Journal of Mathematical Inequalities
Volume 18, Number 1 (2024), 193–205
doi:10.7153/jmi-2024-18-11

# ON FUZZY SPECTRAL RADII FOR FUZZY BOUNDED OPERATORS WITH APPLICATION TO FUZZY VOLTERRA OPERATOR

TUDOR BÎNZAR AND FLAVIUS PATER*

(*Communicated by T. Burić*)

*Abstract.* This paper's aim is to extend and generalize the classic results regarding spectral radii and the corresponding resolvent sets for some different classes of bounded operators acting on fuzzy normed spaces. In this context, the fuzzy norm definition introduced giving shape to a new topology for a fuzzy space, namely a fuzzy topology, also gives the opportunity to study the behavior of various types of operators defined between fuzzy normed spaces, along with their spectral properties. There are several definitions for resolvent sets and consequently, several corresponding definitions of spectral radii that will be considered in this work, since these are non-equivalent ways of defining such notions. Spectral radii are calculated for a fuzzy Volterra type operator acting between fuzzy normed spaces.

## 1. Introduction and preliminaries

Fuzzy operator theory is a relatively new branch of well established studies of operator theory in locally convex spaces or even in the more general topological spaces. In this context, the new fuzzy norm definition introduced giving shape to a new topology for a fuzzy space, namely a fuzzy topology, also gives the opportunity to study the behavior of various types of operators defined between fuzzy normed spaces. Before mentioning our setup, we recall ([27]) that a $t$-norm is a composition law $\diamond : [0,1] \times [0,1] \to [0,1]$ associative, commutative and with $1$ as neutral element, satisfying the monotonicity condition: if $a \leqslant c$ and $b \leqslant d$, where $a,b,c,d \in [0,1]$, then $a \diamond b \leqslant c \diamond d$.

In some situations we will consider the following property ([11]):

$(\mathscr{H})$ : For all $\alpha \in (0,1)$ there exists $\beta \in (0,1)$ such that

$$\underbrace{\beta \diamond \beta \diamond \ldots \diamond \beta}_{n \text{ times}} > \alpha$$

for each $n \in \mathbb{N}, n \geqslant 2$.

The $(\mathscr{H})$ property is verified for example by $t$-norms of Hadžić-Pap type (see [15]). We have approached in this paper the setup of Nădăban and Dzitac ([24]) where

*Mathematics subject classification* (2020): 08A72, 47S40, 46S40.

*Keywords and phrases*: Fuzzy Volterra operator, fuzzy normed space, neighborhood fuzzy bounded linear operator, spectral radius.

* Corresponding author.


Paper JMI-18-11

the fuzzy norm with respect to a continuous $t$-norm is defined on a vector space $\mathscr{X}$ over $\mathbb{K}$ ($\mathbb{K}$ being $\mathbb{R}$ or $\mathbb{C}$) as a function $\mathscr{N}:\mathscr{X}\times[0,\infty)\to[0,1]$ which satisfies:

($\mathscr{N}_1$) $\mathscr{N}(u,0)=0$, for all $u\in\mathscr{X}$;
($\mathscr{N}_2$) $[\mathscr{N}(u,t)=1$, for all $t>0$ $]$ if and only if $u=0$;
($\mathscr{N}_3$) $\mathscr{N}(\delta u,t)=\mathscr{N}\left(u,\frac{t}{|\delta|}\right)$, for all $u\in\mathscr{X}$, all $t\geqslant 0$, and all $\delta\in\mathbb{K}^*$;
($\mathscr{N}_4$) $\mathscr{N}(u+v,t+s)\geqslant\mathscr{N}(u,t)\diamond\mathscr{N}(v,s)$, for any $u,v\in\mathscr{X}$, and any $t,s\geqslant 0$;
($\mathscr{N}_5$) For any $u\in\mathscr{X}$, $\mathscr{N}(u,\cdot)$ is left continuous and $\lim_{t\to\infty}\mathscr{N}(u,t)=1$.

The triple $(\mathscr{X},\mathscr{N},\diamond)$ is called fuzzy normed linear space (FNLS in short).

Few efforts have been made in the direction of developing a spectral theory for different types of bounded operators acting on fuzzy normed linear spaces. Most of the techniques that are working in Banach spaces no longer work in the fuzzy normed spaces setup. Studying the problem of invariant subspaces in Banach spaces, notions like the spectrum, the spectral radius, and the Neumann series have been used extensively.

The Gelfand formula defines a spectral radius for a bounded linear operator $S$ acting on a Banach space: $r(S)=\lim_{n\longrightarrow\infty}\sqrt[n]{\|S^n\|}$. Classic results are that the resolvent $R_\lambda=(\lambda I-S)^{-1}$ equals the Neumann series $\sum_{\iota=0}^{\infty}\frac{S^\iota}{\lambda^{\iota+1}}$ when $|\lambda|>r(S)$. Also the radius of the spectrum $|\sigma(S)|=\sup\{|\lambda|:\lambda\in\sigma(S)\}$ is $r(S)$.

From this point onward, in this direction were developed a large number of papers, spectral theory being approached in much broader contexts, like the one of locally convex spaces and even topological spaces and for linear operators with various types of boundedness. These frameworks and this theory were the subjects of numerous papers such as: ([4, 5, 9, 12, 18, 19, 21, 22, 25, 28, 29, 30, 31]).

All these works were inspirational for getting us closer to the expected results of our context. The present paper's purpose is to present some answers to the question if we can have similar results in a more general framework, e.g., for different types of bounded linear operators defined on fuzzy normed linear spaces.

With the purpose of having these notions valid also in a fuzzy normed spaces context, we extended within our framework some of these results given in the context of locally convex spaces. We managed to go even further by generalizing these results to the context of fuzzy normed spaces endowed with an almost arbitrary $t$-norm. There are several ways authors defined the fuzzy normed spaces setup (see [10], [16], [26]) and, as a natural preoccupation, elements of spectral theory in fuzzy normed spaces and fuzzy Hilbert spaces emerged (see [1, 3, 6]).

Since there are some different classes of bounded operators that should be taken into account, several definitions of resolvent sets and consequently, several corresponding definitions of spectral radii will be considered, since these are non-equivalent ways of defining such notions.

The main results are presented in this paper in three separate sections. The first section is dedicated to proving some properties for a special class of operators acting on fuzzy normed linear spaces, like the *neighborhood locally fuzzy bounded operators* first introduced in ([8]) and also bringing forward the existent connections between this class of operators and the class of continuous ones.

The second section is devoted to defining several types of spectral radii and the corresponding resolvent sets and to prove theorems that show the relations between these radii and characterizations of them. Also, results involving spectral considerations that are true in a Banach spaces setting are proved in this more general setting following in most cases some laborious techniques.

As it is well known, the Volterra operator is compact, acts on Hilbert spaces and its spectrum consists of a single point $\lambda = 0$. Most of its applications are encountered in studying the solutions of ordinary differential equations, especially issues connected with the boundary values for such equations. A great deal of inverse problems can be transformed into Volterra type equations, which are largely used in mathematical physics. This is the reason we approached the study of such an operator in the third section of this paper, where is presented the calculus of several types of fuzzy spectral radii for a Volterra type operator acting on fuzzy normed linear spaces.

