Journal of Mathematical Inequalities
Volume 18, Number 1 (2024), 315–328 doi:10.7153/jmi-2024-18-18

FURTHER IMPROVEMENTS FOR YOUNG'S INEQUALITIES ON THE ARITHMETIC, GEOMETRIC, AND HARMONIC MEAN

XIANGRUN YANG, CHANGSEN YANG AND HAIYING LI

(*Communicated by M. Krnić*)

Abstract. In this paper, we obtain some improvements and generalizations of Young's inequalities on the arithmetic, geometric, and harmonic mean. For example,

(1) If $0 < a < b$, $\beta \geqslant 1$ and $0 < v \leqslant \tau < 1$, then

$$\frac{(a\nabla_v b)^\beta - (a\sharp_v b)^\beta}{(a\nabla_\tau b)^\beta - (a\sharp_\tau b)^\beta} \leqslant \frac{v(1-v)}{\tau(1-\tau)}.$$

(2) If $0 < b < a$, $\beta \geqslant 1$ and $0 < v \leqslant \tau < \frac{1}{2}$, then

$$\frac{(a\nabla_v b)^\beta - K(h,2)^{\beta v}(a\sharp_v b)^\beta}{(a\nabla_\tau b)^\beta - K(h,2)^{\beta\tau}(a\sharp_\tau b)^\beta} \geqslant \frac{v(1-v)}{\tau(1-\tau)};$$

(3) If $0 < a < b$, $\beta \geqslant 1$ and $0 < v \leqslant \tau < 1$, then

$$\frac{(a\nabla_v b)^\beta - (a!_v b)^\beta}{(a\nabla_\tau b)^\beta - (a!_\tau b)^\beta} \leqslant \frac{(a\nabla_v b) - (a!_v b)}{(a\nabla_\tau b) - (a!_\tau b)} \leqslant \frac{v(1-v)}{\tau(1-\tau)}.$$

In addition, we obtain some new results for Young's inequality for operators.

1. Introduction

In the paper, let $\mathbb{N}$ be the set of positive integers. As usual, we denoted the Arithmetic mean, Geometric mean, and Harmonic mean as $a\nabla_v b = (1-v)a + vb$, $a\sharp_v b = a^{1-v}b^v$ and $a!_v b = [(1-v)a^{-1} + vb^{-1}]^{-1}$ for $a, b > 0$ and $v \in [0,1]$. The Young's inequality is well known as the following [7]: If $a, b > 0$ and $0 \leqslant v \leqslant 1$, then

$$a^{1-v}b^v \leqslant (1-v)a + vb, \tag{1.1}$$

where equality holds if and only if $a = b$. And this inequality implies the classical AM-GM-HM inequalities as

$$a!_v b \leqslant a\sharp_v b \leqslant a\nabla_v b. \tag{1.2}$$

Mathematics subject classification (2020): 26D07, 15A15, 15A42, 15A60.
Keywords and phrases: Young's inequality, Kantorovich constant, arithmetic mean, geometric mean, harmonic mean.

Zuo, Shi, Fujii [12] and Liao, Wu, Zhao [6] showed the refinement and reverse inequality of the above Young's inequality in terms of Kantorovich's constant as follows

$$K(h,2)^r a^{1-v} b^v \leqslant (1-v)a + vb \leqslant K(h,2)^R a^{1-v} b^v, \tag{1.3}$$

where $a,\ b \geqslant 0,\ r = \min\{v, 1-v\}, R = \max\{v, 1-v\}$ and $K(h,2) = \frac{(h+1)^2}{4h}$ with $h = \frac{b}{a}$.

It is easy to see that (1.3) implies

$$\left(\frac{1+x}{2}\right)^{2v} \leqslant (1-v) + vx \quad \left(x \geqslant 0,\ \ 0 \leqslant v \leqslant \frac{1}{2}\right) \tag{1.4}$$

and

$$\left(\frac{1+x}{2}\right)^{2v} \geqslant (1-v) + vx \quad \left(x \geqslant 0,\ \ \frac{1}{2} \leqslant v \leqslant 1\right). \tag{1.5}$$

He [2] and Hirzallah [3] refined Young's inequality so that

$$r^2(a-b)^2 \leqslant [(1-v)a + vb]^2 - (a^{1-v}b^v)^2 \leqslant R^2(a-b)^2$$

where $a,\ b \geqslant 0,\ r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.

Alzer, da Fonseca, and Kovačec [1] presented the following Young inequalities

$$\frac{v^m}{\tau^m} \leqslant \frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leqslant \frac{(1-v)^m}{(1-\tau)^m}$$

for $0 < v \leqslant \tau < 1$ and $m \in \mathbb{N}$.

Liao and Wu [5] replicated the above result as follows:

$$\frac{v^m}{\tau^m} \leqslant \frac{(a\nabla_v b)^m - (a!_v b)^m}{(a\nabla_\tau b)^m - (a!_\tau b)^m} \leqslant \frac{(1-v)^m}{(1-\tau)^m} \tag{1.6}$$

for $0 < v \leqslant \tau < 1$ and $m \in \mathbb{N}$.

Sababheh [9] obtained by convexity of function f

$$\frac{v^m}{\tau^m} \leqslant \frac{[(1-v)f(0) + vf(1)]^m - f^m(v)}{[(1-\tau)f(0) + \tau f(1)]^m - f^m(\tau)} \leqslant \frac{(1-v)^m}{(1-\tau)^m} \tag{1.7}$$

for $0 < v \leqslant \tau < 1$ and $m \in \mathbb{N}$.

Ren [8] obtained the following inequalities:

$$\begin{cases} \dfrac{a\nabla_v b - a\sharp_v b}{a\nabla_\tau b - a\sharp_\tau b} \leqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a > 0 \\ \dfrac{a\nabla_v b - a\sharp_v b}{a\nabla_\tau b - a\sharp_\tau b} \geqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a < 0 \end{cases} \tag{1.8}$$

and

$$\begin{cases} \dfrac{(a\nabla_v b)^2 - (a\sharp_v b)^2}{(a\nabla_\tau b)^2 - (a\sharp_\tau b)^2} \leqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a > 0 \\ \dfrac{(a\nabla_v b)^2 - (a\sharp_v b)^2}{(a\nabla_\tau b)^2 - (a\sharp_\tau b)^2} \geqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a < 0 \end{cases} \tag{1.9}$$

for $0 < v \leqslant \tau < 1$ and $a,\ b > 0$.

