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SOME IMPROVEMENTS ABOUT NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. In this paper, we give several numerical radius inequalities for Hilbert space operators by using the Young-type inequalities and the generalization of the Buzano of the Schwarz inequality. These inequalities improve the classical numerical radius inequalities. We also give the upper bound of the numerical radius of 2×2 operator matrices.

1. Introduction

Let $\mathscr{H}$ denote a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the norm induced by the inner product $\langle\cdot,\cdot\rangle$. Let $\mathbb{B}(\mathscr{H})$ denote the collection of all bounded linear operators on $\mathscr{H}$. For $T\in\mathbb{B}(\mathscr{H})$, let $\omega(T)$ and $\|T\|$ denote the numerical radius and usual operator norm of T, respectively. Recall that

$$\omega(T)=\sup\{|\langle Tx,x\rangle| : x\in\mathscr{H}, \|x\|=1\},$$

and

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : x\in\mathscr{H}, \|x\|=1\}\\ &= \sup\{|\langle Tx,y\rangle| : x,y\in\mathscr{H}, \|x\|=\|y\|=1\}.\end{aligned}$$

It is clear that $\omega(\cdot)$ defines an operator norm on $\mathbb{B}(\mathscr{H})$ which is equivalent to the operator norm $\|\cdot\|$, where we have

$$\frac{1}{2}\|T\|\leqslant\omega(T)\leqslant\|T\|, \tag{1.1}$$

for $T\in\mathbb{B}(\mathscr{H})$. The inequalities in (1.1) are sharp. The first inequality becomes an equality if $T^2=0$ and the second inequality becomes an equality if T is normal, i.e. $T^*T=TT^*$. Denote $|T|=(T^*T)^{1/2}$ be the absolute value of $T\in\mathbb{B}(\mathscr{H})$. Then we have

$$\omega(|T|)=\||T|\|=\|T\|.$$

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For $\omega(T)$, an important inequality is the power inequality, which asserts that

$$\omega(T^n) \leqslant \omega^n(T), \tag{1.2}$$

for $n = 1, 2, \cdots$.

In 2003, Kittaneh [14] improved the second inequality in (1.1), and obtained the following result:

$$\omega(T) \leqslant \frac{1}{2}\||T| + |T^*|\| \leqslant \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right). \tag{1.3}$$

In 2005, the same author in [15] proved that if $T \in \mathbb{B}(\mathscr{H})$,

$$\frac{1}{4}\|T^*T + TT^*\| \leqslant \omega^2(T) \leqslant \frac{1}{2}\|T^*T + TT^*\|. \tag{1.4}$$

The left side of the inequality (1.4) has been improved in [19]:

$$\frac{1}{4}\left\||T|^2 + |T^*|^2\right\| \leqslant \frac{1}{2}\sqrt{2\omega^4(T) + \frac{1}{8}\left\|(T+T^*)^2(T-T^*)^2\right\|} \leqslant \omega^2(T).$$

The right side of the inequality (1.4) is a special case of the following more general form [9]:

$$\omega^{2r}(T) \leqslant \frac{1}{2}\left\||T|^{2r} + |T^*|^{2r}\right\|, \tag{1.5}$$

for any $r \geqslant 1$. In 2023, Sheybani et al. [25] presents the general form of the right side of the inequality (1.4):

$$\omega^2(T) \leqslant \min_{0<t<1}\left\|(1-t)|T|^{\frac{1}{1-t}} + t|T^*|^{\frac{1}{t}}\right\|,$$

and

$$\omega^2(T) \leqslant \min_{0\leqslant t\leqslant 1}\left\|\frac{|T|^{4(1-t)} + |T^*|^{4t}}{4} + \frac{(1-t)|T|^2 + t|T^*|^2}{2}\right\|.$$

In [24], Sattari, Moslehian and Yamazaki proved that

$$\omega^{2r}(T) \leqslant \frac{1}{2}\|T\|^{2r} + \frac{1}{4}\left\||T|^{4\alpha r} + |T^*|^{4(1-\alpha)r}\right\| \tag{1.6}$$

for all $0 \leqslant \alpha \leqslant 1$, $r \geqslant 1$. Taking $\alpha = \frac{1}{2}$ in inequality (1.6), we get

$$\omega^{2r}(T) \leqslant \frac{1}{2}\|T\|^{2r} + \frac{1}{4}\left\||T|^{2r} + |T^*|^{2r}\right\|. \tag{1.7}$$

In [11], Heydarbeygi, Sababheh and Moradi proved that

$$\omega^{2r}(T) \leqslant \frac{1}{2}\omega^r(|T||T^*|) + \frac{1}{4}\left\||T|^{2r} + |T^*|^{2r}\right\|. \tag{1.8}$$

Obviously, the inequality (1.8) is better than the inequality (1.7). For more such inequalities, see [8, 17, 21–27] and references therein.

The direct sum of two copies of $\mathscr{H}$ is denoted by $\mathscr{H}^2 = \mathscr{H} \bigoplus \mathscr{H}$. If $A,B,C,D \in \mathbb{B}(\mathscr{H})$, then the operator matrix $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator in $\mathbb{B}(\mathscr{H} \oplus \mathscr{H})$, which is defined by $Tx = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}$ for every vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathscr{H} \oplus \mathscr{H}$.

2. Preliminaries

In this section, we give the following lemmas in order to present our results. These lemmas can be found in the stated references. The first lemma is a simple consequence of the classical Jensen and Young inequalities (see [10, 24]).

LEMMA 2.1. *Let $a,b \geqslant 0$, $0 \leqslant \alpha \leqslant 1$, $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $r \geqslant 1$,*

(i) $a^{\alpha} b^{1-\alpha} \leqslant \alpha a + (1-\alpha) b \leqslant (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}$;

(ii) $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q} \leqslant \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q} \right)^{\frac{1}{r}}$.