In ([24]) it is shown that $\mathscr{X}$ endowed with a fuzzy norm $\mathscr{N}$ is a topological metrizable vector space with respect to the topology $\mathscr{T}_{\mathscr{N}}$ given by the fundamental system of neighborhoods:

$$\mathscr{S}(u,\varepsilon,s) = \{v \in \mathscr{X} \,:\, \mathscr{N}(u-v,s) > 1-\varepsilon\}.$$

The definition of the convergence of a sequence $(z_n)$ in a FNLS $(\mathscr{X},\mathscr{N},\diamond)$ is natural and it is used by all the researchers. Thus, a sequence $(z_n)$ is convergent to $z \in \mathscr{X}$, denoted by $\lim\limits_{n\to\infty} z_n = z$ or $z_n \to z$, if $\lim\limits_{n\to\infty} \mathscr{N}(z_n - z, s) = 1$, for all $s > 0$.

We recall ([13]) that $(z_n) \subset \mathscr{X}$ is a fuzzy Cauchy sequence if $(z_n)$ is a Cauchy sequence in $\mathscr{T}_{\mathscr{N}}$.

A $(\mathscr{X},\mathscr{N},\diamond)$ *FNLS* is called fuzzy complete if any fuzzy Cauchy sequence is fuzzy convergent.

## 2. Properties of various types of bounded operators acting on fuzzy normed spaces

In this part, we recall the definition of the class of neighborhood locally fuzzy bounded operators $(NLFB(\mathscr{X}))$, as was introduced in ([8]), we point out the connection with the class of fuzzy continuous operators $(FC(\mathscr{X}))$ and we show that $NLFB(\mathscr{X})$ is a complex algebra.

DEFINITION 1. ([8]) Let $(\mathscr{X},\mathscr{N},\diamond)$ be a *FNLS* and $T:\mathscr{X}\to\mathscr{X}$ a linear operator. We call $T$ neighborhood locally fuzzy bounded if there exist $\alpha_0 \in (0,1)$ and $s_0 > 0$ such that for all $\alpha \in (0,1)$, it exists $s_\alpha > 0$ with the property that all $u \in \mathscr{X}$ which verify $\mathscr{N}(u,s_0) > \alpha_0$, have also the property $\mathscr{N}(Tu,s_\alpha) > \alpha$.

Denote the class of all neighborhood locally fuzzy bounded operators on $\mathscr{X}$ by $NLFB(\mathscr{X})$.

Recall ([23]) that the linear operator $T:\mathscr{X}\to\mathscr{X}$, where $(\mathscr{X},\mathscr{N},\diamond)$ is *FNLS* is fuzzy continuous if $(\forall)\alpha \in (0,1)$, $(\exists)\beta_\alpha \in (0,1)$, $(\exists)M_\alpha > 0$ such that $(\forall)t > 0$, $(\forall)x \in \mathscr{X}$ with $\mathscr{N}(x,t) > \beta_\alpha \Rightarrow \mathscr{N}(Tx, M_\alpha t) > \alpha$.

Also, $T$ is fuzzy continuous if $(\forall)\alpha \in (0,1)$, $(\forall)s \in (0,1)$, $(\exists)\beta_{\alpha,s} \in (0,1)$, $(\exists)\delta_{\alpha,s} > 0$ such that $(\forall)x \in \mathscr{X} : \mathscr{N}(x, \delta_{\alpha,s}) > \beta_{\alpha,s} \Rightarrow \mathscr{N}(Tx, s) > \alpha$.

We denote by $FC(\mathscr{X})$ the class of fuzzy continuous operators acting on $\mathscr{X}$. The connection between these two classes of operators is given in the next proposition.

PROPOSITION 1. *In a FNLS* $(\mathscr{X}, \mathscr{N}, \diamond)$, *the following inclusion holds:*

$$NLFB(\mathscr{X}) \subsetneq FC(\mathscr{X}).$$

*Proof.* Let $T \in NLFB(\mathscr{X})$ and $\alpha \in (0,1)$. From the hypothesis, $(\exists)\alpha_0 \in (0,1)$, $(\exists)t_0 > 0$, $(\exists)s_\alpha > 0$ such that $(\forall)v \in \mathscr{X}$ with $\mathscr{N}(v, t_0) > \alpha_0$ we have $\mathscr{N}(Tv, s_\alpha) > \alpha$. For $\beta_\alpha = \alpha_0 \in (0,1)$ and $M_\alpha = \dfrac{s_\alpha}{t_0}$, we deduce that $(\forall)u \in \mathscr{X}$, $(\forall)t > 0$ with $\mathscr{N}(u,t) > \beta_\alpha$, that is $\mathscr{N}\left(\dfrac{t_0}{t}u, t_0\right) > \alpha_0$ we have $\mathscr{N}\left(T\left(\dfrac{t_0}{t}u\right), s_\alpha\right) > \alpha$. Hence $\mathscr{N}\left(Tu, \dfrac{ts_\alpha}{t_0}\right) = \mathscr{N}(Tu, M_\alpha t) > \alpha$. Therefore $T \in FC(\mathscr{X})$. $\square$

The inclusion is strict because the operator $I : \mathscr{X} \to \mathscr{X}$, $Iu = u$, where $(\mathscr{X}, \mathscr{N}, \wedge)$ is the fuzzy normed linear space from ([7] Theorem 10), is fuzzy continuous but it is not neighborhood locally fuzzy bounded.

With the purpose of giving conditions for equality to hold in Proposition 1, we introduce the following definition:

DEFINITION 2. Let $(\mathscr{X}, \mathscr{N}, \diamond)$ be a $FNLS$. The space $\mathscr{X}$ is called locally fuzzy bounded if $(\exists)\alpha_0 \in (0,1)$, $(\exists)t_0 > 0$ such that $(\forall)\gamma \in (0,1)$, $(\exists)t_\gamma > 0$, $(\forall)u \in \mathscr{X}$ : $\mathscr{N}(u, t_0) > \alpha_0 \Rightarrow \mathscr{N}(u, t_\gamma) > \gamma$.

REMARK 1. $\mathscr{X}$ is locally fuzzy bounded iff $I \in NLFB(\mathscr{X})$.

Next, a characterization on the occurrence of the equality between the above mentioned classes of operators is presented.

THEOREM 1. *Let* $(\mathscr{X}, \mathscr{N}, \diamond)$ *be a FNLS and* $T : \mathscr{X} \to \mathscr{X}$ *a linear operator. Then* $\mathscr{X}$ *is locally fuzzy bounded iff* $NLFB(\mathscr{X}) = FC(\mathscr{X})$.