Similar to the arithmetic mean and geometric mean, for arithmetic mean and harmonic mean, Sababheh [10] proved that

(i) if $a,b > 0$ and $v,\tau \in [0,1]$ such that $(b-a)(\tau-v) > 0$, then

$$\frac{(a\nabla_v b)^k - (a!_v b)^k}{(a\nabla_\tau b)^k - (a!_\tau b)^k} \leqslant \frac{v(1-v)}{\tau(1-\tau)} \tag{1.10}$$

(ii) if $a,b > 0$ and $v,\tau \in [0,1]$ such that $(b-a)(\tau-v) < 0$, then

$$\frac{(a\nabla_v b)^k - (a!_v b)^k}{(a\nabla_\tau b)^k - (a!_\tau b)^k} \geqslant \frac{v(1-v)}{\tau(1-\tau)} \tag{1.11}$$

for $k = 1,2$.

Yang and Wang [11] improved (1.8) and (1.9) as follows

THEOREM 1.1. *Let $0 < v \leqslant \tau < 1$, $m \in \mathbb{N}$ and $a,\ b$ are real positive numbers. Then*

(1) *If $b > a$, we have*

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leqslant \frac{v(1-v)}{\tau(1-\tau)}; \tag{1.12}$$

(2) *If $b < a$, we have*

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)}. \tag{1.13}$$

In this paper, we point out that the condition $m \in \mathbb{N}$ can be changed into $m \geqslant 1$ in (1.12) and (1.13). Using the same method, we also showed that (1.10) and (1.11) are also valid for any positive number $k \geqslant 1$.

For convenience, in the following, all letters a,b,x designate positive reals with $a \neq b$ unless we state explicitly the contrary. v,τ are always reals in $[0,1]$. By $K(h,2) = \frac{(h+1)^2}{4h}$ we mean the Kantorovich constant.

2. Generalized improvements of Young's inequalities for three mean

In order to show our main results, we firstly give a lemma as follows.

LEMMA 2.1. *Define functions $f,J,K : (0,1) \to \mathbb{R}$ of v, with parameters α,β and x by the formulas*

$$f(v) = \frac{(1-v+vx)^\beta - x^{\beta v}}{(1-v+vx)^\alpha - x^{\alpha v}};$$

$$J(v) = \begin{cases} \frac{(1-v+vx)^\beta - (\frac{1+x}{2})^{2\beta v}}{(1-v+vx)^\alpha - (\frac{1+x}{2})^{2\alpha v}} & v \neq \frac{1}{2} \\ \lim\limits_{v \to \frac{1}{2}} \frac{(1-v+vx)^\beta - (\frac{1+x}{2})^{2\beta v}}{(1-v+vx)^\alpha - (\frac{1+x}{2})^{2\alpha v}} & v = \frac{1}{2} \end{cases};$$

$$K(v) = \frac{(1-v+vx)^{\beta} - (1-v+vx^{-1})^{-\beta}}{(1-v+vx)^{\alpha} - (1-v+vx^{-1})^{-\alpha}}.$$

Then each of these functions is either non-increasing or non-decreasing on $(0,1)$ *according to which of the cases in the following table applies.*

	$0<\alpha<\beta$	$0<\beta<\alpha$
$x<1$	*non-increasing*	*non-decreasing*
$x>1$	*non-decreasing*	*non-increasing*

Proof. Firstly, letting $0<\alpha<\beta$, we can obtain that if $g(u)=\beta-\alpha+\alpha u^{\beta}-\beta u^{\alpha}$, then $g'(u)=\alpha\beta[u^{\beta-1}-u^{\alpha-1}]\leqslant 0$ for $u\in(0,1)$ and $g'(u)\geqslant 0$ for $u\in(1,\infty)$. So we have $g(u)\geqslant g(1)=0$ on $[0,\infty)$. Next, if $h(u)=(\beta-\alpha)u^{\beta}-\beta u^{\beta-\alpha}+\alpha$, then $h'(u)=\beta(\beta-\alpha)[u^{\beta-1}-u^{\beta-\alpha-1}]\leqslant 0$ for $u\in(0,1)$ and $h'(u)\geqslant 0$ for $u\in(1,\infty)$. It also follows that $h(u)\geqslant 0$ on $[0,\infty)$. Now

$$\begin{aligned}
&[(1-v+vx)^{\alpha}-x^{\alpha v}]^2 f'(v)\\
&= [(1-v+vx)^{\alpha}-x^{\alpha v}][\beta(x-1)(1-v+vx)^{\beta-1}-\beta x^{\beta v}\ln x]\\
&\quad -[(1-v+vx)^{\beta}-x^{\beta v}][\alpha(x-1)(1-v+vx)^{\alpha-1}-\alpha x^{\alpha v}\ln x]\\
&= (x-1)(1-v+vx)^{\alpha+\beta-1}\Big\{\beta-\alpha-\beta\Big(\frac{x^v}{1-v+vx}\Big)^{\alpha}+\alpha\Big(\frac{x^v}{1-v+vx}\Big)^{\beta}\Big\}\\
&\quad +x^{\alpha v}(1-v+vx)^{\beta}\ln x\Big\{-\beta\Big(\frac{x^v}{1-v+vx}\Big)^{\beta-\alpha}+(\beta-\alpha)\Big(\frac{x^v}{1-v+vx}\Big)^{\beta}+\alpha\Big\}\\
&= (x-1)(1-v+vx)^{\beta+\alpha-1}g\Big(\frac{x^v}{1-v+vx}\Big)+x^{\alpha v}(1-v+vx)^{\beta}h\Big(\frac{x^v}{1-v+vx}\Big)\ln x.
\end{aligned}$$

