(d,b+1)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1)P_\Phi(2^s \zeta)2^t r + (\eta \cdot z_2)\Psi(2^t r)2^s \zeta]} \frac{dr d\zeta}{r\zeta}, \tag{4.12}$$

and

$$\widehat{\upsilon}_{t,s}^{(d,b)}(\xi,\eta) = \int_1^2 \int_1^2 e^{i[(\xi \cdot z_1)P_\Phi(2^s \zeta)2^t r + (\eta \cdot z_2)P_\Psi(2^t r)2^s \zeta]} \frac{dr d\zeta}{r\zeta}. \tag{4.13}$$

We can easily show that

$$|\widehat{\upsilon}_{t,s}^{(d+1,b+1)}| + |\widehat{\upsilon}_{t,s}^{(d,b+1)}| + |\widehat{\upsilon}_{t,s}^{(d+1,b)}| + |\widehat{\upsilon}_{t,s}^{(d,b)}| \leqslant C. \tag{4.14}$$

By similar argument as that led to (3.67), we get

$$\left| \frac{d^{d+1}}{d\zeta^{d+1}}[(\xi \cdot z_1)\Phi(2^s \zeta)2^t r + (\eta \cdot z_2)\Psi(2^t r)2^s \zeta] \right| \geqslant C|\lambda 2^t \varphi_1(2^s)(\xi \cdot z_1)|, \tag{4.15}$$

and

$$\left| \frac{d^{b+1}}{dr^{b+1}}[(\xi \cdot z_1)\Phi(2^s \zeta)2^t r + (\eta \cdot z_2)\Psi(2^t r)2^s \zeta] \right| \geqslant C|\alpha 2^s \varphi_2(2^t)(\eta \cdot z_2)|. \tag{4.16}$$

Hence, by similar procedure as in the proof of Lemma 3.1, we can obtain

$$|\widehat{\upsilon}_{t,s}^{(d+1,b+1)}(\xi,\eta)|$$
$$\leqslant C|\lambda 2^t \varphi_1(2^s)(\xi \cdot z_1)|^{-\frac{1}{2(d+1)}} |\alpha 2^s \varphi_2(2^t)(\eta \cdot z_2)|^{-\frac{1}{2(b+1)}}, \tag{4.17}$$

$$|\widehat{\upsilon}_{t,s}^{(d+1,b+1)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(d+1,b)}(\xi,\eta)|$$
$$\leqslant C|\lambda 2^t \varphi_1(2^s)(\xi \cdot z_1)|^{-\frac{1}{2(d+1)}} |\alpha 2^s \varphi_2(2^{t+1})(\eta \cdot z_2)|^{\frac{1}{2(b+1)}}, \tag{4.18}$$

$$|\widehat{\upsilon}_{t,s}^{(d+1,b+1)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(d,b+1)}(\xi,\eta)|$$

$$\leqslant C\,|\lambda\,2^t\,\varphi_1(2^{s+1})(\xi\cdot z_1)|^{\frac{1}{2(d+1)}}\,|\alpha\,2^s\,\varphi_2(2^t)(\eta\cdot z_2)|^{-\frac{1}{2(b+1)}}, \qquad (4.19)$$

$$|\widehat{\upsilon}_{t,s}^{(d+1,b+1)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(d,b+1)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(d+1,b)}(\xi,\eta) + \widehat{\upsilon}_{t,s}^{(d,b)}(\xi,\eta)|$$

$$\leqslant C\,|\lambda\,2^t\,\varphi_1(2^{s+1})(\xi\cdot z_1)|^{\frac{1}{2(d+1)}}\,|\alpha\,2^s\,\varphi_2(2^{t+1})(\eta\cdot z_2)|^{\frac{1}{2(b+1)}}, \qquad (4.20)$$

$$|\widehat{\upsilon}_{t,s}^{(d+1,b)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(d,b)}(\xi,\eta)| \leqslant C\,|\lambda\,2^t\,\varphi_1(2^{s+1})(\xi\cdot z_1)|^{\frac{1}{2(d+1)}}, \qquad (4.21)$$

$$|\widehat{\upsilon}_{t,s}^{(d,b+1)}(\xi,\eta) - \widehat{\upsilon}_{t,s}^{(d,b)}(\xi,\eta)| \leqslant C\,|\alpha\,2^s\,\varphi_2(2^{t+1})(\eta\cdot z_2)|^{\frac{1}{2(b+1)}}. \qquad (4.22)$$

Finally, by (4.9), (4.14), (4.17)–(4.22) and Lemma 2.3 along with the remark that follows its statement, the proof is complete. $\square$