*Proof.* "$\Rightarrow$" Suppose that $\mathscr{X}$ is locally fuzzy bounded. According to Proposition 1, it remains to prove that $FC(\mathscr{X}) \subseteq NLFB(\mathscr{X})$. Consider $T \in FC(\mathscr{X})$ and $\alpha \in (0,1)$. Then, $(\forall)s > 0$, $(\exists)\beta_{\alpha,s} \in (0,1)$, $(\exists)u_{\alpha,s} > 0$ such that $(\forall)v \in \mathscr{X} : \mathscr{N}(v, u_{\alpha,s}) > \beta_{\alpha,s} \Rightarrow \mathscr{N}(Tv, s) > \alpha$. For $s = 1$ we deduce $(\exists)\beta_\alpha \in (0,1), (\exists)u_\alpha > 0$ such that $(\forall)v \in \mathscr{X} : \mathscr{N}(v, u_\alpha) > \beta_\alpha \Rightarrow \mathscr{N}(Tv, 1) > \alpha$. From the hypothesis, for $\gamma = \beta_\alpha \in (0,1)$, it results $(\exists)\alpha_0 \in (0,1)$, $(\exists)t_0 > 0$, $(\exists)t_\alpha > 0$ such that $(\forall)x \in \mathscr{X} : \mathscr{N}(x, t_0) > \alpha_0 \Rightarrow \mathscr{N}(x, t_\alpha) > \beta_\alpha$. Hence $(\exists)\alpha_0 \in (0,1)$, $(\exists)t_0 > 0$ such that $(\forall)\alpha \in (0,1)$, $(\exists)s_\alpha = \dfrac{t_\alpha}{u_\alpha} > 0$ such that $(\forall)x \in \mathscr{X} : \mathscr{N}(x, t_0) > \alpha_0 \Rightarrow \mathscr{N}\left(\dfrac{u_\alpha}{t_\alpha}x, u_\alpha\right) = \mathscr{N}(x, t_\alpha) > \beta_\alpha$, whence $\mathscr{N}\left(T(\dfrac{u_\alpha}{t_\alpha}x), 1\right) = \mathscr{N}(Tx, s_\alpha) > \alpha$. Therefore $T \in NLFB(X)$.

"$\Leftarrow$" $I \in FC(\mathscr{X}) = NLFB(\mathscr{X}) \Rightarrow \mathscr{X}$ is locally fuzzy bounded. $\square$

The following two propositions establish some algebraic properties of operators from $NLFB(X)$.

PROPOSITION 2. *Let $(\mathscr{X}, \mathscr{N}, \diamond)$ be a FNLS. If $T \in FC(\mathscr{X})$ and $S \in NLFB(\mathscr{X})$, then $TS \in NLFB(\mathscr{X})$.*

*Proof.* Since $S \in NLFB(\mathscr{X})$, $(\exists)\alpha_0 \in (0,1)$, $(\exists)t_0 > 0$ such that $(\forall)\beta \in (0,1)$, $(\exists)s_\beta > 0$, $(\forall)u \in \mathscr{X}$ with $\mathscr{N}(u,t_0) > \alpha_0 \Rightarrow \mathscr{N}(Su, s_\beta) > \beta$. From $T \in FC(\mathscr{X})$, it results $(\forall)\alpha \in (0,1)$, $(\exists)\beta_\alpha \in (0,1)$, $(\exists)M_\alpha > 0$ such that $(\forall)v \in \mathscr{X}$, $(\forall)t > 0$ with $\mathscr{N}(v,t) > \beta_\alpha \Rightarrow \mathscr{N}(Tv, M_\alpha t) > \alpha \quad (\star)$.

Consider $\alpha \in (0,1)$. Then, for $\beta = \beta_\alpha \in (0,1)$, $(\exists)s_{\beta_\alpha} > 0$ such that $(\forall)u \in \mathscr{X} : \mathscr{N}(u,t_0) > \alpha_0 \Rightarrow \mathscr{N}(Su, s_{\beta_\alpha}) > \beta_\alpha$. Thus, $(\exists)\alpha_0 \in (0,1)$, $(\exists)t_0 > 0$ such that $(\forall)\alpha \in (0,1)$, $(\exists)t_\alpha = M_\alpha s_{\beta_\alpha} > 0$ with the property that $(\forall)u \in \mathscr{X} : \mathscr{N}(u,t_0) > \alpha_0$, we have $\mathscr{N}(Su, s_{\beta_\alpha}) > \beta_\alpha$.

By $(\star)$, it follows $\mathscr{N}((TS)u, t_\alpha) = \mathscr{N}(T(Su), s_{\beta_\alpha} M_\alpha) > \alpha$, hence $TS \in NLFB(\mathscr{X})$. $\square$

PROPOSITION 3. *Let $(\mathscr{X}, \mathscr{N}, \diamond)$ be a FNLS. If $T_1, T_2 \in NLFB(\mathscr{X})$, then $T_1 + T_2 \in NLFB(\mathscr{X})$ and $T_1 T_2 \in NLFB(\mathscr{X})$.*

*Proof.* From Propositions 1 and 2, it results that $T_1 T_2 \in NLFB(\mathscr{X})$. We prove that $T_1 + T_2 \in NLFB(\mathscr{X})$. By hypothesis, $(\exists)\alpha_i^0 \in (0,1)$, $(\exists)t_i^0 > 0$ such that $(\forall)\beta \in (0,1)$, $(\exists)s_\beta^i > 0$, $(\forall)u \in \mathscr{X} :\ \mathscr{N}(u, t_i^0) > \alpha_i^0 \Rightarrow \mathscr{N}(T_i u, s_\beta^i) > \beta$, $i \in \{1,2\}$.

For $\alpha_0 = \max\{\alpha_1^0, \alpha_2^0\} \in (0,1)$ and $t_0 = \min\{t_1^0, t_2^0\} > 0$ we have $(\forall)\alpha \in (0,1)$, $(\exists)\beta_\alpha \in (0,1)$ with $\beta_\alpha \diamond \beta_\alpha > \alpha$, $(\exists)$ $s_\alpha = s^1_{\beta_\alpha} + s^2_{\beta_\alpha} > 0$ such that $(\forall)$ $u \in \mathscr{X}$ : $\mathscr{N}(u,t_0) > \alpha_0 \Rightarrow \mathscr{N}(u, t_i^0) \geqslant \mathscr{N}(u,t_0) > \alpha_0 \geqslant \alpha_i^0$, $i \in \{1,2\}$, hence $\mathscr{N}((T_1+T_2)u, s_\alpha) \geqslant \mathscr{N}(T_1 u, s^1_{\beta_\alpha}) \diamond \mathscr{N}(T_2 u, s^2_{\beta_\alpha}) \geqslant \beta_\alpha \diamond \beta_\alpha > \alpha$.

Therefore $T_1 + T_2 \in NLFB(\mathscr{X})$. $\square$

## 3. Fuzzy spectral radii for fuzzy bounded operators

In this section, within the framework of fuzzy normed spaces with a $t$-norm that is not making this space a locally convex one, a general approach towards spectral sets, resolvent sets and their properties is pursuit. There are several types of resolvent sets and their corresponding spectral sets that we introduce, for each of the operator classes studied in the previous section.

DEFINITION 3. Let $(\mathscr{X}, \mathscr{N}, \diamond)$ be a *FNLS* and $T : \mathscr{X} \to \mathscr{X}$ a linear operator.

*i*) The set $\rho_{fl}(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is invertible in the linear operators algebra $\}$ is called the fuzzy linear resolvent set of $T$;

*ii*) The set $\rho_{fc}(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is invertible in $FC(\mathscr{X})\}$ is called the fuzzy continuous resolvent set of $T$;

*iii*) The set $\rho_{nlfb}(T)=\{\lambda\in\mathbb{C}:(\exists)\gamma\in\mathbb{C},(\exists)S\in NLFB(\mathscr{X})$ such that $(\lambda I-T)^{-1}=\gamma I+S\}$ is called the neighborhood locally fuzzy bounded resolvent set of $T$.