In 2010, Kittaneh and Manasrah [18] obtained the following inequality, which is a refinement of Lemma 2.1:

$$\alpha a + (1-\alpha) b \geqslant a^{\alpha} b^{1-\alpha} + r_0 (\sqrt{a} - \sqrt{b})^2, \tag{2.1}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$. In 2015, a more general extension of the power form of (2.1) established in [2] is

$$(\alpha a + (1-\alpha) b)^m \geqslant (a^{\alpha} b^{1-\alpha})^m + r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}} \right)^2, \tag{2.2}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$, m is a positive integer. By using the limit property and the monotony of the sequence $\alpha_m = \left(\frac{a^m - c^m}{b^m - d^m} \right)^{\frac{1}{m}}$ with positive real numbers a,b,c,d satisfying $a > c$ and $b > d$, Choi [7] in 2018 proved that

$$(\alpha a + (1-\alpha) b)^m \geqslant (a^{\alpha} b^{1-\alpha})^m + (2r_0)^m \left[\left(\frac{a+b}{2} \right)^m - \left(\sqrt{ab} \right)^m \right], \tag{2.3}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$, m is a positive integer. Recently, by using the theory of weak sub-majorization, the authors in [13] extended (2.3) to an arbitrary real power $r \geqslant 1$ of the form:

$$(\alpha a + (1-\alpha) b)^r \geqslant (a^{\alpha} b^{1-\alpha})^r + (2r_0)^r \left[\left(\frac{a+b}{2} \right)^r - \left(\sqrt{ab} \right)^r \right], \tag{2.4}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$. Inequality (2.4) can also be written

$$(ab)^r \leqslant \left(\frac{a^p}{p} + \frac{b^q}{q} \right)^r - \left(2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \right)^r \left[\left(\frac{a^p + b^q}{2} \right)^r - \left(a^{\frac{p}{2}} b^{\frac{q}{2}} \right)^r \right], \tag{2.5}$$

where $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$.

The second lemma can be found in [16].

LEMMA 2.2. *Let $T\in\mathbb{B}(\mathscr{H})$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then*

$$|\langle Tx,y\rangle|^2\leqslant\left\langle f^2(|T|)x,x\right\rangle\left\langle g^2(|T^*|)y,y\right\rangle,$$

for all $x,y\in\mathscr{H}$.

In particular, when $f(t)=g(t)=t^{\frac{1}{2}}$, we can get

$$|\langle Tx,y\rangle|^2\leqslant\langle|T|x,x\rangle\langle|T^*|y,y\rangle,$$

for all $x,y\in\mathscr{H}$.

The third lemma follows from the spectral theorem for positive operators and Jensen's inequality (see [16]).

LEMMA 2.3. *Let $T\in\mathbb{B}(\mathscr{H})$ be positive. Then*

$$\langle Tx,x\rangle^r\leqslant\langle T^rx,x\rangle$$

for all $r\geqslant1$ and for all $x\in\mathscr{H}$ with $\|x\|=1$.

If $0\leqslant r\leqslant1$,

$$\langle T^rx,x\rangle\leqslant\langle Tx,x\rangle^r.$$

Using the same strategy as in [20, Lemma 2.2], we get the fourth lemma, which depends on the Buzano extension of Schwarz inequality.

LEMMA 2.4. *Let $x,y,e\in\mathscr{H}$ with $\|e\|=1$, $0\leqslant\alpha\leqslant1$, $r\geqslant1$. Then*

$$|\langle x,e\rangle\langle e,y\rangle|^{2r}\leqslant\frac{1+\alpha}{4}\|x\|^{2r}\|y\|^{2r}+\frac{1-\alpha}{4}|\langle x,y\rangle|^{2r}+\frac{1}{2}|\langle x,y\rangle|^r\|x\|^r\|y\|^r.$$

Proof. In [6], Buzano shows the following extension of the Schwarz inequality:

$$|\langle x,e\rangle\langle e,y\rangle|\leqslant\frac{\|x\|\|y\|+|\langle x,y\rangle|}{2},$$

where $x,y,e\in\mathscr{H}$ with $\|e\|=1$.

By using the convexity of the function $f(t)=t^r$, it can be obtained

$$|\langle x,e\rangle\langle e,y\rangle|^r\leqslant\left(\frac{\|x\|\|y\|+|\langle x,y\rangle|}{2}\right)^r\leqslant\frac{\|x\|^r\|y\|^r+|\langle x,y\rangle|^r}{2}.$$

Thus, we have

$$\begin{aligned}
&|\langle x,e\rangle\langle e,y\rangle|^{2r}\\
&\leqslant \left(\frac{\|x\|^r\|y\|^r+|\langle x,y\rangle|^r}{2}\right)^2\\
&= \frac{1}{4}\left(\|x\|^{2r}\|y\|^{2r}+2|\langle x,y\rangle|^r\|x\|^r\|y\|^r+|\langle x,y\rangle|^{2r}\right)\\
&= \frac{1}{4}\left(\|x\|^{2r}\|y\|^{2r}+2|\langle x,y\rangle|^r\|x\|^r\|y\|^r+\alpha|\langle x,y\rangle|^{2r}+(1-\alpha)|\langle x,y\rangle|^{2r}\right)\\
&\leqslant \frac{1}{4}\left(\|x\|^{2r}\|y\|^{2r}+2|\langle x,y\rangle|^r\|x\|^r\|y\|^r+\alpha\|x\|^{2r}\|y\|^{2r}+(1-\alpha)|\langle x,y\rangle|^{2r}\right)\\
&= \frac{1+\alpha}{4}\|x\|^{2r}\|y\|^{2r}+\frac{1-\alpha}{4}|\langle x,y\rangle|^{2r}+\frac{1}{2}|\langle x,y\rangle|^r\|x\|^r\|y\|^r. \quad \square
\end{aligned}$$

The fifth lemma is well known and it was obtained in [5,12].