Proof of Theorem 1.4. The proof of Theorem 1.4 based on Minkowski's inequality and Theorem 1.3 with z_1 and z_2 are replaced by u' and v' respectively. Notice

$$\upsilon_{\Omega,\Lambda_{\Phi,\Psi}}(f)(x,y)$$

$$\leqslant \sup_{t,s\in\mathbb{R}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{m-1}} |\Omega(u',v')| \int_{2^t}^{2^{t+1}} \int_{2^s}^{2^{s+1}} |f(\Phi(\zeta)ru',\Psi(r)\zeta v')|\frac{drd\zeta}{r\zeta}d\sigma(u')\,d\sigma(v')$$

$$\leqslant C\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{m-1}} |\Omega(u',v')|\,\upsilon_{\Phi,\Psi}^{(u',v')}(f)(x,y)\,d\sigma(u')\,d\sigma(v').$$

Thus,

$$\|\upsilon_{\Omega,\Lambda_{\Phi,\Psi}}(f)\|_{L^p} \leqslant C\,\|\Omega\|_{L^1}\,\|\upsilon_{\Phi,\Psi}^{(u',v')}\|_{L^p}$$

$$\leqslant C_p\,\|\Omega\|_{L^1}\,\|f\|_{L^p}$$

for all $1 < p < \infty$ and $C_p > 0$. This ends the proof of Theorem 1.4. $\square$

5. Proof of Theorem 1.2

This section is devoted to present the proof of Theorem 1.2.

Proof of Theorem 1.2. For $d,b \geqslant 0$, assume that Ω, Φ, Ψ are as in the statement of Theorem 1.2. We decompose the function Ω as in [7]. Let $\{\theta_\iota : \iota \in \mathbb{N} \cup \{0\}\}$ be a sequence of numbers and $\{\Omega_\iota : \iota \in \mathbb{N} \cup \{0\}\}$ be a sequence of functions on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ such that

$$\int_{\mathbb{S}^{n-1}} \Omega_\iota(u',.)d\sigma(u') = \int_{\mathbb{S}^{m-1}} \Omega_\iota(.,v')d\sigma(v') = 0, \qquad (5.1)$$

$$\Omega_\iota(tx,sy) = \Omega_\iota(x,y), \qquad\qquad \forall t,s > 0, \qquad (5.2)$$

$$\|\Omega_\iota\|_1 \leqslant 4, \ \|\Omega_\iota\|_2 \leqslant 4(w_\iota)^2, \qquad (5.3)$$

$$\Omega(x,y) = \sum_{\iota=0}^{\infty} \theta_\iota\Omega_\iota(x,y), \qquad (5.4)$$

$$\sum_{\iota=0}^{\infty} (\iota+1)^2 \, \theta_\iota \leqslant \|\Omega\|_{L(\log L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}, \tag{5.5}$$

where $\theta_0 = 1$ and $w_\iota = 2^{\iota+1}$. By (5.4), we get

$$T_{\Omega,\Lambda}(f) = \sum_{\iota=0}^{\infty} \theta_\iota \, T_{\Omega_\iota,\Lambda}(f)(x,y), \tag{5.6}$$

where

$$T_{\Omega_\iota,\Lambda}f = \int_{\mathbb{R}^n\times\mathbb{R}^m} f(x-\Phi(|v|)u', y-\Psi(|u|)v')\frac{\Omega_\iota(u',v')}{|u|^n\,|v|^m}\,du\,dv.$$

Thus, by (5.5) and (5.6), we only need to prove that

$$\|T_{\Omega_\iota,\Lambda}f\|_p \leqslant C_p(\iota+1)^2\|f\|_p. \tag{5.7}$$

For $\Phi,\Psi \in \mathscr{H}(d,b,\lambda,\alpha)$, let P_Φ, P_Ψ be two polynomials satisfying one of the conditions $(i)-(iv)$ in Definition 1.2 such that

$$\Phi(t) = P_\Phi(t) + \lambda_1\varphi_1(t) \qquad \text{and} \qquad \Psi(t) = P_\Psi(t) + \alpha\varphi_2(t) \tag{5.8}$$

where

$$P_\Phi(t) = \sum_{i=2}^{d} a_i t^i \qquad \text{and} \qquad P_\Psi(t) = \sum_{i=2}^{b} c_i t^i. \tag{5.9}$$

For $2 \leqslant l \leqslant d$ and $2 \leqslant s \leqslant b$, let

$$P_\Phi^l(t) = \sum_{i=2}^{l} a_i t^i \qquad \text{and} \qquad P_\Psi^s(t) = \sum_{i=2}^{s} c_i t^i. \tag{5.10}$$

Notice that, we are the convinced that

$$P_\Phi^1(t) = P_\Psi^1(t) = 0.$$

For $\iota \in \mathbb{N}\cup\{0\}$ and $j,k \in \mathbb{Z}$, let $\{\sigma_{\iota,j,k}^{(d+1,b+1)} : j,k \in \mathbb{Z}\}$ be the family of measures defined by