We see if $x>1$ then both of the last two terms connected by the '+' in the middle are nonnegative since h and g are nonnegative; so, as the initial expression is of from $[(1-v+vx)^{\alpha}-x^{\alpha v}]^2 f'(v)$, we find $f'(v)\geqslant 0$, and so f is non-decreasing. If $x<1$ the first term is evidently negative and the second is so because of the occurrence of $\ln x$; so $f'(v)\leqslant 0$, and so f is non-increasing. We proceed with examining J' and K' in a similar manner. Namely, for $v\neq\frac{1}{2}$, we have

$$\begin{aligned}
&\Big[(1-v+vx)^{\alpha}-\Big(\frac{1+x}{2}\Big)^{2\alpha v}\Big]^2 J'(v)\\
&= \Big[(1-v+vx)^{\alpha}-\Big(\frac{1+x}{2}\Big)^{2\alpha v}\Big]\Big[\beta(x-1)(1-v+vx)^{\beta-1}-2\beta\Big(\frac{1+x}{2}\Big)^{2\beta v}\ln\frac{1+x}{2}\Big]\\
&\quad -\Big[(1-v+vx)^{\beta}-\Big(\frac{1+x}{2}\Big)^{2\beta v}\Big]\Big[\alpha(x-1)(1-v+vx)^{\alpha-1}-2\alpha\Big(\frac{1+x}{2}\Big)^{2\alpha v}\ln\frac{1+x}{2}\Big]\\
&= (x-1)(1-v+vx)^{\alpha+\beta-1}\Big\{\beta-\alpha-\beta\Big(\frac{(\frac{1+x}{2})^{2v}}{1-v+vx}\Big)^{\alpha}+\alpha\Big(\frac{(\frac{1+x}{2})^{2v}}{1-v+vx}\Big)^{\beta}\Big\}\\
&\quad +2\Big(\frac{1+x}{2}\Big)^{2\alpha v}(1-v+vx)^{\beta}\\
&\quad \times\ln\frac{1+x}{2}\Big\{-\beta\Big(\frac{(\frac{1+x}{2})^{2v}}{1-v+vx}\Big)^{\beta-\alpha}+(\beta-\alpha)\Big(\frac{(\frac{1+x}{2})^{2v}}{1-v+vx}\Big)^{\beta}+\alpha\Big\}
\end{aligned}$$

$$= (x-1)(1-v+vx)^{\alpha+\beta-1} g\Big(\frac{(\frac{1+x}{2})^{2v}}{1-v+vx}\Big)$$
$$+2\Big(\frac{1+x}{2}\Big)^{2\alpha v}(1-v+vx)^{\beta} h\Big(\frac{(\frac{1+x}{2})^{2v}}{1-v+vx}\Big)\ln\frac{1+x}{2},$$

and

$$\begin{aligned}
&[(1-v+vx)^{\alpha}-(1-v+vx^{-1})^{-\alpha}]^2K'(v)\\
&=[(1-v+vx)^{\alpha}-(1-v+vx^{-1})^{-\alpha}]\\
&\quad\times[\beta(x-1)(1-v+vx)^{\beta-1}-\beta(1-v+vx^{-1})^{-\beta-1}(1-x^{-1})]\\
&\quad-[(1-v+vx)^{\beta}-(1-v+vx^{-1})^{-\beta}]\\
&\quad\times[\alpha(x-1)(1-v+vx)^{\alpha-1}-\alpha(1-v+vx^{-1})^{-\alpha-1}(1-x^{-1})]\\
&=(x-1)(1-v+vx)^{\alpha+\beta-1}\\
&\quad\times\Big\{\beta-\alpha-\beta\Big(\frac{(1-v+vx^{-1})^{-1}}{1-v+vx}\Big)^{\alpha}+\alpha\Big(\frac{(1-v+vx^{-1})^{-1}}{1-v+vx}\Big)^{\beta}\Big\}\\
&\quad+\frac{(x-1)}{x}(1-v+vx^{-1})^{-\alpha-\beta-1}\\
&\quad\times\Big\{-\alpha+\alpha\Big(\frac{(1-v+vx)}{(1-v+vx^{-1})^{-1}}\Big)^{\beta}+\beta-\beta\Big(\frac{(1-v+vx)}{(1-v+vx^{-1})^{-1}}\Big)^{\alpha}\Big\}\\
&=(x-1)(1-v+vx)^{\alpha+\beta-1}g\Big(\frac{(1-v+vx^{-1})^{-1}}{1-v+vx}\Big)\\
&\quad+\frac{(x-1)}{x}(1-v+vx^{-1})^{-\alpha-\beta-1}g\Big(\frac{(1-v+vx)}{(1-v+vx^{-1})^{-1}}\Big).
\end{aligned}$$

We have that $J'(v),K'(v)\geqslant 0$ if $x>1$ and $J'(v),K'(v)\leqslant 0$ under the condition $x\in(0,1)$, which completes the proof of (i). Next, if $0<\beta<\alpha$, then $h(u),g(u)\leqslant 0$, and this implies that $f'(v),J'(v),K'(v)\geqslant 0$ if $x\in(0,1)$ and $f'(v),J'(v),K'(v)\leqslant 0$ under the condition $x>1$. Hence (ii) is also valid. □

THEOREM 2.2. *Let $0<v\leqslant\tau<1$, $0<\alpha<\beta$ and a, b are real positive numbers. Then*

(1) *If $b>a$, we can get*

$$\frac{(a\nabla_v b)^{\beta}-(a\sharp_v b)^{\beta}}{(a\nabla_\tau b)^{\beta}-(a\sharp_\tau b)^{\beta}}\leqslant\frac{(a\nabla_v b)^{\alpha}-(a\sharp_v b)^{\alpha}}{(a\nabla_\tau b)^{\alpha}-(a\sharp_\tau b)^{\alpha}};\tag{2.1}$$

(2) *If $b<a$, then the reverse inequality is valid.*

Proof. Let $f(v)=\frac{(1-v+vx)^{\beta}-x^{\beta v}}{(1-v+vx)^{\alpha}-x^{\alpha v}}$. By Lemma 2.1 (i), we have

(1) if $x>1$, then $f'(v)\geqslant 0$, meaning that $f(v)$ is increasing on $(0,1)$, that is to

say $\frac{f(v)}{f(\tau)} \leqslant 1$. Therefore

$$\frac{(1-v+vx)^\beta - x^{\beta v}}{(1-\tau+\tau x)^\beta - x^{\beta\tau}} = \frac{((1-v+vx)^\alpha - x^{\alpha v})f(v)}{((1-\tau+\tau x)^\alpha - x^{\alpha\tau})f(\tau)} \leqslant \frac{(1-v+vx)^\alpha - x^{\alpha v}}{(1-\tau+\tau x)^\alpha - x^{\alpha\tau}}.$$