LEMMA 2.5. *Let $A,B\in\mathbb{B}(\mathscr{H})$. Then*

(i) $\omega\left(\begin{bmatrix}A & O\\ O & B\end{bmatrix}\right)=\max\{\omega(A),\omega(B)\}$;

(ii) $\omega\left(\begin{bmatrix}A & B\\ B & A\end{bmatrix}\right)=\max\{\omega(A+B),\omega(A-B)\}$.

In particular, $\omega\left(\begin{bmatrix}O & B\\ B & O\end{bmatrix}\right)=\omega(B)$.

The last needed lemma, which can be found in [3], gives a norm inequality involving convex functions of positive operators.

LEMMA 2.6. *Let f be a non-negative, convex function on $[0,\infty)$, and let $A,B\in\mathbb{B}(\mathscr{H})$ be positive operators. Then*

$$\left\|f\left(\frac{A+B}{2}\right)\right\|\leqslant\left\|\frac{f(A)+f(B)}{2}\right\|.$$

Particularly, for $r\geqslant 1$, it holds

$$\left\|\left(\frac{A+B}{2}\right)^r\right\|\leqslant\left\|\frac{A^r+B^r}{2}\right\|.$$

3. Some improvements of numerical radius inequalities

The main goal of this section is to derive several upper bounds for numerical radius which are improvements of some existing ones.

THEOREM 3.1. *Let $T \in \mathbb{B}(\mathscr{H})$, $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, and let f and g are non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then for $r\geqslant 1$,*

$$\omega^{2r}(T)\leqslant\left\|\frac{1}{2p}\left(f^{2pr}(|T|)+g^{2pr}(|T^*|)\right)+\frac{1}{2q}\left(f^{2qr}(|T^*|)+g^{2qr}(|T|)\right)\right\|-\inf_{\|x\|=1}\eta(x),$$

where

$$\eta(x)=\left(2\min\left\{\frac{1}{p},\frac{1}{q}\right\}\right)^r\left[\left(\frac{|\langle Tx,x\rangle|^p+|\langle T^*x,x\rangle|^q}{2}\right)^r-\left(|\langle Tx,x\rangle|^{\frac{p}{2}}|\langle T^*x,x\rangle|^{\frac{q}{2}}\right)^r\right].$$

Proof. Let $x\in\mathscr{H}$ with $\|x\|=1$. Then

$$\begin{aligned}
&|\langle Tx,x\rangle|^{2r}\\
&=|\langle Tx,x\rangle\langle x,Tx\rangle|^r\\
&=|\langle Tx,x\rangle\langle T^*x,x\rangle|^r\\
&\leqslant\left(\frac{1}{p}|\langle Tx,x\rangle|^p+\frac{1}{q}|\langle T^*x,x\rangle|^q\right)^r-\eta(x)\\
&\quad\text{(by inequality (2.5))}\\
&\leqslant\frac{1}{p}|\langle Tx,x\rangle|^{pr}+\frac{1}{q}|\langle T^*x,x\rangle|^{qr}-\eta(x)\\
&\quad\text{(by Lemma 2.1(ii))}\\
&\leqslant\frac{1}{p}\left\langle f^2(|T|)x,x\right\rangle^{\frac{pr}{2}}\left\langle g^2(|T^*|)x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle f^2(|T^*|)x,x\right\rangle^{\frac{qr}{2}}\left\langle g^2(|T|)x,x\right\rangle^{\frac{qr}{2}}-\eta(x)\\
&\quad\text{(by Lemma 2.2)}\\
&\leqslant\frac{1}{2p}\left(\left\langle f^2(|T|)x,x\right\rangle^{pr}+\left\langle g^2(|T^*|)x,x\right\rangle^{pr}\right)\\
&\quad+\frac{1}{2q}\left(\left\langle f^2(|T^*|)x,x\right\rangle^{qr}+\left\langle g^2(|T|)x,x\right\rangle^{qr}\right)-\eta(x)\\
&\quad\text{(by the arithmetic-geometic mean inequality)}\\
&\leqslant\frac{1}{2p}\left(\left\langle f^{2pr}(|T|)x,x\right\rangle+\left\langle g^{2pr}(|T^*|)x,x\right\rangle\right)\\
&\quad+\frac{1}{2q}\left(\left\langle f^{2qr}(|T^*|)x,x\right\rangle+\left\langle g^{2qr}(|T|)x,x\right\rangle\right)-\eta(x)\\
&\quad\text{(by Lemma 2.3)}\\
&=\left\langle\left[\frac{1}{2p}\left(f^{2pr}(|T|)+g^{2pr}(|T^*|)\right)+\frac{1}{2q}\left(f^{2qr}(|T^*|)+g^{2qr}(|T|)\right)\right]x,x\right\rangle-\eta(x).
\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$\omega^{2r}(T) \leqslant \left\| \frac{1}{2p} \left(f^{2pr}(|T|) + g^{2pr}(|T^*|) \right) + \frac{1}{2q} \left(f^{2qr}(|T^*|) + g^{2qr}(|T|) \right) \right\| - \inf_{\|x\|=1} \eta(x). \quad \square$$

Considering $f(t) = g(t) = t^{\frac{1}{2}}$, $p = q = 2$ in Theorem 3.1, we get the following corollary.

COROLLARY 3.2. *Let $T \in \mathbb{B}(\mathscr{H})$, $r \geqslant 1$. Then*

$$\omega^{2r}(T) \leqslant \frac{1}{2} \left\| |T|^{2r} + |T^*|^{2r} \right\| - \inf_{\|x\|=1} \zeta(x),$$

where

$$\zeta(x) = \left[\left(\frac{|\langle Tx, x \rangle|^2 + |\langle T^*x, x \rangle|^2}{2} \right)^r - (|\langle Tx, x \rangle| |\langle T^*x, x \rangle|)^r \right].$$

REMARK 3.3. Indeed, since $\zeta(x) \geqslant 0$, then

$$\begin{aligned} \omega^{2r}(T) &\leqslant \frac{1}{2} \left\| |T|^{2r} + |T^*|^{2r} \right\| - \inf_{\|x\|=1} \zeta(x) \\ &\leqslant \frac{1}{2} \left\| |T|^{2r} + |T^*|^{2r} \right\|. \end{aligned}$$

Therefore, Corollary 3.2 improves the inequality (1.5).