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f(x,y)d\sigma_{\iota,j,k}^{(d+1,b+1)}(x,y) = \iint_{\substack{w_\iota^j<|v|<w_\iota^{j+1}\\ w_\iota^k<|u|<w_\iota^{k+1}}} f(\Phi(|v|)u', \Psi(|u|)v')\frac{\Omega_\iota(u',v')}{|u|^n\,|v|^m}du\,dv.$$

$$\tag{5.11}$$

Also, we define the family of measures $\{\sigma_{\iota,j,k}^{(l,s)} : 1 \leqslant l \leqslant d,\ 1 \leqslant s \leqslant b\}$ by

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f(x,y)d\sigma_{\iota,j,k}^{(l,s)}(x,y) = \iint_{\substack{w_\iota^j<|v|<w_\iota^{j+1}\\ w_\iota^k<|u|<w_\iota^{k+1}}} f(P_\Phi^l(|v|)u', P_\Psi^s(|u|)v')\frac{\Omega_\iota(u',v')}{|u|^n\,|v|^m}du\,dv.$$

$$\tag{5.12}$$

By (5.11) and (5.12), we have

$$T_{\Omega_\iota,\Lambda}(f)(x,y) = \sum_{j,k\in\mathbb{Z}} \sigma_{\iota,j,k}^{(d+1,b+1)} * f(x,y). \tag{5.13}$$

The maximal function corresponding the measures $\sigma_{\iota,j,k}^{(d+1,b+1)}$ is given by

$$(\sigma_\iota^{(d+1,b+1)})^*(f)(x,y) = \sup_{j,k} \left| |\sigma_{\iota,j,k}^{(d+1,b+1)}| * f(x,y) \right|. \tag{5.14}$$

Thus, by Theorem 1.3 and the first inequality in (5.3), we get

$$\|(\sigma_\iota^{(d+1,b+1)})^*(f)\|_p \leqslant C_p (\iota+1)^2 \|f\|_p \tag{5.15}$$

for all $1 < p < \infty$ with constant C_p independent of ι.

Now, notice that

$$\widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) = \iint_{\substack{w_\iota^j<|v|<w_\iota^{j+1} \\ w_\iota^k<|u|<w_\iota^{k+1}}} e^{i(\xi,\eta)(\Phi(|v|)u',\Psi(|u|)v')} \frac{\Omega_\iota(u',v')}{|u|^n |v|^m} du\,dv, \tag{5.16}$$

and that

$$\widehat{\sigma}_{\iota,j,k}^{(l,s)}(\xi,\eta) = \iint_{\substack{w_\iota^j<|v|<w_\iota^{j+1} \\ w_\iota^k<|u|<w_\iota^{k+1}}} e^{i(\xi,\eta)(P_\Phi^l(|v|)u',P_\Psi^s(|u|)v')} \frac{\Omega_\iota(u',v')}{|u|^n |v|^m} du\,dv, \tag{5.17}$$

for $1 \leqslant l \leqslant d$ and $1 \leqslant s \leqslant b$. Notice that

$$\widehat{\sigma}_{\iota,j,k}^{(d+1,1)} = \widehat{\sigma}_{\iota,j,k}^{(1,b+1)} = 0.$$

It is clear that

$$|\widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}| + |\widehat{\sigma}_{\iota,j,k}^{(d+1,b)}| + |\widehat{\sigma}_{\iota,j,k}^{(d,b+1)}| + |\widehat{\sigma}_{\iota,j,k}^{(d,b)}| \leqslant C. \tag{5.18}$$

Next

$$\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) \right|$$

$$\leqslant \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\Omega_\iota(u',v')| \left| \int_1^{w_\iota} \int_1^{w_\iota} e^{i[(\xi\cdot u')\Phi(w_\iota^k\zeta)w_\iota^j r+(\eta\cdot v')\Psi(w_\iota^j r)w_\iota^k\zeta]} \frac{dr\,d\zeta}{r\zeta} \right| d\sigma(u')d\sigma(v').$$