(2) If $0 < x \leqslant 1$, then $f'(v) \leqslant 0$, meaning that $f(v)$ is decreasing on $(0,1)$, that is to say $\frac{f(v)}{f(\tau)} \geqslant 1$. Therefore

$$\frac{(1-v+vx)^\beta - x^{\beta v}}{(1-\tau+\tau x)^\beta - x^{\beta\tau}} = \frac{((1-v+vx)^\alpha - x^{\alpha v})f(v)}{((1-\tau+\tau x)^\alpha - x^{\alpha\tau})f(\tau)} \geqslant \frac{(1-v+vx)^\alpha - x^{\alpha v}}{(1-\tau+\tau x)^\alpha - x^{\alpha\tau}}.$$

One deduces (2.1) by noting facts like this: if we substitute in $(1-v+vx)^\beta - x^{\beta v}$, x by $\frac{b}{a}$ and then multiply with a^β we get $(a\nabla_v b)^\beta - (a\sharp_v b)^\beta$.

Using (1.8), and Theorem 2.2, we have the following result. □

COROLLARY 2.3. *Let $0 < v \leqslant \tau < 1$, $\beta \geqslant 1$ and a, b are real positive numbers. Then*

(1) *If $b > a$, we have*

$$\frac{(a\nabla_v b)^\beta - (a\sharp_v b)^\beta}{(a\nabla_\tau b)^\beta - (a\sharp_\tau b)^\beta} \leqslant \frac{(a\nabla_v b) - (a\sharp_v b)}{(a\nabla_\tau b) - (a\sharp_\tau b)} \leqslant \frac{v(1-v)}{\tau(1-\tau)}; \tag{2.2}$$

(2) *If $b < a$, then the reverse inequality is valid.*

REMARK 2.4. (1) Let $\beta = 2$ or $\beta = m \in \mathbb{N}$, we can get [9, Theorem 2.3] and [11, Theorem 2.1], respectively.

(2) Let $a = b$, $b = a$, $v = 1-\tau$, $\tau = 1-v$ in inequality (2.2), we can also obtain the reverse inequality of (2.2) directly for $b < a$.

(3) Let $0 < v \leqslant \tau < 1$, so $\frac{1-v}{1-\tau} \geqslant 1$, therefore

(i) If $b > a$, then

$$\frac{(a\nabla_v b)^\beta - (a\sharp_v b)^\beta}{(a\nabla_\tau b)^\beta - (a\sharp_\tau b)^\beta} \leqslant \frac{v(1-v)}{\tau(1-\tau)} \leqslant \frac{v(1-v)^\beta}{\tau(1-\tau)^\beta} \leqslant \frac{(1-v)^\beta}{(1-\tau)^\beta};$$

(ii) If $b < a$, then

$$\frac{(a\nabla_v b)^\beta - (a\sharp_v b)^\beta}{(a\nabla_\tau b)^\beta - (a\sharp_\tau b)^\beta} \geqslant \frac{v(1-v)}{\tau(1-\tau)} \geqslant \frac{v^\beta(1-v)}{\tau^\beta(1-\tau)} \geqslant \frac{v^\beta}{\tau^\beta}.$$

Using Lemma 2.1, we can also obtain the following results.

THEOREM 2.5. *Let* $0<\alpha<\beta$, $0<a<b$ *and let* $h=\frac{b}{a}$. *Then*
(a) If $\frac{1}{2}<v\leqslant\tau\leqslant 1$ *or* $0<v\leqslant\tau<\frac{1}{2}$, *then*

$$\frac{K(h,2)^{\beta v}(a\sharp_v b)^{\beta}-(a\nabla_v b)^{\beta}}{K(h,2)^{\beta\tau}(a\sharp_\tau b)^{\beta}-(a\nabla_\tau b)^{\beta}}\leqslant\frac{K(h,2)^{\alpha v}(a\sharp_v b)^{\alpha}-(a\nabla_v b)^{\alpha}}{K(h,2)^{\alpha\tau}(a\sharp_\tau b)^{\alpha}-(a\nabla_\tau b)^{\alpha}} \tag{2.3}$$

(b) If $0<v<\frac{1}{2}<\tau<1$, *then we have the reverse inequality of* (2.3).
On the other hand, if $0<b<a$, *then the reverse inequality of above results is true under their other conditions, respectively.*

Proof. Let $J(v)=\frac{(1-v+vx)^{\beta}-(\frac{1+x}{2})^{2\beta v}}{(1-v+vx)^{\alpha}-(\frac{1+x}{2})^{2\alpha v}}$, then $J(v)\leqslant J(\tau)$ for $0<v<\tau\leqslant 1$ under the condition $x\geqslant 1$, and this implies that

$$\frac{(1-v+vx)^{\beta}-(\frac{1+x}{2})^{2\beta v}}{(1-v+vx)^{\alpha}-(\frac{1+x}{2})^{2\alpha v}}\leqslant\frac{(1-\tau+\tau x)^{\beta}-(\frac{1+x}{2})^{2\beta\tau}}{(1-\tau+\tau x)^{\alpha}-(\frac{1+x}{2})^{2\alpha\tau}}$$

holds for $x>1$. With evident notation this inequality is of form $\frac{c}{d}\leqslant\frac{e}{f}$. Now by (1.4) and (1.5) d and e have the same sign and hence $\frac{d}{e}$ is nonnegative. So multiplying the fraction with $\frac{d}{e}$ we can get the inequality $\frac{c}{e}\leqslant\frac{d}{f}$, that is,

$$\frac{(1-v+vx)^{\beta}-(\frac{1+x}{2})^{2\beta v}}{(1-\tau+\tau x)^{\beta}-(\frac{1+x}{2})^{2\beta\tau}}\leqslant\frac{(1-v+vx)^{\alpha}-(\frac{1+x}{2})^{2\alpha v}}{(1-\tau+\tau x)^{\alpha}-(\frac{1+x}{2})^{2\alpha\tau}}$$

for $x>1$ and $\frac{1}{2}<v\leqslant\tau\leqslant 1$ or $0<v\leqslant\tau<\frac{1}{2}$; and

$$\frac{(1-v+vx)^{\beta}-(\frac{1+x}{2})^{2\beta v}}{(1-\tau+\tau x)^{\beta}-(\frac{1+x}{2})^{2\beta\tau}}\geqslant\frac{(1-v+vx)^{\alpha}-(\frac{1+x}{2})^{2\alpha v}}{(1-\tau+\tau x)^{\alpha}-(\frac{1+x}{2})^{2\alpha\tau}}$$

for $x>1$ and $0<v<\frac{1}{2}<\tau<1$.
By taking $x=\frac{b}{a}$, we can get our desired results directly. □