THEOREM 3.4. *Let $T \in \mathbb{B}(\mathscr{H})$, $r \geqslant 2$, $0 \leqslant \alpha \leqslant 1$. Then*

$$\begin{aligned} \omega^{2r}(T) \leqslant{}& \frac{1+\alpha}{4} \left\| \alpha |T|^{\frac{2}{\alpha}} + (1-\alpha)|T^*|^{\frac{2}{1-\alpha}} \right\|^{\frac{r}{2}} + \frac{1-\alpha}{4} \omega^r(T^2) \\ &+ \frac{1}{4} \omega^{\frac{r}{2}}(T^2) \, \| |T|^r + |T^*|^r \|. \end{aligned}$$

Proof. Let $x \in \mathscr{H}$ with $\|x\| = 1$. Then

$$\begin{aligned} & |\langle Tx, x \rangle|^{2r} \\ &= |\langle Tx, x \rangle \langle x, T^*x \rangle|^r \\ &\leqslant \frac{1+\alpha}{4} \|Tx\|^r \|T^*x\|^r + \frac{1-\alpha}{4} |\langle Tx, T^*x \rangle|^r + \frac{1}{2} |\langle Tx, T^*x \rangle|^{\frac{r}{2}} \|Tx\|^{\frac{r}{2}} \|T^*x\|^{\frac{r}{2}}. \\ &\quad \text{(by Lemma 2.4)} \end{aligned}$$

Since

$$\begin{aligned}
&\frac{1+\alpha}{4}\|Tx\|^{r}\|T^{*}x\|^{r}\\
=&\frac{1+\alpha}{4}\left\langle |T|^{2}x,x\right\rangle^{\frac{r}{2}}\left\langle |T^{*}|^{2}x,x\right\rangle^{\frac{r}{2}}\\
=&\frac{1+\alpha}{4}\left\langle \left(|T|^{\frac{2}{\alpha}}\right)^{\alpha}x,x\right\rangle^{\frac{r}{2}}\left\langle \left(|T^{*}|^{\frac{2}{1-\alpha}}\right)^{1-\alpha}x,x\right\rangle^{\frac{r}{2}}\\
\leqslant&\frac{1+\alpha}{4}\left(\left\langle |T|^{\frac{2}{\alpha}}x,x\right\rangle^{\alpha}\left\langle |T^{*}|^{\frac{2}{1-\alpha}}x,x\right\rangle^{1-\alpha}\right)^{\frac{r}{2}}\\
&\text{(by Lemma 2.3)}\\
\leqslant&\frac{1+\alpha}{4}\left(\alpha\left\langle |T|^{\frac{2}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle |T^{*}|^{\frac{2}{1-\alpha}}x,x\right\rangle\right)^{\frac{r}{2}},\\
&\text{(by the Young inequality)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\left|\langle Tx,T^{*}x\rangle\right|^{\frac{r}{2}}\|Tx\|^{\frac{r}{2}}\|T^{*}x\|^{\frac{r}{2}}\\
\leqslant&\frac{1}{4}\left|\left\langle T^{2}x,x\right\rangle\right|^{\frac{r}{2}}\left(\|Tx\|^{r}+\|T^{*}x\|^{r}\right)\\
&\text{(by the arithmetic-geometic mean inequality)}\\
=&\frac{1}{4}\left|\left\langle T^{2}x,x\right\rangle\right|^{\frac{r}{2}}\left(\left\langle |T|^{2}x,x\right\rangle^{\frac{r}{2}}+\left\langle |T^{*}|^{2}x,x\right\rangle^{\frac{r}{2}}\right)\\
\leqslant&\frac{1}{4}\left|\left\langle T^{2}x,x\right\rangle\right|^{\frac{r}{2}}\left(\langle |T|^{r}x,x\rangle+\langle |T^{*}|^{r}x,x\rangle\right)\\
&\text{(by Lemma 2.3)}\\
=&\frac{1}{4}\left|\left\langle T^{2}x,x\right\rangle\right|^{\frac{r}{2}}\left\langle (|T|^{r}+|T^{*}|^{r})x,x\right\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\langle Tx,x\rangle\right|^{2r}\\
\leqslant&\frac{1+\alpha}{4}\left(\alpha\left\langle |T|^{\frac{2}{\alpha}}x,x\right\rangle+(1-\alpha)\left\langle |T^{*}|^{\frac{2}{1-\alpha}}x,x\right\rangle\right)^{\frac{r}{2}}\\
&+\frac{1-\alpha}{4}\left|\left\langle T^{2}x,x\right\rangle\right|^{r}+\frac{1}{4}\left|\left\langle T^{2}x,x\right\rangle\right|^{\frac{r}{2}}\left\langle (|T|^{r}+|T^{*}|^{r})x,x\right\rangle.
\end{aligned}$$

Taking the supremum over $x\in\mathscr{H}$ with $\|x\|=1$, we get

$$\begin{aligned}
\omega^{2r}(T)\leqslant&\frac{1+\alpha}{4}\left\|\alpha|T|^{\frac{2}{\alpha}}+(1-\alpha)|T^{*}|^{\frac{2}{1-\alpha}}\right\|^{\frac{r}{2}}+\frac{1-\alpha}{4}\omega^{r}(T^{2})\\
&+\frac{1}{4}\omega^{\frac{r}{2}}(T^{2})\,\||T|^{r}+|T^{*}|^{r}\|.\quad\square
\end{aligned}$$

Considering $\alpha=\frac{1}{2}$, $r=2$ in Theorem 3.4, we get the following corollary.