The fuzzy spectral sets $\sigma_{fl}(T)$, $\sigma_{fc}(T)$, $\sigma_{nlfb}(T)$ are defined as being the complements in $\mathbb{C}$ of the corresponding fuzzy resolvent sets.

We will denote by $R_\lambda$ the inverse ( left or right ) of $\lambda I-T$, provided it exists. Also, we call $R_\lambda$ the resolvent of $T$.

REMARK 2. *i*) It is obvious from Proposition 1 that $\sigma_{fl}(T)\subseteq\sigma_{fc}(T)\subseteq\sigma_{nlfb}(T)$.

*ii*) If $(\mathscr{X},N,\diamond)$ is locally fuzzy bounded and $T:\mathscr{X}\to\mathscr{X}$ is linear operator, then by Theorem 1, $\sigma_{fc}(T)=\sigma_{nlfb}(T)$.

In the next proposition, the equality of the fuzzy spectral sets given in Remark 2, ii), is also valid for neighborhood locally fuzzy bounded operators.

PROPOSITION 4. *Let $(\mathscr{X},N,\diamond)$ be a FNLS. If $T\in NLFB(\mathscr{X})$, then $\sigma_{fc}(T)=\sigma_{nlfb}(T)$.*

*Proof.* If $\mathscr{X}$ is locally fuzzy bounded, then the result is given in Remark 2, ii).

Suppose now that $\mathscr{X}$ is not locally fuzzy bounded. Since $\sigma_{fc}(T)\subseteq\sigma_{nlfb}(T)$, according to Remark 2,$i$), it remains to prove that $\rho_{fc}(T)\subseteq\rho_{nlfb}(T)$. Let $\lambda\in\rho_{fc}(T)$, $\lambda\neq 0$. As $R_\lambda=\dfrac{1}{\lambda}R_\lambda T+\dfrac{1}{\lambda}I$ and $R_\lambda\in FC(\mathscr{X})$, it results from Proposition 2 that $\lambda\in\rho_{nfb}(T)$. It is left to prove that $0\in\sigma_{fc}(T)$, which necessarily yields $0\in\sigma_{nfb}(T)$. If, by absurd $0\notin\sigma_{fc}(T)$, i.e. $R_0=T^{-1}\in FC(\mathscr{X})$, then according to Proposition 2, $I=T^{-1}T\in NLFB(\mathscr{X})$ that is impossible in a non-locally fuzzy bounded space. Consequently, $0\in\sigma_{fc}(T)$. $\square$

In the sequel, two natural convergence definitions for the sequences from the considered classes of operators are introduced.

DEFINITION 4. Let $(\mathscr{X},\mathscr{N},\diamond)$ be a *FNLS* and $(S_n)_n\subset FC(\mathscr{X})$. We say that the sequence $(S_n)$ converges fuzzy equicontinuously to null operator denoted $S_n\xrightarrow{\text{feq}}0$ if $(\forall)\alpha\in(0,1)$, $(\forall)t>0$, $(\exists)\beta_{\alpha,t}\in(0,1)$, $(\exists)s_{\alpha,t}>0$ such that $(\forall)\varepsilon>0$, $(\exists)n_0(\varepsilon,\alpha,t)\in\mathbb{N}$ with the property that $(\forall)n\geqslant n_0(\varepsilon,\alpha,t)$, $(\forall)u\in\mathscr{X}:\mathscr{N}(u,s_{\alpha,t})>\beta_{\alpha,t}\Rightarrow\mathscr{N}(S_nu,\varepsilon t)>\alpha$

DEFINITION 5. Let $(\mathscr{X},\mathscr{N},\diamond)$ be a *FNLS* and $(S_n)_n\subset NLFB(\mathscr{X})$. We say that the sequence $(S_n)$ converges fuzzy uniformly to null operator denoted $S_n\xrightarrow{\text{fu}}0$ if $(\exists)\alpha_0\in(0,1)$, $(\exists)t_0>0$, such that $(\forall)\beta\in(0,1)$, $(\forall)s>0$, $(\exists)n_0(\beta,s)\in\mathbb{N}$ with the property that $(\forall)n\geqslant n_0(\beta,s)$, $(\forall)u\in\mathscr{X}:\mathscr{N}(u,t_0)>\alpha_0\Rightarrow\mathscr{N}(S_nu,s)>\beta$.

Taking into account the newly defined concepts of convergence, we arrive to the next definition.

DEFINITION 6. Let $(\mathscr{X},\mathscr{N},\diamond)$ be a *FNLS* and $T:\mathscr{X}\to\mathscr{X}$ a linear operator. The numbers:

*i)* $r_{fl}(T)=\inf\{\mu>0:\ (\forall)u\in\mathscr{X},(\forall)t>0,\lim\limits_{n\to\infty}\mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right)=1\}$;

*ii)* $r_{fc}(T)=\inf\{\mu>0:\ \frac{T^n}{\mu^n}\xrightarrow{\text{feq}}0\}$;

*iii)* $r_{nlfb}(T)=\inf\{\mu>0:\ \frac{T^n}{\mu^n}\xrightarrow{\text{fu}}0\}$

are called fuzzy linear radius of $T$, fuzzy continuous radius of $T$, neighborhood locally fuzzy bounded radius of $T$, respectively.

In the next result, relations between the freshly introduced radii are provided.

THEOREM 2. *If $(\mathscr{X},\mathscr{N},\diamond)$ is a FNLS and $T:\mathscr{X}\to\mathscr{X}$ is a linear operator, then $r_{fl}(T)\leqslant r_{fc}(T)\leqslant r_{nlfb}(T)$.*

*Proof.* We show that

$$\left\{\mu>0:\frac{T^n}{\mu^n}\xrightarrow{\text{feq}}0\right\}\subset\left\{\mu>0:(\forall)u\in\mathscr{X},(\forall)t>0,\lim_{n\to\infty}\mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right)=1\right\}.$$

Fix $\mu>0$, $\alpha\in(0,1)$, $u\in\mathscr{X}$, $t>0$. From $\frac{T^n}{\mu^n}\xrightarrow{\text{feq}}0$, it results that $(\exists)\beta_{\alpha,t}\in(0,1)$, $(\exists)s_{\alpha,t}>0$ such that $(\forall)\varepsilon>0$, $(\exists)n_0(\varepsilon,\alpha,t)\in\mathbb{N}:(\forall)n\geqslant n_0(\varepsilon,\alpha,t)$, $(\forall)v\in\mathscr{X}$ with $\mathscr{N}(v,s_{\alpha,t})>\beta_{\alpha,t}\ \Rightarrow\ \mathscr{N}\left(\frac{T^n}{\mu^n}v,\varepsilon t\right)>\alpha$. Using $(\mathscr{N}5)$ we deduce that $(\exists)s>0$ such that $\mathscr{N}(u,s)>\beta_{\alpha,t}$. From the above, for $\varepsilon=\frac{s_{\alpha,t}}{s}>0$ and $v=\frac{s_{\alpha,t}}{s}u$ we have $\mathscr{N}(v,s_{\alpha,t})=\mathscr{N}(u,s)>\beta_{\alpha,t}$, whence

$$\mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right)=\mathscr{N}\left(\frac{T^n}{\mu^n}\left(\frac{s}{s_{\alpha,t}}v\right),t\right)=\mathscr{N}\left(\frac{T^n}{\mu^n}v,\frac{s_{\alpha,t}}{s}t\right)=\mathscr{N}\left(\frac{T^n}{\mu^n}v,\varepsilon t\right)>\alpha,$$

that is $\lim\limits_{n\to\infty}\mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right)=1$.