LEMMA 2.6. *Let* a, b *be real positive numbers and let* $h=\frac{b}{a}$. *Then*
(a) If $\frac{1}{2}<v\leqslant\tau<1$, *then*

$$\frac{K(h,2)^{v}a\sharp_v b-a\nabla_v b}{K(h,2)^{\tau}a\sharp_\tau b-a\nabla_\tau b}\leqslant\frac{v}{\tau}\leqslant\frac{v(1-v)}{\tau(1-\tau)} \tag{2.4}$$

(b) If $0<v\leqslant\tau<\frac{1}{2}$, *then*

$$\frac{K(h,2)^{v}a\sharp_v b-a\nabla_v b}{K(h,2)^{\tau}a\sharp_\tau b-a\nabla_\tau b}\geqslant\frac{v(1-v)}{\tau(1-\tau)}\geqslant\frac{v}{\tau}. \tag{2.5}$$

Proof. Firstly we let for any $x>0$ and $0<v\leqslant 1$,

$$f(v)=\frac{\left(\frac{x+1}{2}\right)^{2v}-(1-v+vx)}{v}.$$

Then

$$f'(v) = \frac{\left(\frac{x+1}{2}\right)^{2v}\left[2v\ln\left(\frac{x+1}{2}\right)-1\right]+1}{v^2} \equiv \frac{h(x)}{v^2}$$

and

$$h'(x) = 2v^2\left(\frac{x+1}{2}\right)^{2v-1}\ln\left(\frac{x+1}{2}\right).$$

It means that $h'(x) \leqslant 0$ for $x \in (0,1]$ and $h'(x) \geqslant 0$ for $x \in [1,\infty)$. So $h(x) \geqslant h(1) = 0$ and $f'(v) \geqslant 0$. Therefore $f(v)$ is increasing on $(0,1)$, which implies that $\frac{f(v)}{1-v}$ is also increasing on $(0,1)$, that is to say

$$\frac{\left(\frac{x+1}{2}\right)^{2v}-(1-v+vx)}{v} \leqslant \frac{\left(\frac{x+1}{2}\right)^{2\tau}-(1-\tau+\tau x)}{\tau}$$

and

$$\frac{\left(\frac{x+1}{2}\right)^{2v}-(1-v+vx)}{v(1-v)} \leqslant \frac{\left(\frac{x+1}{2}\right)^{2\tau}-(1-\tau+\tau x)}{\tau(1-\tau)}$$

for any $0 < v \leqslant \tau < 1$.

Therefore,

$$\frac{\left(\frac{x+1}{2}\right)^{2v}-(1-v+vx)}{\left(\frac{x+1}{2}\right)^{2\tau}-(1-\tau+\tau x)} \leqslant \frac{v}{\tau}$$

for $\frac{1}{2} < v \leqslant \tau \leqslant 1$ by (1.5); and

$$\frac{\left(\frac{x+1}{2}\right)^{2v}-(1-v+vx)}{\left(\frac{x+1}{2}\right)^{2\tau}-(1-\tau+\tau x)} \geqslant \frac{v(1-v)}{\tau(1-\tau)}$$

for $0 < v \leqslant \tau < \frac{1}{2}$ by (1.4).

Taking $x = \frac{b}{a}$, we can get our desired results directly. □

THEOREM 2.7. *Let a, b be real positive numbers, $h = \frac{b}{a}$, and $\beta \geqslant 1$. Then*
(a) If $0 < a < b$ and $\frac{1}{2} < v \leqslant \tau \leqslant 1$, then

$$\frac{K(h,2)^{\beta v}(a\sharp_v b)^\beta - (a\nabla_v b)^\beta}{K(h,2)^{\beta\tau}(a\sharp_\tau b)^\beta - (a\nabla_\tau b)^\beta} \leqslant \frac{v}{\tau} \leqslant \frac{v(1-v)}{\tau(1-\tau)} \tag{2.6}$$

(b) If $0 < b < a$ and $0 < v \leqslant \tau < \frac{1}{2}$, then

$$\frac{K(h,2)^{\beta v}(a\sharp_v b)^\beta - (a\nabla_v b)^\beta}{K(h,2)^{\beta\tau}(a\sharp_\tau b)^\beta - (a\nabla_\tau b)^\beta} \geqslant \frac{v(1-v)}{\tau(1-\tau)} \geqslant \frac{v}{\tau}. \tag{2.7}$$

Proof. Let $J(v)=\frac{(1-v+vx)^{\beta}-(\frac{1+x}{2})^{2\beta v}}{1-v+vx-(\frac{1+x}{2})^{2v}}$.