COROLLARY 3.5. *Let $T \in \mathbb{B}(\mathscr{H})$. Then*

$$\omega^4(T) \leqslant \frac{3}{16}\left\||T|^4+|T^*|^4\right\| + \frac{1}{8}\omega^2(T^2) + \frac{1}{4}\omega(T^2)\left\||T|^2+|T^*|^2\right\|.$$

REMARK 3.6. Corollary 3.5 improves the inequality (1.5). We remark that

$$\begin{aligned}
\omega^4(T) &\leqslant \frac{3}{16}\left\||T|^4+|T^*|^4\right\| + \frac{1}{8}\omega^2(T^2) + \frac{1}{4}\omega(T^2)\left\||T|^2+|T^*|^2\right\| \\
&\leqslant \frac{3}{16}\left\||T|^4+|T^*|^4\right\| + \frac{1}{8}\omega^4(T) + \frac{1}{4}\omega^2(T)\left\||T|^2+|T^*|^2\right\| \\
&\leqslant \frac{3}{16}\left\||T|^4+|T^*|^4\right\| + \frac{1}{32}\left\||T|^2+|T^*|^2\right\|^2 + \frac{1}{8}\left\||T|^2+|T^*|^2\right\|^2 \\
&= \frac{3}{16}\left\||T|^4+|T^*|^4\right\| + \frac{5}{32}\left\||T|^2+|T^*|^2\right\|^2 \\
&\leqslant \frac{3}{16}\left\||T|^4+|T^*|^4\right\| + \frac{5}{16}\left\||T|^4+|T^*|^4\right\| \\
&= \frac{1}{2}\left\||T|^4+|T^*|^4\right\|,
\end{aligned}$$

which follows from inequality (1.2), inequality (1.4) and the fact that $\frac{1}{2}\left\||T|^2+|T^*|^2\right\|^2 \leqslant \left\||T|^4+|T^*|^4\right\|$.

Recall that if $T \in M_2(\mathbb{R})$, then $\|T\| = \max_{1\leqslant i\leqslant n}\sigma_i$, where $\sigma_i's$ are the square roots of the eigenvalues of T^*T, which are called the singular values of T, and the quantity $\omega(T)$, for a matrix of the form

$$T = \begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix} \quad \text{or} \quad T = \begin{bmatrix} a_1 & 0 \\ b & a_2 \end{bmatrix},$$

is defined by

$$\omega(T) = \frac{1}{2}|a_1+a_2| + \frac{1}{2}\sqrt{|a_1-a_2|^2+|b|^2},$$

where $a_1, a_2, b \in \mathbb{R}$.

REMARK 3.7. In [20, Theorem 2.1], Omidvar and Moradi proved that

$$\omega^4(T) \leqslant \frac{3}{8}\left\||T|^4+|T^*|^4\right\| + \frac{1}{8}\omega(T^2)\left\||T|^2+|T^*|^2\right\|. \tag{3.1}$$

We note that Corollary 3.5 is an improvement of inequality (3.1) in some cases. To see this, let $T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, then $|T|^2 = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$, $|T^*|^2 = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$, $|T|^4 = \begin{bmatrix} 2 & 6 \\ 6 & 26 \end{bmatrix}$, $|T^*|^4 = \begin{bmatrix} 8 & 12 \\ 12 & 20 \end{bmatrix}$, and $T^2 = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$. So, after simple calculations, we get

$$\omega(T) \approx 2.2071,\ \omega(T^2) \approx 4.6213,$$

$$\left\||T|^2+|T^*|^2\right\| \approx 10.2426,\ \left\||T|^4+|T^*|^4\right\| \approx 53.4558.$$

Substituting the above results into Corollary 3.5 and inequality (3.1), we have

$$\omega^4(T) \approx 23.7295,$$

$$\begin{aligned}
&\frac{3}{16}\left\||T|^4+|T^*|^4\right\|+\frac{1}{8}\omega^2(T^2)+\frac{1}{4}\omega(T^2)\left\||T|^2+|T^*|^2\right\| \\
\approx\ &\frac{3}{16}\times 53.4558+\frac{1}{8}\times 4.6213^2+\frac{1}{4}\times 4.6213\times 10.2426 \\
\approx\ &24.5261,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{3}{8}\left\||T|^4+|T^*|^4\right\|+\frac{1}{8}\omega(T^2)\left\||T|^2+|T^*|^2\right\| \\
\approx\ &\frac{3}{8}\times 53.4558+\frac{1}{8}\times 4.6213\times 10.2426 \\
\approx\ &25.9627.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\omega^4(T) &< \frac{3}{16}\left\||T|^4+|T^*|^4\right\|+\frac{1}{8}\omega^2(T^2)+\frac{1}{4}\omega(T^2)\left\||T|^2+|T^*|^2\right\| \\
&< \frac{3}{8}\left\||T|^4+|T^*|^4\right\|+\frac{1}{8}\omega(T^2)\left\||T|^2+|T^*|^2\right\|.
\end{aligned}$$

THEOREM 3.8. *Let $T\in\mathbb{B}(\mathscr{H})$, $r\geqslant 2$, $0\leqslant\alpha\leqslant 1$. Then*

$$\begin{aligned}
\omega^{2r}(T) \leqslant\ &\frac{1+\alpha}{8}\left\|f^{2r}(|T|^2)+f^{2r}(|T^*|^2)\right\|^{\frac{1}{2}}\left\|g^{2r}(|T|^2)+g^{2r}(|T^*|^2)\right\|^{\frac{1}{2}} \\
&+\frac{1-\alpha}{4}\omega^r(T^2)+\frac{1}{4}\omega^{\frac{r}{2}}(T^2)\left\||T|^r+|T^*|^r\right\|.
\end{aligned}$$