By similar argument as that led to (4.17) with z_1 and z_2 replaced by u' and v' respectively, and Hölder's inequality, we get

$$\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) \right|$$

$$\leqslant C(\iota+1)^2 \|\Omega_\iota\|_{L^q}$$

$$\times \left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\mathscr{A}_d\,\varphi_1(w_\iota^k)(\xi\cdot u')|^{-\frac{1}{(d+1)}} |\mathscr{B}_b\,\varphi_2(w_\iota^j)(\eta\cdot v')|^{-\frac{1}{(b+1)}} d\sigma(u')d\sigma(v') \right)^{\frac{1}{q'}} \tag{5.19}$$

where $\mathscr{A}_d = \lambda\, w_\iota^j$ and $\mathscr{B}_b = \alpha\, w_\iota^k$. Hence, by (5.3) with $q' = 2$, we obtain

$$
\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) \right|
$$
$$
\leqslant C\,(\iota+1)^2\, 4(w_\iota)^2
$$
$$
\times \left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\mathscr{A}_d\, \varphi_1(w_\iota^k)(\xi\cdot u')|^{-\frac{1}{(d+1)}} |\mathscr{B}_b\, \varphi_2(w_\iota^j)(\eta\cdot v')|^{-\frac{1}{(b+1)}} d\sigma(u')d\sigma(v') \right)^{\frac{1}{2}}
$$
$$
\leqslant C\,(\iota+1)^2\, 4(w_\iota)^2\, \mathscr{G}_{d,b} |\mathscr{A}_d\, \varphi_1(w_\iota^k)\xi|^{-\frac{1}{2(d+1)}} |\mathscr{B}_b\, \varphi_2(w_\iota^j)\eta|^{-\frac{1}{2(b+1)}}
$$
$$
\leqslant C\,(\iota+1)^2\, 4(w_\iota)^2\, |\mathscr{A}_d\, \varphi_1(w_\iota^k)\xi|^{-\frac{1}{2(d+1)}} |\mathscr{B}_b\, \varphi_2(w_\iota^j)\eta|^{-\frac{1}{2(b+1)}},
\tag{5.20}
$$

where

$$
\mathscr{G}_{d,b} = \left(\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} |\xi'\cdot u'|^{-\frac{1}{d+1}} |\eta'\cdot v'|^{-\frac{1}{b+1}} d\sigma(u')d\sigma(v') \right)^{\frac{1}{2}}
$$
$$
= \sup_{\xi'\in\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} |\xi'\cdot u'|^{-\frac{1}{d+1}} d\sigma(u') \right)^{\frac{1}{2}} \sup_{\eta'\in\mathbb{S}^{m-1}} \left(\int_{\mathbb{S}^{m-1}} |\eta'\cdot v'|^{-\frac{1}{b+1}} d\sigma(v') \right)^{\frac{1}{2}}
$$
$$
< \infty.
$$

On the other hand, it can be shown that

$$
\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) \right| \leqslant C\,(\iota+1)^2\|\Omega_\iota\|_1 \leqslant C\,(\iota+1)^2,
\tag{5.21}
$$

where the last inequality obtained by (5.3). Notice that

$$
\left(4\,(w_\iota)^2 \right)^{\frac{1}{\iota+1}} \leqslant C.
\tag{5.22}
$$

Finally, by interpolation between (5.20) and (5.21) with $0 < \varepsilon = \dfrac{1}{4(\iota+1)} < 1$, we get

$$
\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) \right|
$$
$$
\leqslant C\,(\iota+1)^2\, |\lambda\, w_\iota^j\, \varphi_1(w_\iota^k)\,\xi|^{-\frac{1}{8(\iota+1)(d+1)}} |\alpha\, w_\iota^k\, \varphi_2(w_\iota^j)\,\eta|^{-\frac{1}{8(\iota+1)(b+1)}}.
\tag{5.23}
$$

Next,

$$
\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b+1)}(\xi,\eta) \right|
$$
$$
= \left| \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \int_{w_\iota^j}^{w_\iota^{j+1}} \int_{w_\iota^k}^{w_\iota^{k+1}} \Omega_\iota(u',v')\, e^{-i\Psi(r)\,\zeta\,(\eta\cdot v')} \right.
$$
$$
\left. \times \left(e^{-i\Phi(\zeta)\,r\,(\xi.u')} - e^{-iP_\Phi(\zeta)\,r\,(\xi.u')} \right) \frac{dr\, d\zeta}{r\ \zeta} d\sigma(u')\, d\sigma(v') \right|.
$$

Then, by Fubini's Theorem, the fact that $w_\iota^j < \zeta < w_\iota^{j+1}$, and φ_1 is increasing , we get

$$\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b+1)}(\xi,\eta) \right|$$

$$\leqslant \left| \lambda\, w_\iota^{j+1}\, \varphi_1(w_\iota^{k+1})\, \xi \right| \int_{\mathbb{S}^{n-1}} \int_{w_\iota^k}^{w_\iota^{k+1}} \left| \int_{\mathbb{S}^{m-1}} \int_{w_\iota^j}^{w_\iota^{j+1}} \Omega_\iota(u',v')\, e^{-i\Psi(r)\,\zeta(\eta\cdot v')} d\sigma(v') \frac{dr}{r} \right| \frac{d\zeta}{\zeta}\, d\sigma(u')$$

$$\leqslant \ln(w_\iota) \left| \lambda\, w_\iota^{j+1}\, \varphi_1(w_\iota^{k+1})\, \xi \right| \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{m-1}} \left| \Omega_\iota(u',v') \right| \mathscr{L}_{\iota,j,\Psi}(\eta,v')\, d\sigma(u')\, d\sigma(v'), \tag{5.24}$$

where

$$\mathscr{L}_{\iota,j,\Psi}(\eta,v') = \left| \int_{w_\iota^j}^{w_\iota^{j+1}} e^{-i\Psi(r)\,\zeta(\eta\cdot v')} \frac{dr}{r} \right|.$$