(i) If $x>1$ and $\frac{1}{2}<v\leqslant\tau\leqslant 1$, using Lemma 2.1 and Lemma 2.6, we have

$$\begin{aligned}\frac{(1-v+vx)^{\beta}-(\frac{1+x}{2})^{2\beta v}}{(1-\tau+\tau x)^{\beta}-(\frac{1+\tau}{2})^{2\beta\tau}}&=\frac{J(v)}{J(\tau)}\frac{1-v+vx-(\frac{1+x}{2})^{2v}}{1-\tau+\tau x-(\frac{1+x}{2})^{2\tau}}\\&\leqslant\frac{1-v+vx-(\frac{1+x}{2})^{2v}}{1-\tau+\tau x-(\frac{1+x}{2})^{2\tau}}\\&\leqslant\frac{v}{\tau}\leqslant\frac{v(1-v)}{\tau(1-\tau)}\end{aligned}$$

(ii) If $x\in(0,1)$ and $0<v\leqslant\tau<\frac{1}{2}$, using Lemma 2.1 and Lemma 2.6, we also have

$$\begin{aligned}\frac{(1-v+vx)^{\beta}-(\frac{1+x}{2})^{2\beta v}}{(1-\tau+\tau x)^{\beta}-(\frac{1+\tau}{2})^{2\beta\tau}}&=\frac{J(v)}{J(\tau)}\frac{1-v+vx-(\frac{1+x}{2})^{2v}}{1-\tau+\tau x-(\frac{1+x}{2})^{2\tau}}\\&\geqslant\frac{1-v+vx-(\frac{1+x}{2})^{2v}}{1-\tau+\tau x-(\frac{1+x}{2})^{2\tau}}\\&\geqslant\frac{v(1-v)}{\tau(1-\tau)}\geqslant\frac{v}{\tau}\end{aligned}$$

Taking $x=\frac{b}{a}$, we can get our desired results directly. □

Now using (1.10), (1.11) and Lemma 2.1, by the same method as above, we can easily obtain the following result.

THEOREM 2.8. *Let $0<v\leqslant\tau<1$, $\beta\geqslant 1$ and a, b real positive numbers. Then*
(1) *If $b>a$, then*

$$\frac{(a\nabla_v b)^{\beta}-(a!_v b)^{\beta}}{(a\nabla_\tau b)^{\beta}-(a!_\tau b)^{\beta}}\leqslant\frac{(a\nabla_v b)-(a!_v b)}{(a\nabla_\tau b)-(a!_\tau b)}\leqslant\frac{v(1-v)}{\tau(1-\tau)};\tag{2.8}$$

(2) *If $b<a$, then*

$$\frac{(a\nabla_v b)^{\beta}-(a!_v b)^{\beta}}{(a\nabla_\tau b)^{\beta}-(a!_\tau b)^{\beta}}\geqslant\frac{(a\nabla_v b)-(a!_v b)}{(a\nabla_\tau b)-(a!_\tau b)}\geqslant\frac{v(1-v)}{\tau(1-\tau)}.\tag{2.9}$$

Proof. (1) For $\alpha=1$, the function $K(v)=\frac{(1\nabla_v x)^{\beta}-(1\nabla_v x^{-1})^{-\beta}}{(1\nabla_v x)-(1\nabla_v x^{-1})^{-1}}$. We consider the numerator, put $x=\frac{b}{a}$ and multiply with a^{β}. This yields

$$\begin{aligned}\left(\left(1\nabla_v\frac{b}{a}\right)^{\beta}-\left(1\nabla_v\frac{a}{b}\right)^{-\beta}\right)a^{\beta}&=\left(1-v+v\frac{b}{a}\right)^{\beta}a^{\beta}-\left(1-v+v\frac{a}{b}\right)^{-\beta}(a^{-1})^{-\beta}\\&=((1-v)a+vb)^{\beta}-((1-v)a^{-1}+vb^{-1})^{-\beta})\\&=(a\nabla_v b)^{\beta}-(a!b)^{\beta}.\end{aligned}$$

We will below use similar equations for τ in place of v and for 1 in place of β. Since $b > a$, we have $x > 1$ and the hypothesis tells us $\beta \geqslant \alpha > 0$. So if $0 < v \leqslant \tau \leqslant 1$ we have by Lemma 2.1 the inequality $K(v) \leqslant K(\tau)$, that is

$$\frac{(1\nabla_v x)^\beta - (1\nabla_v x^{-1})^{-\beta}}{(1\nabla_v x) - (1\nabla_v x^{-1})^{-1}} \leqslant \frac{(1\nabla_\tau x)^\beta - (1\nabla_\tau x^{-1})^{-\beta}}{(1\nabla_\tau x) - (1\nabla_\tau x^{-1})^{-1}}.$$

As $x > 1$, for any $v \in (0,1)$ there holds $1\nabla_v x > 1\nabla_v x^{-1}$. So we can interchange in above inequality the left lower with the right upper expression and we get

$$\frac{(1\nabla_v x)^\beta - (1\nabla_v x^{-1})^{-\beta}}{(1\nabla_\tau x)^\beta - (1\nabla_\tau x^{-1})^{-\beta}} \leqslant \frac{(1\nabla_v x) - (1\nabla_v x^{-1})^{-1}}{(1\nabla_\tau x) - (1\nabla_\tau x^{-1})^{-1}}.$$

Here now we substitute $x = \frac{b}{a}$, then multiply both the parts of the left fraction with a^β and of the right fraction with a and get the left of (2.8), while the right part follows from Sababheh's inequality (1.11) for $k = 1$.

(2) The proof of part (2) is similar. □

3. Applications

Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices and $M_n^+(\mathbb{C})$ denote the space of all $n \times n$ positive semidefinite matrices in $M_n(\mathbb{C})$. We recall that $X \in M_n^+(\mathbb{C})$ implies $\mathrm{tr}X \geqslant 0$ and $\det X \geqslant 0$, see [12, Corollary 7.1.5] and the definition of the Loewner or positive semidefinite ordering, see [12, Definition 7.7.1]. A matrix norm $|||.|||$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. For $A = [a_{ij}] \in M_n(\mathbb{C})$, the trace norm of A is defined by

$$\|A\|_1 = \mathrm{tr}|A| = \sum_{i=1}^{n} s_i(A)$$

where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity and tr is the usual trace function.