Proof. Let $x\in\mathscr{H}$ with $\|x\|=1$, we have

$$\begin{aligned}
&|\langle Tx,x\rangle|^{2r} \\
=\ &|\langle Tx,x\rangle\langle x,T^*x\rangle|^r \\
\leqslant\ &\frac{1+\alpha}{4}\|Tx\|^r\|T^*x\|^r+\frac{1-\alpha}{4}|\langle Tx,T^*x\rangle|^r+\frac{1}{2}|\langle Tx,T^*x\rangle|^{\frac{r}{2}}\|Tx\|^{\frac{r}{2}}\|T^*x\|^{\frac{r}{2}} \\
&\text{(by Lemma 2.4)} \\
\leqslant\ &\frac{1+\alpha}{8}\left(\|Tx\|^{2r}+\|T^*x\|^{2r}\right)+\frac{1-\alpha}{4}\left|\langle T^2x,x\rangle\right|^r \\
&+\frac{1}{4}\left|\langle T^2x,x\rangle\right|^{\frac{r}{2}}\left(\|Tx\|^r+\|T^*x\|^r\right) \\
&\text{(by the arithmetic-geometic mean inequality)} \\
=\ &\frac{1+\alpha}{8}\left(\langle|T|^2x,x\rangle^r+\langle|T^*|^2x,x\rangle^r\right)+\frac{1-\alpha}{4}\left|\langle T^2x,x\rangle\right|^r \\
&+\frac{1}{4}\left|\langle T^2x,x\rangle\right|^{\frac{r}{2}}\left(\langle|T|^2x,x\rangle^{\frac{r}{2}}+\langle|T^*|^2x,x\rangle^{\frac{r}{2}}\right)
\end{aligned}$$

$$\leqslant \frac{1+\alpha}{8}\left(\left\langle f^2\left(|T|^2\right)x,x\right\rangle^{\frac{r}{2}}\left\langle g^2\left(|T|^2\right)x,x\right\rangle^{\frac{r}{2}}+\left\langle f^2\left(|T^*|^2\right)x,x\right\rangle^{\frac{r}{2}}\left\langle g^2\left(|T^*|^2\right)x,x\right\rangle^{\frac{r}{2}}\right)$$
$$+\frac{1-\alpha}{4}\left|\left\langle T^2x,x\right\rangle\right|^r+\frac{1}{4}\left|\left\langle T^2x,x\right\rangle\right|^{\frac{r}{2}}\left(\left\langle |T|^2x,x\right\rangle^{\frac{r}{2}}+\left\langle |T^*|^2x,x\right\rangle^{\frac{r}{2}}\right)$$
(by Lemma 2.2)
$$\leqslant \frac{1+\alpha}{8}\left(\left\langle f^2\left(|T|^2\right)x,x\right\rangle^{r}+\left\langle f^2\left(|T^*|^2\right)x,x\right\rangle^{r}\right)^{\frac{1}{2}}$$
$$\times\left(\left\langle g^2\left(|T|^2\right)x,x\right\rangle^{r}+\left\langle g^2\left(|T^*|^2\right)x,x\right\rangle^{r}\right)^{\frac{1}{2}}$$
$$+\frac{1-\alpha}{4}\left|\left\langle T^2x,x\right\rangle\right|^r+\frac{1}{4}\left|\left\langle T^2x,x\right\rangle\right|^{\frac{r}{2}}\left(\left\langle |T|^2x,x\right\rangle^{\frac{r}{2}}+\left\langle |T^*|^2x,x\right\rangle^{\frac{r}{2}}\right)$$
(by the Cauchy-Schwarz inequality)
$$\leqslant \frac{1+\alpha}{8}\left\langle\left(f^{2r}\left(|T|^2\right)+f^{2r}\left(|T^*|^2\right)\right)x,x\right\rangle^{\frac{1}{2}}\left\langle\left(g^{2r}\left(|T|^2\right)+g^{2r}\left(|T^*|^2\right)\right)x,x\right\rangle^{\frac{1}{2}}$$
$$+\frac{1-\alpha}{4}\left|\left\langle T^2x,x\right\rangle\right|^r+\frac{1}{4}\left|\left\langle T^2x,x\right\rangle\right|^{\frac{r}{2}}\langle(|T|^r+|T^*|^r)x,x\rangle$$
(by Lemma 2.3).

Taking the supremum over $x\in\mathscr{H}$ with $\|x\|=1$, we get

$$\omega^{2r}(T)\leqslant \frac{1+\alpha}{8}\left\|f^{2r}(|T|^2)+f^{2r}(|T^*|^2)\right\|^{\frac{1}{2}}\left\|g^{2r}(|T|^2)+g^{2r}(|T^*|^2)\right\|^{\frac{1}{2}}$$
$$+\frac{1-\alpha}{4}\omega^r(T^2)+\frac{1}{4}\omega^{\frac{r}{2}}(T^2)\,\||T|^r+|T^*|^r\|. \quad \square$$

Considering $f(t)=g(t)=t^{\frac{1}{2}}$ in Theorem 3.8, we get the following corollary.

COROLLARY 3.9. *Let* $T\in\mathbb{B}(\mathscr{H})$, $r\geqslant 2$, $0\leqslant\alpha\leqslant 1$. *Then*

$$\omega^{2r}(T)\leqslant \frac{1+\alpha}{8}\left\||T|^{2r}+|T^*|^{2r}\right\|+\frac{1-\alpha}{4}\omega^r(T^2)+\frac{1}{4}\omega^{\frac{r}{2}}(T^2)\,\||T|^r+|T^*|^r\|.$$

REMARK 3.10. Corollary 3.9 improves the inequality (1.5) for $r=2$. In fact, taking $\alpha=\frac{1}{2}$, $r=2$ in Corollary 3.9, we reobtain Corollary 3.5, which improves $r=2$ in inequality (1.5).