Hence the inclusion is proved and this implies $r_{fl}(T)\leqslant r_{fc}(T)$. We prove that $\frac{T^n}{\mu^n}\xrightarrow{\text{fu}}0\ \Rightarrow\ \frac{T^n}{\mu^n}\xrightarrow{\text{feq}}0$.

Consider $\mu>0$ fixed, $\alpha\in(0,1)$, $t>0$, $\varepsilon>0$. Following the hypothesis, $(\exists)\alpha_0\in(0,1)$, $(\exists)t_0>0$ such that, for $s=\varepsilon t>0$, $(\exists)n_0(\varepsilon,\alpha,t)\in\mathbb{N}$ with the property $(\forall)n\geqslant n_0(\varepsilon,\alpha,t)$, $(\forall)u\in\mathscr{X}:\mathscr{N}(u,t_0)>\alpha_0\ \Rightarrow\ \mathscr{N}\left(\frac{T^n}{\mu^n}u,\varepsilon t\right)>\alpha$.

Consequently, $(\forall)\alpha\in(0,1)$, $(\forall)t>0$ $(\exists)\beta_{\alpha,t}=\alpha_0\in(0,1)$, $(\exists)s_{\alpha,t}=t_0>0$ such that $(\forall)\varepsilon>0$, $(\exists)n_0(\varepsilon,\alpha,t)\in\mathbb{N}$ with the property $(\forall)n\geqslant n_0(\varepsilon,\alpha,t)$, $(\forall)u\in\mathscr{X}$: $\mathscr{N}(u,s_{\alpha,t})>\beta_{\alpha,t}\ \Rightarrow\ \mathscr{N}\left(\frac{T^n}{\mu^n}u,\varepsilon t\right)>\alpha$. Therefore $r_{fc}(T)\leqslant r_{nlfb}(T)$. $\square$

The last inequality in Theorem 2 becomes equality in the case of a $NLFB(\mathscr{X})$ operator.

THEOREM 3. *If* $(\mathscr{X},\mathscr{N},\diamond)$ *is a FNLS and* $T\in NLFB(\mathscr{X})$*, then* $r_{fc}(T)=r_{nlfb}(T)$.

*Proof.* From Theorem 2 we know that $r_{fc}(T)\leqslant r_{nlfb}(T)$. We prove that $r_{nlfb}(T)\leqslant r_{fc}(T)$. Since $T\in NLFB(\mathscr{X})$ it results $(\exists)\alpha_0\in(0,1)$ $(\exists)t_0>0$ such that $(\forall)\alpha\in(0,1)$, $(\exists)s_\alpha>0$ with the property $(\forall)u\in\mathscr{X}$, having $\mathscr{N}(u,t_0)>\alpha_0 \Rightarrow \mathscr{N}(Tu,s_\alpha)>\alpha$ $(\star\star)$.

Let $\mu>r_{fc}(T)$. Then $\frac{T^n}{\mu^n}\xrightarrow{\text{feq}}0$, so $(\forall)\alpha\in(0,1)$, $(\forall)t>0$, $(\exists)\beta_{\alpha,t}\in(0,1)$, $(\exists)s_{\alpha,t}>0$ such that $(\forall)\varepsilon>0$, $(\exists)$ $n_0(\varepsilon,\alpha,t)\in\mathbb{N}$, $(\forall)$ $n\geqslant n_0(\varepsilon,\alpha,t)$, $(\forall)$ $v\in\mathscr{X}$ with $\mathscr{N}(v,s_{\alpha,t})>\beta_{\alpha,t} \Rightarrow \mathscr{N}\left(\frac{T^n}{\mu^n}v,\varepsilon t\right)>\alpha$. $(\star\star\star)$

Considering $\alpha\in(0,1)$, $t>0$ and taking $\beta_{\alpha,t}$ instead of $\alpha$ in $(\star\star)$ we get that $(\exists)$ $s_{\beta_{\alpha,t}}>0$ such that $(\forall)u\in\mathscr{X}$ with $\mathscr{N}(u,t_0)>\alpha_0 \Rightarrow \mathscr{N}(Tu,s_{\beta_{\alpha,t}})>\beta_{\alpha,t}$. For $\varepsilon=\frac{\mu\cdot s_{\alpha,t}}{s_{\beta_{\alpha,t}}}>0$ and $v=T\left(\frac{s_{\alpha,t}}{s_{\beta_{\alpha,t}}}u\right)$ in $(\star\star\star)$ we deduce that $(\exists)$ $\alpha_0\in(0,1)$, $(\exists)t_0>0$ such that $(\forall)$ $\alpha\in(0,1)$, $(\forall)t>0$, $(\exists)$ $n_0(\alpha,t)=n_0\left(\frac{\mu\cdot s_{\alpha,t}}{s_{\beta_{\alpha,t}}},\alpha,t\right)\in\mathbb{N}$ such that $(\forall)n\geqslant n_0(\alpha,t)$, $(\forall)u\in\mathscr{X}$ : $\mathscr{N}(u,t_0)>\alpha_0 \Rightarrow \mathscr{N}(Tu,s_{\beta_{\alpha,t}})=\mathscr{N}(v,s_{\alpha,t})>\beta_{\alpha,t}$ which implies $\mathscr{N}\left(\frac{T^{n+1}}{\mu^{n+1}}u,t\right)=\mathscr{N}\left(\frac{T^n}{\mu^n}v,\varepsilon t\right)>\alpha$. So $\frac{T^{n+1}}{\mu^{n+1}}\xrightarrow{\text{feq}}0$, which yields to $\mu\geqslant r_{nlfb}(T)$.

Therefore $r_{nlfb}(T)\leqslant r_{fc}(T)$. $\square$

The following definition is useful for computing $r_{fc}(T)$.

DEFINITION 7. Let $(\mathscr{X},\mathscr{N},\diamond)$ be a *FNLS* and $\mathscr{G}(\mathscr{X})$ a family of linear operators on $\mathscr{X}$. We say that the family $\mathscr{G}(\mathscr{X})$ is fuzzy equicontinuous if $(\forall)\alpha\in(0,1)$, $(\forall)t>0$, $(\exists)\beta_{\alpha,t}\in(0,1)$, $(\exists)s_{\alpha,t}>0$ such that $(\forall)u\in\mathscr{X}$ : $\mathscr{N}(u,s_{\alpha,t})>\beta_{\alpha,t} \Rightarrow \mathscr{N}(Su,t)>\alpha$, $(\forall)S\in\mathscr{G}(\mathscr{X})$.

Alternative calculi for fuzzy radii are given.