LEMMA 3.1. (Minkowski's inequality, [12, Theorem 7.8.8]) *Let $A, B \in M_n^+(\mathbb{C})$, then*

$$\det(A+B)^{\frac{1}{n}} \geqslant \det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}.$$

LEMMA 3.2. ([5]) *Let $A, B, X \in M_n(\mathbb{C})$ and $A, B \in M_n^+(\mathbb{C})$. If $0 \leqslant v \leqslant 1$, then*

$$|||A^v X B^{1-v}||| \leqslant |||AX|||^v |||XB|||^{1-v}.$$

THEOREM 3.1. *Let* $A,B \in M_n^+(\mathbb{C})$, $\beta \geqslant 1$ *and* $0 < v \leqslant \tau < 1$. *Then*
(i) If $B \geqslant A \geqslant 0$, *we have*

$$\frac{\|(1-v)A+vB\|_1^\beta - (\|A\|_1^{1-v}\|B\|_1^v)^\beta}{v(1-v)} \leqslant \frac{\|(1-\tau)A+\tau B\|_1^\beta - (\|A\|_1^{1-\tau}\|B\|_1^\tau)^\beta}{\tau(1-\tau)};$$

(ii) If $A \geqslant B \geqslant 0$, *we have*

$$\frac{\|(1-v)A+vB\|_1^\beta - (\|A\|_1^{1-v}\|B\|_1^v)^\beta}{v(1-v)} \geqslant \frac{\|(1-\tau)A+\tau B\|_1^\beta - (\|A\|_1^{1-\tau}\|B\|_1^\tau)^\beta}{\tau(1-\tau)}.$$

Proof. Suppose $B \geqslant A$. Then putting $a = \mathrm{tr}(A)$ and $b = \mathrm{tr}(B)$ we have $b \geqslant a$ and using Corollary 2.3 we deduce

$$\begin{aligned}
&\|(1-v)A+vB\|_1^\beta \\
&= (\mathrm{tr}((1-v)A)+\mathrm{tr}(vB))^\beta \\
&= ((1-v)\mathrm{tr}(A)+v\mathrm{tr}(B))^\beta \\
&\leqslant (\mathrm{tr}(A)^{1-v}\mathrm{tr}(B)^v)^\beta + \frac{v(1-v)}{\tau(1-\tau)}[((1-\tau)\mathrm{tr}(A)+\tau\mathrm{tr}(B))^\beta - (\mathrm{tr}(A)^{1-\tau}\mathrm{tr}(B)^\tau)^\beta] \\
&= (\|A\|_1^{1-v}\|B\|_1^v)^\beta + \frac{v(1-v)}{\tau(1-\tau)}[\|(1-\tau)A+\tau B\|_1^\beta - (\|A\|_1^{1-\tau}\|B\|_1^\tau)^\beta].
\end{aligned}$$

Using the same method we can get (ii) similarly, so we omit it. □

THEOREM 3.2. *Let* $A,B \in M_n^+(\mathbb{C})$, $n\beta \geqslant 1$ *and* $0 < v \leqslant \tau < 1$. *Then*
(i) If $B \geqslant A \geqslant 0$, *we can get*

$$\begin{aligned}
&\det((1-\tau)A+\tau B)^\beta \\
&\geqslant \frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)\det A^{\frac{1}{n}} + v\det B^{\frac{1}{n}}]^{\beta n} - \det(A^{1-v}B^v)^\beta\right] + \det\left(A^{1-\tau}B^\tau\right)^\beta;
\end{aligned}$$

(ii) If $A \geqslant B \geqslant 0$, *we can get*

$$\begin{aligned}
&\det((1-v)A+vB)^\beta \\
&\geqslant \frac{v(1-v)}{\tau(1-\tau)}\left[[(1-\tau)\det A^{\frac{1}{n}} + \tau\det B^{\frac{1}{n}}]^{\beta n} - \det(A^{1-\tau}B^\tau)^\beta\right] + \det(A^{1-v}B^v)^\beta.
\end{aligned}$$

Proof. Suppose $B \geqslant A$. Then putting $b = \det B^{\frac{1}{n}}$ and $a = \det A^{\frac{1}{n}}$, we have $b \geqslant a$ and again by Corollary 2.3 and Lemma 3.1, we have

$$\begin{aligned}
&\det((1-\tau)A+\tau B)^\beta \\
&= \left[\det((1-\tau)A+\tau B)^{\frac{1}{n}}\right]^{\beta n} \\
&\geqslant \left[(1-\tau)\det A^{\frac{1}{n}} + \tau\det B^{\frac{1}{n}}\right]^{\beta n}
\end{aligned}$$

$$\geqslant \frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)\det A^{\frac{1}{n}}+v\det B^{\frac{1}{n}}]^{\beta n}-[\det A^{\frac{1-v}{n}}\det B^{\frac{v}{n}}]^{\beta n}\right]$$
$$+\left[\det A^{\frac{1-\tau}{n}}\det B^{\frac{\tau}{n}}\right]^{\beta n}$$
$$=\frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)\det A^{\frac{1}{n}}+v\det B^{\frac{1}{n}}]^{\beta n}-\det(A^{1-v}B^{v})^{\beta}\right]+\det\left(A^{1-\tau}B^{\tau}\right)^{\beta}.$$

Using the same method we can get (ii) similarly, so we omit it. □

THEOREM 3.3. *Let* $A,B,X\in M_n(\mathbb{C})$ *with* $A,B\in M_n^+(\mathbb{C})$, $\beta\geqslant 1$ *and* $0<v\leqslant\tau<1$. *Then for any unitarily invariant norm* $|||\cdot|||$

(i) If $|||XB|||\geqslant|||AX|||$, *we get*

$$[(1-\tau)|||AX|||+\tau|||XB|||]^{\beta}$$
$$\geqslant\frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)|||AX|||+v|||XB|||]^{\beta}-(|||AX|||^{1-v}|||XB|||^{v})^{\beta}\right]+|||A^{1-\tau}XB^{\tau}|||^{\beta};$$

(ii) If $|||AX|||\geqslant|||XB|||$, *we get*

$$[(1-v)|||AX|||+v|||XB|||]^{\beta}$$
$$\geqslant\frac{v(1-v)}{\tau(1-\tau)}\left[[(1-\tau)|||AX|||+\tau|||XB|||]^{\beta}-(|||AX|||^{1-\tau}|||XB|||^{\tau})^{\beta}\right]+|||A^{1-v}XB^{v}|||^{\beta}.$$

Proof. Suppose $|||XB|||\geqslant|||AX|||$ and by Corollary 2.3 and Lemma 3.2, we have

$$[(1-\tau)|||AX|||+\tau|||XB|||]^{\beta}-|||A^{1-\tau}XB^{\tau}|||^{\beta}$$
$$\geqslant[(1-\tau)|||AX|||+\tau|||XB|||]^{\beta}-(|||AX|||^{1-\tau}|||XB|||^{\tau})^{\beta}$$
$$\geqslant\frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)|||AX|||+v|||XB|||]^{\beta}-(|||AX|||^{1-v}|||XB|||^{v})^{\beta}\right].$$