Therefore, by (4.16) and Van der Corput lemma in [13], we get

$$\mathscr{L}_{\iota,j,\Psi}(\eta,v') \leqslant \left| \alpha\, w_\iota^k\, \varphi_2(w_\iota^j)\,(\eta\cdot v') \right|^{-\frac{1}{(b+1)}}. \tag{5.25}$$

Also, $\mathscr{L}_{\iota,j,\Psi}$ satisfies

$$\mathscr{L}_{\iota,j,\Psi}(\eta,v') \leqslant C(\iota+1). \tag{5.26}$$

By interpolation between (5.25) and (5.26) with $0 < \varepsilon = \dfrac{1}{q'} < 1$, we get

$$\mathscr{L}_{\iota,j,\Psi}(\eta,v') \leqslant C(\iota+1) \left| \alpha\, w_\iota^k\, \varphi_2(w_\iota^j)\,(\eta\cdot v') \right|^{-\frac{1}{q'(b+1)}}. \tag{5.27}$$

Thus, by (5.24), (5.27), Hölder's inequality and the fact that

$$\sup_{\eta' \in \mathbb{S}^{m-1}} \left(\int_{\mathbb{S}^{m-1}} \left| \eta'\cdot v' \right|^{-\frac{1}{b+1}} d\sigma(v') \right)^{\frac{1}{q'}} < \infty, \tag{5.28}$$

we obtain

$$\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b+1)}(\xi,\eta) \right|$$

$$\leqslant C(\iota+1)^2\, \|\Omega_\iota\|_{L^q} \left| \lambda\, w_\iota^{j+1}\, \varphi_1(w_\iota^{k+1})\, \xi \right| \left| \alpha\, w_\iota^k\, \varphi_2(w_\iota^j)\, \eta \right|^{-\frac{1}{q'(b+1)}}. \tag{5.29}$$

On the other hand, we have

$$\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b+1)}(\xi,\eta) \right| \leqslant C(\iota+1)^2\, \|\Omega_\iota\|_{L^1}. \tag{5.30}$$

Finally, by (5.3), (5.22), and interpolation between (5.29) and (5.30), we get

$$\left| \widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b+1)}(\xi,\eta) \right|$$

$$\leqslant C(\iota+1)^2 \left| \lambda\, w_\iota^{j+1}\, \varphi_1(w_\iota^{k+1})\, \xi \right|^{\frac{1}{4(\iota+1)}} \left| \alpha\, w_\iota^k\, \varphi_2(w_\iota^j)\, \eta \right|^{-\frac{1}{8(\iota+1)(b+1)}}. \tag{5.31}$$

Similarly, we obtain that

$$
\begin{aligned}
&\left|\widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d+1,b)}(\xi,\eta)\right| \\
&\leqslant C(\iota+1)^2 \left|\alpha\, w_\iota^{k+1}\varphi_2(w_\iota^{j+1})\,\eta\right|^{\frac{1}{4(\iota+1)}} |\lambda\, w_\iota^j\,\varphi_1(w_\iota^k)\,\xi|^{-\frac{1}{8(\iota+1)(d+1)}}.
\end{aligned}
\tag{5.32}
$$

By similar steps as in (3.30)–(3.31), we can obtain that

$$
\begin{aligned}
&\left|\widehat{\sigma}_{\iota,j,k}^{(d+1,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d+1,b)}(\xi,\eta) + \widehat{\sigma}_{\iota,j,k}^{(d,b)}(\xi,\eta)\right| \\
&\leqslant C(\iota+1)^2 \left|\lambda\, w_\iota^{j+1}\,\varphi_1(w_\iota^{k+1})\,\xi\right|^{\frac{1}{4(\iota+1)}} \left|\alpha\, w_\iota^{k+1}\,\varphi_2(w_\iota^{j+1})\,\eta\right|^{\frac{1}{4(\iota+1)}}.
\end{aligned}
\tag{5.33}
$$