THEOREM 4. *If* $(\mathscr{X},\mathscr{N},\diamond)$ *is a FNLS and* $T:\mathscr{X}\to\mathscr{X}$ *is a linear operator, then:*

*i)* $r_{fl}=\inf\left\{\mu>0:(\forall)\alpha\in(0,1),(\forall)u\in\mathscr{X},(\exists)t_{\alpha,u}>0 \text{ such that } (\forall)n\in\mathbb{N}^*\Rightarrow \mathscr{N}\left(\frac{T^n}{\mu^n}u,t_{\alpha,u}\right)>\alpha\right\}$;

*ii) If* $T\in FC(\mathscr{X})$*, then* $r_{fc}(T)=\inf\left\{\mu>0:\ \left\{\frac{T^n}{\mu^n}\right\}_{n\in\mathbb{N}^*} \text{ is fuzzy equicontinuous}\right\}$;

*iii) If* $T \in NLFB(\mathscr{X})$, *then* $r_{nlfb}(T) = \inf\Big\{\mu > 0 : (\exists)\alpha_0 \in (0,1), (\exists)t_0 > 0$ *such that* $(\forall)\alpha \in (0,1), (\forall)t > 0, (\forall)u \in \mathscr{X} : \ \mathscr{N}(u,t_0) > \alpha_0, (\forall)n \in \mathbb{N}^* \Rightarrow \mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right) > \alpha\Big\}$.

*Proof.* *i*) Denote by $r'_{fl} = \inf\Big\{\mu > 0 : (\forall)\alpha \in (0,1), (\forall)u \in \mathscr{X}, (\exists)t_{\alpha,u} > 0$ such that $(\forall)n \in \mathbb{N}^* \Rightarrow \mathscr{N}\left(\frac{T^n}{\mu^n}u,t_{\alpha,u}\right) > \alpha\Big\}$. We show that $\Big\{\mu > 0 : (\forall)u \in \mathscr{X}, (\forall)t > 0 \Rightarrow \lim\limits_{n\to\infty} \mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right) = 1\Big\} \subset \Big\{\mu > 0 : (\forall)\alpha \in (0,1), (\forall)u \in \mathscr{X}, (\exists)t_{\alpha,u} > 0$ such that $(\forall)n \in \mathbb{N}^* \Rightarrow \mathscr{N}\left(\frac{T^n}{\mu^n}u,t_{\alpha,u}\right) > \alpha\Big\}$.

Fix $\mu > 0$ from the left side set of the previous inclusion relation. Let $\alpha \in (0,1)$, $u \in \mathscr{X}$. For $t = s > 0$ fixed, we deduce that $(\exists)n_1(\alpha,u) \in \mathbb{N}^* \setminus \{1\}$ such that $(\forall)n \geqslant n_1(\alpha,u)$ we have $\mathscr{N}\left(\frac{T^n}{\mu^n}u,s\right) > \alpha$. It is clear from $(\mathscr{N}5)$ that $(\exists)t_i(u) > 0$, $i \in \{1,2,\dots,n_1(\alpha,x)-1\}$ such that $\mathscr{N}\left(\frac{T^n}{\mu^n}u,t_i(u)\right) > \alpha$. Hence $(\exists)t_{\alpha,u} = \max\{s,t_1(u), t_2(u),\dots,t_{n_1(\alpha,u)-1}(u)\}$ with the property that $(\forall)n \in \mathbb{N}^* \Rightarrow \mathscr{N}\left(\frac{T^n}{\mu^n}u,t_{\alpha,u}\right) > \alpha$, which proves the proposed inclusion. So $r_{fl}(T) \geqslant r'_{fl}(T)$.

Conversely, suppose $\mu > r'_{fl}(T)$ fixed. Choose $\eta > 0$ with $\mu > \eta > r'_{fl}(T)$. Then, $(\forall)u \in \mathscr{X}$, $(\forall)\alpha \in (0,1)$, $(\exists)t_{\alpha,u} > 0$ such that $(\forall) \in \mathbb{N}^* \Rightarrow \mathscr{N}\left(\frac{T^n}{\eta^n}u,t_{\alpha,u}\right) > \alpha$. For all $t > 0$, we have $\mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right) = \mathscr{N}\left(\left(\frac{\eta}{\mu}\right)^n\frac{T^n}{\eta^n}u,t\right) = \mathscr{N}\left(\frac{T^n}{\eta^n}u,\left(\frac{\mu}{\eta}\right)^n t\right) > \alpha$, $(\forall)n \in \mathbb{N}^*$ with $\left(\frac{\mu}{\eta}\right)^n t > t_{\alpha,u}$, so $\lim\limits_{n\to\infty} \mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right) = 1$, $(\forall)u \in \mathscr{X}$, $(\forall)t > 0$. Thus $r'_{fl}(T) \geqslant r_{fl}(T)$. Therefore $r_{fl}(T) = r'_{fl}(T)$.

*ii*) Consider $T \in FC(\mathscr{X})$ and $r'_{fc}(T) = \inf\Big\{\mu > 0 : \ \left\{\frac{T^n}{\mu^n}\right\}_{n\in\mathbb{N}^*}$ is fuzzy equicontinuous$\Big\}$. If $\mu > 0$ is fixed such that $\frac{T^n}{\mu^n} \xrightarrow{\text{feq}} 0$, then $(\forall)\alpha \in (0,1)$, $(\forall)t > 0$, $(\exists)\beta_{\alpha,t} \in (0,1)$, $(\exists)s_{\alpha,t} > 0$ such that $(\forall)\varepsilon > 0$, $(\exists)n_0(\varepsilon,\alpha,t) \in \mathbb{N}^*$ such that $(\forall)n \geqslant n_0(\varepsilon,\alpha,t)$, $(\forall)u : \ \mathscr{N}(u,s_{\alpha,t}) > \beta_{\alpha,t} \ \Rightarrow \ \mathscr{N}\left(\frac{T^n}{\mu^n}u,\varepsilon t\right) > \alpha$. Thus, taking $\varepsilon = 1$, we deduce that $(\forall)\alpha \in (0,1)$, $(\forall)t > 0$, $(\exists)n_0(\alpha,t) \in \mathbb{N}^*$ such that $(\forall)n \geqslant n_0(\alpha,t)$, $(\forall)u \in \mathscr{X} : \ \mathscr{N}(u,s_{\alpha,t}) > \beta_{\alpha,t} \ \Rightarrow \ \mathscr{N}\left(\frac{T^n}{\mu^n}u,t\right) > \alpha$.

Since $T \in FC(\mathscr{X})$ we obtain that $(\forall)\alpha \in (0,1)$, $(\forall)t > 0$, $(\forall)n < n_0(\alpha,t)$, $(\exists)\beta_{\alpha,t,n} \in (0,1)$, $(\exists)s_{\alpha,t,n} > 0$ such that $(\forall)u \in \mathscr{X} : \ \mathscr{N}(u,s_{\alpha,t,n}) > \beta_{\alpha,t,n} \ \Rightarrow$

$\mathcal{N}\left(\frac{T^n}{\mu^n}u,t\right) > \alpha$.