REMARK 3.11. In [22, Corollary 3.1], Qiao, Hai and Bai proved that

$$\omega^4(T)\leqslant \frac{1+\alpha}{4}\left\||T|^4+|T^*|^4\right\|+\frac{1-\alpha}{2}\omega^2(T^2), \tag{3.2}$$

for $0\leqslant\alpha\leqslant 1$.

If taking $r=2$ Corollary 3.9, we will obtain

$$\omega^4(T)\leqslant \frac{1+\alpha}{8}\left\||T|^4+|T^*|^4\right\|+\frac{1-\alpha}{4}\omega^2(T^2)+\frac{1}{4}\omega(T^2)\left\||T|^2+|T^*|^2\right\|, \tag{3.3}$$

for $0 \leqslant \alpha \leqslant 1$. We point out that (3.3) is an improvement of (3.2) in some cases. For instance, let $T = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then $|T|^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|T^*|^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $|T|^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $|T^*|^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $T^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So, after simple calculations, we get

$$\omega(T) \approx 1.1180,\ \omega(T^2) = 1,$$

$$\left\| |T|^2 + |T^*|^2 \right\| = 5,\ \left\| |T|^4 + |T^*|^4 \right\| = 17.$$

Substituting the above results into inequality (3.2) and (3.3), we have

$$\omega^4(T) \approx 1.5623,$$

$$\frac{1+\alpha}{8} \left\| |T|^4 + |T^*|^4 \right\| + \frac{1-\alpha}{4} \omega^2(T^2) + \frac{1}{4} \omega(T^2) \left\| |T|^2 + |T^*|^2 \right\| = \frac{29+15\alpha}{8},$$

and

$$\frac{1+\alpha}{4} \left\| |T|^4 + |T^*|^4 \right\| + \frac{1-\alpha}{2} \omega^2(T^2) = \frac{19+15\alpha}{4}.$$

Therefore,

$$\begin{aligned} \omega^4(T) &< \frac{1+\alpha}{8} \left\| |T|^4 + |T^*|^4 \right\| + \frac{1-\alpha}{4} \omega^2(T^2) + \frac{1}{4} \omega(T^2) \left\| |T|^2 + |T^*|^2 \right\| \\ &< \frac{1+\alpha}{4} \left\| |T|^4 + |T^*|^4 \right\| + \frac{1-\alpha}{2} \omega^2(T^2) \end{aligned}$$

for any $\alpha \in [0,1]$.

THEOREM 3.12. *Let $A, B, C, D \in \mathbb{B}(\mathscr{H})$, $0 \leqslant \alpha \leqslant 1$, $r \geqslant 2$. Then*

$$\begin{aligned} \omega^{2r}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &\leqslant 2^{2r-1} \max\left\{ \omega^{2r}(A), \omega^{2r}(D) \right\} \\ &\quad + (1+\alpha) 2^{2r-4} \max\left\{ \left\| |C|^{2r} + |B^*|^{2r} \right\|, \left\| |B|^{2r} + |C^*|^{2r} \right\| \right\} \\ &\quad + (1-\alpha) 2^{2r-3} \max\left\{ \omega^r(BC), \omega^r(CB) \right\} \\ &\quad + 2^{2r-3} \max\left\{ \omega^{\frac{r}{2}}(BC), \omega^{\frac{r}{2}}(CB) \right\} \\ &\quad \times \max\left\{ \left\| |C|^r + |B^*|^r \right\|, \left\| |B|^r + |C^*|^r \right\| \right\}. \end{aligned}$$

Proof. Let $x \in \mathscr{H}$ with $\|x\| = 1$, and $T = \begin{bmatrix} O & B \\ C & O \end{bmatrix}$, we have

$$\begin{aligned}
& |\langle Tx, x\rangle|^{2r} \\
&= |\langle Tx, x\rangle \langle x, T^*x\rangle|^r \\
&\leqslant \frac{1+\alpha}{4}\|Tx\|^r\|T^*x\|^r + \frac{1-\alpha}{4}|\langle Tx, T^*x\rangle|^r + \frac{1}{2}|\langle Tx, T^*x\rangle|^{\frac{r}{2}}\|Tx\|^{\frac{r}{2}}\|T^*x\|^{\frac{r}{2}} \\
&\quad \text{(by Lemma 2.4)} \\
&\leqslant \frac{1+\alpha}{8}(\|Tx\|^{2r} + \|T^*x\|^{2r}) + \frac{1-\alpha}{4}\left|\left\langle T^2x, x\right\rangle\right|^r \\
&\quad + \frac{1}{4}\left|\left\langle T^2x, x\right\rangle\right|^{\frac{r}{2}}(\|Tx\|^r + \|T^*x\|^r) \\
&\quad \text{(by the arithmetic-geometic mean inequality)} \\
&= \frac{1+\alpha}{8}\left(\left\langle |T|^2x, x\right\rangle^r + \left\langle |T^*|^2x, x\right\rangle^r\right) + \frac{1-\alpha}{4}\left|\left\langle T^2x, x\right\rangle\right|^r \\
&\quad + \frac{1}{4}\left|\left\langle T^2x, x\right\rangle\right|^{\frac{r}{2}}\left(\left\langle |T|^2x, x\right\rangle^{\frac{r}{2}} + \left\langle |T^*|^2x, x\right\rangle^{\frac{r}{2}}\right) \\
&\leqslant \frac{1+\alpha}{8}\left\langle \left(|T|^{2r} + |T^*|^{2r}\right)x, x\right\rangle + \frac{1-\alpha}{4}\left|\left\langle T^2x, x\right\rangle\right|^r \\
&\quad + \frac{1}{4}\left|\left\langle T^2x, x\right\rangle\right|^{\frac{r}{2}}\langle(|T|^r + |T^*|^r)x, x\rangle. \\
&\quad \text{(by Lemma 2.3)} \\
&= \frac{1+\alpha}{8}\left\langle \begin{bmatrix} |C|^{2r} + |B^*|^{2r} & O \\ O & |B|^{2r} + |C^*|^{2r} \end{bmatrix} x, x\right\rangle + \frac{1-\alpha}{4}\left|\left\langle \begin{bmatrix} BC & O \\ O & CB \end{bmatrix} x, x\right\rangle\right|^r \\
&\quad + \frac{1}{4}\left|\left\langle \begin{bmatrix} BC & O \\ O & CB \end{bmatrix} x, x\right\rangle\right|^{\frac{r}{2}}\left\langle \begin{bmatrix} |C|^r + |B^*|^r & O \\ O & |B|^r + |C^*|^r \end{bmatrix} x, x\right\rangle \\
&\leqslant \frac{1+\alpha}{8}\omega\left(\begin{bmatrix} |C|^{2r} + |B^*|^{2r} & O \\ O & |B|^{2r} + |C^*|^{2r} \end{bmatrix}\right) + \frac{1-\alpha}{4}\omega^r\left(\begin{bmatrix} BC & O \\ O & CB \end{bmatrix}\right) \\
&\quad + \frac{1}{4}\omega^{\frac{r}{2}}\left(\begin{bmatrix} BC & O \\ O & CB \end{bmatrix}\right)\omega\left(\begin{bmatrix} |C|^r + |B^*|^r & O \\ O & |B|^r + |C^*|^r \end{bmatrix}\right).
\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ and by Lemma 2.5, we get