In addition, we have

$$
\begin{aligned}
&\left|\widehat{\sigma}_{\iota,j,k}^{(d+1,b)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b)}(\xi,\eta)\right| \\
&\leqslant \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} \int_{w_\iota^j}^{w_\iota^{j+1}} \int_{w_\iota^k}^{w_\iota^{k+1}} |\Omega_\iota(u',v')| \left|e^{-i\lambda\varphi_1(\zeta)r(\xi\cdot u')}-1\right| \frac{dr}{r}\frac{d\zeta}{\zeta}\,d\sigma(u')\,d\sigma(v').
\end{aligned}
$$

By the properties of φ_1 and $w_\iota^k < \zeta < w_\iota^{k+1}$, we get

$$
\left|\widehat{\sigma}_{\iota,j,k}^{(d+1,b)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b)}(\xi,\eta)\right| \leqslant C(\iota+1)^2 \|\Omega_\iota\|_{L^1} |\lambda\, w_\iota^{j+1}\,\varphi_1(w_\iota^{k+1})\,\xi|.
\tag{5.34}
$$

Also, we have

$$
\left|\widehat{\sigma}_{\iota,j,k}^{(d+1,b)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b)}(\xi,\eta)\right| \leqslant C(\iota+1)^2 \|\Omega_\iota\|_{L^1}.
\tag{5.35}
$$

Finally, by (5.3) and interpolation between (5.34) and (5.35), we get

$$
\left|\widehat{\sigma}_{\iota,j,k}^{(d+1,b)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b)}(\xi,\eta)\right| \leqslant C(\iota+1)^2 |\lambda\, w_\iota^{j+1}\,\varphi_1(w_\iota^{k+1})\,\xi|^{\frac{1}{4(\iota+1)}}.
\tag{5.36}
$$

Similarly, we have

$$
\left|\widehat{\sigma}_{\iota,j,k}^{(d,b+1)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(d,b)}(\xi,\eta)\right| \leqslant C(\iota+1)^2 |\alpha\, w_\iota^{k+1}\,\varphi_2(w_\iota^{j+1})\,\eta|^{\frac{1}{4(\iota+1)}}.
\tag{5.37}
$$

Now, by similar argument as that led to (5.17) we can prove the following:

$$
|\widehat{\sigma}_{\iota,j,k}^{(l,s)}(\xi,\eta)| \leqslant C(\iota+1)^2 |a_l\, w_\iota^j\, w_\iota^{lk}\, l!\, \xi|^{-\frac{1}{8l(\iota+1)}} |c_s\, w_\iota^k\, w_\iota^{sj}\, s!\, \eta|^{-\frac{1}{8s(\iota+1)}},
\tag{5.38}
$$

$$
|\widehat{\sigma}_{\iota,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(l-1,s)}(\xi,\eta)| \leqslant C(\iota+1)^2 |a_l\, w_\iota^{j+1}\, w_\iota^{l(k+1)}\, \xi|^{\frac{1}{8l(\iota+1)}} |c_s\, w_\iota^k\, w_\iota^{sj}\, s!\, \eta|^{-\frac{1}{8s(\iota+1)}},
\tag{5.39}
$$

$$
|\widehat{\sigma}_{\iota,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(l,s-1)}(\xi,\eta)| \leqslant C(\iota+1)^2 |a_l\, w_\iota^j\, w_\iota^{lk}\, l!\, \xi|^{-\frac{1}{8l(\iota+1)}} |c_s\, w_\iota^{k+1}\, w_\iota^{s(j+1)}\, \eta|^{\frac{1}{8s(\iota+1)}},
\tag{5.40}
$$

$$|\widehat{\sigma}_{\iota,j,k}^{(l,s)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(l-1,s)}(\xi,\eta) - \widehat{\sigma}_{\iota,j,k}^{(l,s-1)}(\xi,\eta) + \widehat{\sigma}_{\iota,j,k}^{(l,s)}(\xi,\eta)|$$
$$\leqslant C(\iota+1)^2 |a_l\, w_\iota^{j+1}\, w_\iota^{l(k+1)}\, \xi|^{\frac{1}{4l(\iota+1)}} |c_s\, w_\iota^{k+1}\, w_\iota^{s(j+1)}\, \eta|^{\frac{1}{4s(\iota+1)}}, \tag{5.41}$$

$$|\widehat{\sigma}_{\iota,j,k}^{(l,s-1)} - \widehat{\sigma}_{\iota,j,k}^{(l-1,s-1)}(\xi,\eta)| \leqslant C(\iota+1)^2 |a_l\, w_\iota^{j+1}\, w_\iota^{l(k+1)}\, \xi|^{\frac{1}{4l(\iota+1)}}, \tag{5.42}$$

$$|\widehat{\sigma}_{\iota,j,k}^{(l-1,s)} - \widehat{\sigma}_{\iota,j,k}^{(l-1,s-1)}(\xi,\eta)| \leqslant C(\iota+1)^2 |c_s\, w_\iota^{k+1}\, w_\iota^{s(j+1)}\, \eta|^{\frac{1}{4s(\iota+1)}}. \tag{5.43}$$