Using the same method we can get (ii) similarly, so we omit it. □

THEOREM 3.4. *Let* $A,B\in M_n^+(\mathbb{C})$ *such that* $0\leqslant A\leqslant B,\beta\geqslant 1$ *and* $\frac{1}{2}<v\leqslant\tau\leqslant 1$. *Then*

$$\frac{K(h,2)^{\beta v}\|A\|_1^{\beta(1-v)}\|B\|_1^{\beta v}-\|(1-v)A+vB\|_1^{\beta}}{v}$$
$$\leqslant\frac{K(h,2)^{\beta\tau}\|A\|_1^{\beta(1-\tau)}\|B\|_1^{\beta\tau}-\|(1-\tau)A+\tau B\|_1^{\beta}}{\tau}$$

where $h=\frac{\operatorname{tr}(B)}{\operatorname{tr}(A)}$.

Proof. According to (2.6), we have

$$
\begin{aligned}
&\|(1-v)A+vB\|_1^{\beta} \\
&= [(1-v)\mathrm{tr}(A)+v\mathrm{tr}(B)]^{\beta} \\
&\geqslant K(h,2)^{\beta v}\mathrm{tr}(A)^{\beta(1-v)}\mathrm{tr}(B)^{\beta v} \\
&\quad -\frac{v}{\tau}[K(h,2)^{\beta\tau}\mathrm{tr}(A)^{\beta(1-\tau)}\mathrm{tr}(B)^{\beta\tau}-((1-\tau)\mathrm{tr}(A)+\tau\mathrm{tr}(B))^{\beta}] \\
&= K(h,2)^{\beta v}\|A\|_1^{\beta(1-v)}\|B\|_1^{\beta v}-\frac{v}{\tau}[K(h,2)^{\beta\tau}\|A\|_1^{\beta(1-\tau)}\|B\|_1^{\beta\tau}-\|(1-\tau)A+\tau B\|_1^{\beta}].
\end{aligned}
$$

This completes the proof. □

Applying Theorem 2.8, we also have

THEOREM 3.5. *Let* $A,B\in M_n^+(\mathbb{C})$, $\beta\geqslant 1$ *and* $0<v\leqslant\tau<1$. *Then*
(1) *If* $B\geqslant A\geqslant 0$, *we obtain*

$$\frac{\|(1-v)A+vB\|_1^{\beta}-(\|A\|_1!_v\|B\|_1)^{\beta}}{v(1-v)}\leqslant\frac{\|(1-\tau)A+\tau B\|_1^{\beta}-(\|A\|_1!_\tau\|B\|_1)^{\beta}}{\tau(1-\tau)};$$

(2) *If* $A\geqslant B\geqslant 0$, *we obtain*

$$\frac{\|(1-v)A+vB\|_1^{\beta}-(\|A\|_1!_v\|B\|_1)^{\beta}}{v(1-v)}\geqslant\frac{\|(1-\tau)A+\tau B\|_1^{\beta}-(\|A\|_1!_\tau\|B\|_1)^{\beta}}{\tau(1-\tau)}.$$

Proof. (1) We can write

$$\|(1-v)A+vB\|_1^{\beta}-(\|A\|_1!_v\|B\|)^{\beta}=((1-v)\mathrm{tr}A+v\mathrm{tr}B)^{\beta}-(\mathrm{tr}A!_v\mathrm{tr}B)^{\beta},$$

and a similar expression for τ in place of v. So, with the substitutions $a=\mathrm{tr}A$ and $b=\mathrm{tr}B$, we see the inequality claimed can be written

$$\frac{(a\nabla_v b)^{\beta}-(a!_v b)^{\beta}}{v(1-v)}\leqslant\frac{(a\nabla_\tau b)-(a!_\tau b)}{\tau(1-\tau)}.$$

Here again because of $a\nabla_v b\geqslant a!_v b$ we can interchange the left lower expression with the right upper and get this way an inequality which follows directly from Theorem 2.8 as $b\geqslant a$.

Using the same method we can get (2) similarly, so we omit it. □

Acknowledgement. This work was partially supported by the NNSF of China (No. 11771126; 12371139).

REFERENCES

[1] H. Alzer, C. M. da Fonseca and A. Kovačec, *Young-type inequalities and their matrix analogues*, Linear and Multilinear Algebra, **63**, 3 (2015), 622–635.

[2] C. He and L. Zou, *Some inequalities involoving unitarily invariant norms*, Math. Inequal. Appl., **15** (2012), 767–776.

[3] O. Hirzallah and F. Kittaneh, *Matrix Young inequalities for the Hilbert-Schmidt norm*, Linear Algebra Appl, **308** (2000), 77–84.

[4] R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge UP 1985.

[5] W. Liao and J. Wu, *Matrix inequalities for the difference between arithmetic mean and harmonic mean*, Ann. Funct. Anal., **6**, 3 (2015), 191–202.

[6] W. Liao, J. Wu and J. Zhao, *New version of reverse Young and Heinz mean inequalities with the Kantorovich constant*, Taiwanese Journal of Mathematics, **19** (2015), 467–479.

[7] D. S. Mitrinovic, *Analytic inequalities*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[8] Y. H. Ren, *Some results of Young-type inequalities*, RACSAM, **114**, article number 143 (2020).

[9] M. Sababheh, *Convexity and matrix means*, Linear Algebra Appl., **506** (2016), 588–602.

[10] M. Sababheh, *On the matrix harmonic mean*, J. Math. Inequal., **12**, 4 (2018), 901–920.

[11] C. Yang and Z. Wang, *Some new improvements of Young's inequalities*, J. Math. Inequal., **17**, 1 (2023), 205–217.

[12] H. Zuo, G. Shi and M. Fujii, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., **5** (2011), 551–556.

(Received October 6, 2023)

Xiangrun Yang
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007, P. R. China
e-mail: doss65130206@163.com

Changsen Yang
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007, P. R. China
e-mail: yangchangsen0991@sina.com

Haiying Li
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007, P. R. China
e-mail: lihaiying@htu.edu.cn