$$\begin{aligned}
\omega^{2r}\left(\begin{bmatrix} O & B \\ C & O \end{bmatrix}\right) \leqslant\ & \frac{1+\alpha}{8}\max\left\{\||C|^{2r} + |B^*|^{2r}\|, \||B|^{2r} + |C^*|^{2r}\|\right\} \\
& + \frac{1-\alpha}{4}\max\{\omega^r(BC), \omega^r(CB)\} \\
& + \frac{1}{4}\max\left\{\omega^{\frac{r}{2}}(BC), \omega^{\frac{r}{2}}(CB)\right\} \\
& \times \max\{\||C|^r + |B^*|^r\|, \||B|^r + |C^*|^r\|\}.
\end{aligned}$$

Now by Lemma 2.6, we have

$$\begin{aligned}
&\left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} x,x \right\rangle\right|^{2r} \\
&\leqslant \left(\left|\left\langle \begin{bmatrix} A & O \\ O & D \end{bmatrix} x,x \right\rangle\right| + \left|\left\langle \begin{bmatrix} O & B \\ C & O \end{bmatrix} x,x \right\rangle\right|\right)^{2r} \\
&\leqslant 2^{2r-1}\left(\left|\left\langle \begin{bmatrix} A & O \\ O & D \end{bmatrix} x,x \right\rangle\right|^{2r} + \left|\left\langle \begin{bmatrix} O & B \\ C & O \end{bmatrix} x,x \right\rangle\right|^{2r}\right) \\
&\leqslant 2^{2r-1}\omega^{2r}\left(\begin{bmatrix} A & O \\ O & D \end{bmatrix}\right) + 2^{2r-1}\omega^{2r}\left(\begin{bmatrix} O & B \\ C & O \end{bmatrix}\right) \\
&= 2^{2r-1}\max\left\{\omega^{2r}(A),\omega^{2r}(D)\right\} \\
&\quad +(1+\alpha)2^{2r-4}\max\left\{\||C|^{2r}+|B^*|^{2r}\|,\||B|^{2r}+|C^*|^{2r}\|\right\} \\
&\quad +(1-\alpha)2^{2r-3}\max\left\{\omega^{r}(BC),\omega^{r}(CB)\right\} \\
&\quad +2^{2r-3}\max\left\{\omega^{\frac{r}{2}}(BC),\omega^{\frac{r}{2}}(CB)\right\} \\
&\quad \times\max\left\{\||C|^{r}+|B^*|^{r}\|,\||B|^{r}+|C^*|^{r}\|\right\}.
\end{aligned}$$

This completes the proof. □

Considering $A=B=C=D$ in Theorem 3.12, we get the following corollary.

COROLLARY 3.13. *Let* $A\in\mathbb{B}(\mathscr{H})$, $0\leqslant\alpha\leqslant1$, $r\geqslant2$. *Then*

$$\omega^{2r}(A)\leqslant\frac{1+\alpha}{8}\||A|^{2r}+|A^*|^{2r}\|+\frac{1-\alpha}{4}\omega^{r}(A^2)+\frac{1}{4}\omega^{\frac{r}{2}}(A^2)\||A|^{r}+|A^*|^{r}\|.$$

REMARK 3.14. In [1, Corollary 2.10], Ammar, Frakis and Kittaneh proved that

$$\omega^{4}(A)\leqslant\frac{3+\alpha}{8}\||A|^{4}+|A^*|^{4}\|+\frac{1-\alpha}{4}\omega^{2}(A^2), \tag{3.4}$$

for $0\leqslant\alpha\leqslant1$.

If taking $r=2$ Corollary 3.13, we will obtain

$$\omega^{4}(A)\leqslant\frac{1+\alpha}{8}\left\||A|^{4}+|A^*|^{4}\right\|+\frac{1-\alpha}{4}\omega^{2}(A^2)+\frac{1}{4}\omega(A^2)\left\||A|^{2}+|A^*|^{2}\right\|, \tag{3.5}$$

for $0\leqslant\alpha\leqslant1$. Inequality (3.5) is an improvement of inequality (3.4). In fact, we only note that

$$\omega(A^2)\left\||A|^{2}+|A^*|^{2}\right\|\leqslant\frac{1}{2}\left\||A|^{2}+|A^*|^{2}\right\|^{2}\leqslant\left\||A|^{4}+|A^*|^{4}\right\|.$$

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