So, $(\forall)\alpha \in (0,1)$, $(\forall)t > 0$, $(\exists)\gamma_{\alpha,t} = \max\{\beta_{\alpha,t}, \beta_{\alpha,t,1}, \dots, \beta_{\alpha,t,n_0(\alpha,t)-1}\} \in (0,1)$, $(\exists)$ $\delta_{\alpha,t} = \min\{s_{\alpha,t}, s_{\alpha,t,1}, \dots, s_{\alpha,t,n_0(\alpha,t)-1}\}$
$> 0$ such that $(\forall)n \in \mathbb{N}^*$ we have $(\forall)u \in \mathcal{X} : \mathcal{N}(u, \delta_{\alpha,t}) > \gamma_{\alpha,t} \Rightarrow \mathcal{N}\left(\frac{T^n}{\mu^n}u,t\right) > \alpha$, meaning $\left\{\frac{T^n}{\mu^n}\right\}_{n\in\mathbb{N}^*}$ is fuzzy equicontinuous . Hence $r_{fc}(T) \geqslant r'_{fc}(T)$.

Now, take $\mu > r'_{fc}(T)$. Then, $(\exists)\eta > 0$ such that $\mu > \eta > r'_{fc}(T)$. As $\left\{\frac{T^n}{\eta^n}\right\}_{n\in\mathbb{N}^*}$ is fuzzy equicontinuous, it results that $(\forall)\alpha \in (0,1)$, $(\forall)t > 0$, $(\exists)\beta_{\alpha,t} \in (0,1)$, $(\exists)s_{\alpha,t} > 0$, such that $(\forall)u \in \mathcal{X} : \mathcal{N}(u, s_{\alpha,t}) > \beta_{\alpha,t} \Rightarrow \mathcal{N}\left(\frac{T^n}{\eta^n}u,t\right) > \alpha$, $(\forall)n \in \mathbb{N}^*$.

So, $(\forall)\alpha \in (0,1)$, $(\forall)t > 0$, $(\exists)\beta_{\alpha,t} \in (0,1)$, $(\exists)s_{\alpha,t} > 0$, with the property $(\forall)\varepsilon > 0$, $(\exists)n_0(\varepsilon,\alpha,t) \in \mathbb{N}^*$ such that $\frac{\mu^{n_0}}{\eta^{n_0}} > \frac{1}{\varepsilon}$ and $(\forall)u \in \mathcal{X} : \mathcal{N}(u, s_{\alpha,t}) > \beta_{\alpha,t}$, $(\forall)n \geqslant n_0(\varepsilon,\alpha,t)$ we obtain $\mathcal{N}\left(\frac{T^n}{\mu^n}u, \varepsilon t\right) = \mathcal{N}\left(\frac{T^n}{\eta^n}u, \frac{\mu^n}{\eta^n}\varepsilon t\right) \geqslant \mathcal{N}\left(\frac{T^n}{\eta^n}u,t\right) > \alpha$ which proves that $\frac{T^n}{\mu^n} \xrightarrow{\text{feq}} 0$.

Therefore $r'_{fc}(T) \geqslant r_{fc}(T)$, hence $r_{fc}(T) = r'_{fc}(T)$.

*iii*) It can be proved similarly to *ii*). □

## 4. Volterra operator in fuzzy context

Being one of the most useful operator in a good number of fields, from ordinary differential equations to inverse problems of mathematical-physics, appearing in a wide range of setups (from abstract operator spaces to different types of function spaces suited for solving differential equations), the Volterra operator represents one of the most effective methods for solving one-dimensional inverse problems. It also helps dealing with transformation and factorization operator method. Over the years, since its first appearance, many specialists dedicated their time to find methods to characterize as much as possible this operator's properties (see [2, 14, 20]).

We consider in this section the classical Volterra operator acting on $L^2[0,1]$ endowed with fuzzy norm derived from the usual norm of $L^2[0,1]$, denoted by $||.||_2$. We calculate the spectral radii of this type of Volterra operator in this new framework who was described in detail in the previous sections.

Suppose $V : L^2[0,1] \to L^2[0,1]$ is defined by:

$$Vf(u) = \int_0^u f(s)ds$$

Recall the definition of the operator norm $||.||$ on $L^2[0,1]$:

$$||V|| = \sup_{||f||_2=1} ||Vf||_2$$

It's a known fact that operator norm of $V$ is $2/\pi$ and

$$\lim_{m\to\infty} ||m!V^m|| = 1/2 \qquad (1)$$

(see [17]). Now, if we consider $\mathscr{X} := L^2[0,1]$, and put

$$\mathscr{N}(f,s) = \begin{cases} \dfrac{s}{s+||f||_2}, & s>0, \\ 0, & s=0, \end{cases}$$

then $\mathscr{N}$ is a fuzzy norm on $L^2[0,1]$ making $(L^2[0,1],\mathscr{N},\diamond)$ a fuzzy normed space. We intend to study the fuzzy boundedness and the spectral radii of the fuzzy Volterra operator $F_V : (L^2[0,1],\mathscr{N},\diamond) \to (L^2[0,1],\mathscr{N},\diamond)$, $F_V f(u) = \int\limits_0^u f(s)ds$. Consider $\alpha_0 = \dfrac{1}{2}$ and $s_0 = 1$. It is easy to see that for $f \in \mathscr{X}$, the inequalities $\mathscr{N}(f,s_0) > \alpha_0$ and $||f||_2 < 1$ are equivalent. Thus, $(\forall)\alpha \in (0,1), (\exists)\ s_\alpha = \dfrac{2\alpha}{\pi(1-\alpha)} > 0$ such that $(\forall) f \in \mathscr{X}$ with $\mathscr{N}(f,s_0) > \alpha_0$, we have

$$\mathscr{N}(F_V f, s_\alpha) = \frac{s_\alpha}{s_\alpha + ||F_V f||_2} \geqslant \frac{s_\alpha}{s_\alpha + ||F_V||||f||_2} = \frac{s_\alpha}{s_\alpha + \frac{2}{\pi}||f||_2} > \frac{s_\alpha}{s_\alpha + \frac{2}{\pi}} = \alpha$$

Therefore $F_V \in NLFB(\mathscr{X})$.

According to Theorem 3, the equality $r_{fc}(F_V) = r_{nlfb}(F_V)$ holds. It is possible to prove that $r_{fl}(F_V) = r_{fc}(F_V) = r_{nlfb}(F_V)$.

Hence, for calculating the spectral radius of $F_V$, it is sufficient to work with the formula:

$$\begin{aligned} r_{fl}(F_V) &= \inf\{\mu > 0:\ (\forall) f \in L^2[0,1], (\forall) t > 0, \lim_{n\to\infty} \mathscr{N}\left(\frac{F_V^n}{\mu^n} f, t\right) = 1\} \\ &= \inf\left\{\mu > 0:\ (\forall) f \in L^2[0,1], (\forall) t > 0, \lim_{n\to\infty} \frac{t}{t + \frac{||F_V^n f||_2}{\mu^n}} = 1\right\} \\ &= \inf\left\{\mu > 0:\ (\forall) f \in L^2[0,1], \lim_{n\to\infty} \frac{||F_V^n f||_2}{\mu^n} = 0\right\}. \end{aligned}$$

Since $\dfrac{||F_V^n f||_2}{\mu^n} \leqslant \dfrac{||F_V^n||||f||_2}{\mu^n}$, we have:

$$\begin{aligned} r_{fl}(F_V) &\leqslant \inf\left\{\mu > 0:\ (\forall) f \in L^2[0,1], \lim_{n\to\infty} \frac{||F_V^n||||f||_2}{\mu^n} = 0\right\} \\ &= \inf\left\{\mu > 0:\ (\forall) f \in L^2[0,1], \lim_{n\to\infty} \frac{||n!F_V^n||||f||_2}{n!\mu^n} = 0\right\}. \end{aligned}$$

But, following (1), $\lim_{m\to\infty} ||n!F_V^n|| = 1/2$, whence the limit $\lim_{n\to\infty} \frac{||n!F_V^n|| ||f||_2}{n!\mu^n} = 0$ yields $\lim_{n\to\infty} \frac{||f||_2}{2n!\mu^n} = 0$ which is satisfied for all $\mu > 0$. So, $r_{fl}(F_V) \leqslant 0$ therefore $r_{fc}(F_V) = r_{nlfb}(F_V) = r_{fl}(F_V) = 0$.