Now, we choose and fix a function $\phi(t) \in C_0^\infty(\mathbb{R})$ such that $\phi(t) \equiv 1$ for $|t| \leqslant \dfrac{1}{2}$ and $\phi(t) \equiv 0$ for $|t| \geqslant 1$. For $j,k \in \mathbb{Z}$, let $(\widehat{\psi}_{j,k}^{(l)})(\xi) = (\widehat{\psi}^{(l)})(|w_\iota^j\, w_\iota^{lk}\, a_l\, \xi|^2)$ and $(\widehat{\phi}_{j,k}^{(s)})(\eta) = (\widehat{\phi}^{(s)})(|w_\iota^k\, w_\iota^{sj}\, c_s\, \eta|^2)$. Thus, for $1 \leqslant l \leqslant d+1$ and $1 \leqslant s \leqslant b+1$, we define the family of measures $\{\vartheta_{\iota,j,k}^{(l,s)} : j,k \in \mathbb{Z}\}$ by

$$\begin{aligned}
\widehat{\vartheta}_{\iota,j,k}^{(l,s)}(\xi,\eta) = &\; \widehat{\sigma}_{\iota,j,k}^{(l,s)}(\xi,\eta) \prod_{l<r\leqslant d+1}(\widehat{\psi}_{j,k}^{(r)})|\xi|) \prod_{s<m\leqslant b+1}(\widehat{\phi}_{j,k}^{(m)})|\eta|) \\
&- \widehat{\sigma}_{\iota,j,k}^{(l-1,s)}(\xi,\eta) \prod_{l-1<r\leqslant d+1}(\widehat{\psi}_{j,k}^{(r)})|\xi|) \prod_{s<m\leqslant b+1}(\widehat{\phi}_{j,k}^{(m)})|\eta|) \\
&- \widehat{\sigma}_{\iota,j,k}^{(l,s-1)}(\xi,\eta) \prod_{l<r\leqslant d+1}(\widehat{\psi}_{j,k}^{(r)})|\xi|) \prod_{s-1<m\leqslant b+1}(\widehat{\phi}_{j,k}^{(s)})|\eta|) \\
&+ \widehat{\sigma}_{\iota,j,k}^{(l-1,s-1)}(\xi,\eta) \prod_{l-1<r\leqslant d+1}(\widehat{\psi}_{j,k}^{(r)})|\xi|) \prod_{s-1<m\leqslant b+1}(\widehat{\phi}_{j,k}^{(s)})|\eta|)
\end{aligned} \tag{5.44}$$

where we use the convention $\prod_{i\in\emptyset} A_i = 1$. By the definition of $\vartheta_{\iota,j,k}$, we can show that

$$\|\vartheta_{\iota,j,k}^{(l,s)}\| \leqslant C(\iota+1)^2 \tag{5.45}$$

and

$$|\widehat{\vartheta}_{\iota,j,k}^{(l,s)}(\xi,\eta)| \leqslant C(\iota+1)^2 |A_{\iota,l}\, L_l(\xi)|^{\pm\frac{1}{8l(\iota+1)}} |B_{\iota,s}\, H_s(\eta)|^{\pm\frac{1}{8s(\iota+1)}}, \tag{5.46}$$

where

$$L_l(\xi) = \begin{cases} \lambda\,\xi, & l=d+1 \\ a_l\,\xi, & l\neq d+1 \end{cases},$$

$$H_s(\eta) = \begin{cases} \alpha\,\eta, & s=b+1 \\ c_s\,\eta, & s\neq b+1 \end{cases},$$

$$A_{\iota,l} = \begin{cases} w_\iota^j\,\varphi_1(w_\iota^k), & l=d+1 \\ C w_\iota^j\, w_\iota^{lk}, & l\neq d+1 \end{cases}$$

and

$$B_{\iota,s} = \begin{cases} w_\iota^k\,\varphi_2(w_\iota^j), & l=b+1 \\ C w_\iota^k\, w_\iota^{sj}, & l\neq b+1. \end{cases}$$

Also, we can observe that

$$\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\vartheta_{\iota,j,k}^{(l,s)} = \sigma_{\iota,j,k}^{(d+1,b+1)}.$$

Thus,

$$
\begin{aligned}
T_{\Omega_\iota,\Lambda}(f)(x,y) &= \sum_{j,k\in\mathbb{Z}}\sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\vartheta_{\iota,j,k}^{(l,s)}*f \\
&= \sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\left(\sum_{j,k\in\mathbb{Z}}\vartheta_{\iota,j,k}^{(l,s)}*f\right).
\end{aligned}
\tag{5.47}
$$