*Acknowledgement.* This work was supported by a grant of the Romanian Ministry of Research, Innovation and Digitalization, project number PFE 26/30.12.2021, PERFORM-CDI@UPT100. The increasing of the performance of the Polytechnic University of Timişoara by strengthening the research, development and technological transfer capacity in the field of "Energy, Environment and Climate Change" at the beginning of the second century of its existence, within "Program 1 – Development of the national system of Research and Development, Subprogram 1.2 – Institutional Performance – Institutional Development Projects – Excellence Funding Projects in RDI, PNCDI III".

## REFERENCES

[1] A. ABDUL-SAMEE, K. AHLAM JAMEEL AND C. FARIA ALI, *Spectral Theory in Fuzzy Normed Spaces*, Journal of Al-Nahrain University, **14** (2) (2011), 178–185.
[2] R. P. AGARWAL, D. O'REGAN AND V. LAKSHMIKANTHAM, *Fuzzy Volterra Integral Equations: A Stacking Theorem Approach*, Applicable Analysis, **83** (5), (2004).
[3] N. F. AL-MAYAHI, A. M. ABBAS, *Some Properties of Spectral Theory in Fuzzy Hilbert Spaces*, Journal of Al-Qadisiyah for Computer Science and Mathematics, **8** (2), 1–7.
[4] G. R. ALLAN, *A spectral theory for locally convex algebras*, Proc. London Math. Soc. **15**, (1965), 399–421.
[5] H. ARIZMENDI, K. JAROSZ, *Extended spectral radius in topological algebras*, Rocky Mountain Journal of Mathematics, **3** (4), (1993), 1179–1195.
[6] T. BÎNZAR, *Almost strongly fuzzy bounded operators with applications to fuzzy spectral theory*, Fuzzy Sets and Systems, **444** (C), (2022), 131–143.
[7] T. BÎNZAR, F. PATER, S. NĂDĂBAN, *A study of boundedness in fuzzy normed linear spaces*, Symmetry-Basel, **11** (2019), 923.
[8] T. BÎNZAR, F. PATER, S. NĂDĂBAN, *Fuzzy bounded operators with application to Radon transform*, **141** (110359), 2020.
[9] F. G. BONALES, R. V. MENDOZA, *Extending the formula to calculate the spectral radius of an operator*, Proc. Amer. Math. Soc, **126** (1), (1998), 97–103.
[10] S. C. CHENG, J. L. MORDENSON, *Fuzzy linear operator and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., **86** (1994), 429–436.
[11] Y. J. CHO, T. RASSIAS, R. SAADATI, *Fuzzy Operator Theory in Mathematical Analysis*, Springer Science and Business Media LLC, (2018).
[12] H. R. DOWSON, *Spectral theory of linear operators*, Academic Press, (1978).
[13] A. GEORGE, P. VEERAMANI, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64** (3), (1994) 395–399.
[14] I. C. GOHBERG, M. G. KREIN, *Theory and Applications of Volterra Operators in Hilbert Space*, Translation of Mathematical Monographs, **24**, (1970).
[15] O. HADŽIĆ, E. PAP, *Fixed Point Theory in Probabilistic Metric Spaces*, vol. 536 of Mathematics and Its Applications, Kluwer Academic, Dordrecht, (2001).
[16] A. K. KATSARAS, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems, **12** (1984), 143–154.
[17] D. KERSHAW, *Operator norms of powers of the Volterra operator*, Journal of Integral Equations and Applications, **11** (3) (1999), 351–362.
[18] E. KRAMAR, *On numerical ranges of operators on locally convex spaces*, Journal of the Australian Mathematical Society **20** (04) 1975, 468–482.

[19] S. LAHRECH, W. JADDAR, *Banach-Steinhaus type theorems in locally convex spaces for bounded convex processes*, Int. Journal of Math. Analysis, **1** (9), 2007, 437–441.

[20] N. LAO, R. WHITLEY, *Norms of powers of the Volterra operator*, Integral Equations Operator Theory **27** (1997), 419–425.

[21] C. MARTÍNEZ, M. SANZ, *Spectral mapping theorem for fractional powers in locally convex spaces*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, **24** (4), (1997), 685–702.

[22] R. T. MOORE, *Adjoints, numerical ranges and spectra of operators on locally convex spaces*, Bull. Am. Math. Soc. **75** (1969), 85–90.

[23] S. NĂDĂBAN, *Fuzzy continuous mappings in fuzzy normed linear spaces*, Int. J. of Computers Communications & Control, **10** (6), (2015), 834–842.

[24] S. NĂDĂBAN, I. DZITAC, *Atomic Decompositions of Fuzzy Normed Linear Spaces for Wavelet Applications*, Informatica, **25** (4) (2014), 643–662.

[25] R. I. OPREA, F. PATER, A. JURATONI AND O. BUNDĂU, *An introduction to spectral theory in fuzzy normed linear spaces*, AIP Conference Proceedings **2293**, 350008, (2020), https://doi.org/10.1063/5.0026609.

[26] R. SAADATI, S. M. VAEZPOUR, *Some results on fuzzy Banach spaces*, J. Appl. Math. Comput. **17** (1–2) (2005), 475–484.

[27] B. SCHWEIZER, A. SKLAR, *Statistical metric spaces*, Pacific J. Math., **10** (1960), 314–334.

[28] S. M. STOIAN, *Spectral radius of quotient bounded operators*, Studia Univ., Babes-Bolyai, Mathematica, **XLIX** 4, (2004), 115–126.

[29] V. G. TROITSKY, *Spectral radii of bounded operators on topological vector spaces*, Panamerican Math. J., **11** (3), (2001), 1–35.

[30] L. WAELBROECK, *Étude spectrale des algébres complétes*, Acad.Roy. Belg. Cl. Sci. Mem. Coll., **8** (2) 7, (1960).

[31] V. WROBEL, *Spectral approximation theorems in locally convex spaces*, Pacific Journal of Mathematics, **64** (1), (1976), 273–281.

(Received January 30, 2023)

*Tudor Bînzar*
*Politehnica University of Timişoara*
*Department of Mathematics*
*Regina Maria 1, RO-300004 Timişoara, Romania*
*e-mail:* tudor.binzar@upt.ro

*Flavius Pater*
*Politehnica University of Timişoara*
*Department of Mathematics*
*Regina Maria 1, RO-300004 Timişoara, Romania*
*e-mail:* flavius.pater@upt.ro