Therefore, we have

$$\|T_{\Omega_\iota,\Lambda}(f)\|_p \leqslant \sum_{l=1}^{d+1}\sum_{s=1}^{b+1}\|T_{\Omega_\iota,\Lambda}^{(l,s)}f\|_p, \tag{5.48}$$

where

$$T_{\Omega_\iota,\Lambda}^{(l,s)}f(x,y) = \sum_{j,k\in\mathbb{Z}}\vartheta_{\iota,j,k}^{(l,s)}*f.$$

On the other hand, let

$$(\vartheta_\iota^{(l,s)})^*(f) = \sup_{j,k}\left|(\vartheta_{\iota,j,k}^{(l,s)}*f)(x,y)\right|.$$

Thus, by Theorem 1.4 and (5.3), we have

$$\|(\vartheta_\iota^{(l,s)})^*(f)\|_p \leqslant C(\iota+1)^2\|f\|_p, \tag{5.49}$$

for all $1 < p < \infty$ with a constant C_p independent of ι.

Hence, by (5.46), (5.49), and Proposition 2.2 with $\rho_1 = l$, $\rho_2 = s$, and $\varepsilon_1 = \varepsilon_2 = 1$, we obtain

$$\left\|T_{\Omega_\iota,\Lambda}^{(l,s)}f\right\|_p = \left\|\sum_{j,k\in\mathbb{Z}}\vartheta_{\iota,j,k}^{(l,s)}*f\right\|_p \leqslant C_p(\iota+1)^2\|f\|_p. \tag{5.50}$$

Therefore, by (5.13), (5.47), (5.48), and (5.50), we get

$$\left\|T_{\Omega_\iota,\Lambda}f\right\|_p = \left\|\sum_{j,k\in\mathbb{Z}}\sigma_{\iota,j,k}^{(d+1,b+1)}*f\right\|_p \leqslant C_p(\iota+1)^2\|f\|_p. \tag{5.51}$$

where $C_p > 0$ is a constant independent of ι. Finally, by (5.5), (5.48), and (5.51), the proof of Theorem 1.2 is complete. $\square$

REFERENCES

[1] B. AL-AZRIYAH, A. AL-SALMAN, *Singular and Marcinkiewicz Integral Operators on Product Domains*, to appear in Commun. Korean. Math. Soc. **38** (2), 401–430.

[2] H. AL-QASSEM, Y. PAN, *L^p boundedness for singular integrals with rough kernels on product domains*, Hokkaido Math. J. vol. **31** (2002) pp. 555–613.

[3] A. AL-SALMAN, *Marcinkiewicz functions with hardy space kernels*, Math. Inequalities Appl. vol. **21**, no. 2 (2018), 553–567.

[4] A. AL-SALMAN, *Marcinkiewicz integral operators along twisted surfaces*, Commun. Pure Appl. Anal. 2022, **21** (1): 159–181.

[5] A. AL-SALMAN, *Maximal functions along twisted surfaces on product domains*, Bull. Korean Math. Soc. **58** (2021), no. 4, pp. 1003–1019.

[6] A. AL-SALMAN, *Singular integral operators on product domains along twisted surfaces*, Front. Math. China 2021, **16** (1): 13–28.

[7] A. AL-SALMAN, H. AL-QASSEM, Y. PAN, *Singular integrals on product domains*, Indiana Univ. Math. j. 2006; **55**: 369–387.

[8] J. DUOANDIKOETXEA, *Multiple singular integrals and maximal functions along hypersurfaces*, Ann. inst. Fourier, Ggrenoble **36**, 4 (1986), 185–206.

[9] D. FAN, K. GUO, Y. PAN, *Singular integrals with rough kernels on product spaces*, Hokkaido Math. J. vol. **28** (1999) pp. 435–460.

[10] R. FEFFERMAN, *Singular integrals on product domains*, Bull. Am. Math. Soc. **4** (1981), 196–201.

[11] R. FEFFERMAN, E. M. STEIN, *Singular integrals on product spaces*, Adv. in Math. **45**, 117–143 (1982).

[12] Y. JIANG, S. LU, *A class of singular integral operators with rough kernel on product domains*, Hokkaido Math. J. vol. **24** (1995) pp. 1–7.

[13] E. M. STEIN, *Harmonic analysis: Real-Variable Methods, Orthogonality and Oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.

Badriya Al-Azri
Sultan Qaboos University
College of Science, Department Mathematics
Al-Khod Muscat, Sultanate of Oman
e-mail: badriyaalazri6@gmail.com

Ahmad Al-Salman
Sultan Qaboos University
College of Science, Department of Mathematics
Al-Khod Muscat, Sultanate of Oman
and
Department of Mathematics
Yarmouk University
Irbid, Jordan
e-mail: alsalman@squ.edu.om
alsalman@yu.edu.jo

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com