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RELATIVE GROWTH OF A COMPLEX POLYNOMIAL WITH RESTRICTED ZEROS

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Abstract. Let $p(z)$ be a polynomial of degree n with zero of multiplicity s at the origin and the remaining zeros be in $|z| \geqslant k$ or in $|z| \leqslant k$, $k>0$. In this paper, we investigate the relative growth of a polynomial $p(z)$ with respect to two circles $|z|=r$ and $|z|=R$ and obtain inequalities about the dependence of $|p(rz)|$ on $|p(Rz)|$, where $|z|=1$, for $0<r\leqslant R\leqslant k$ or $0<k\leqslant R\leqslant r$ while taking into account the placement of the zeros of the underlying polynomial. Our results improve as well as generalize certain well-known polynomial inequalities. Some numerical examples are also given in order to illustrate and compare graphically the obtained inequalities with some recent results.

1. Introduction

Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and $p'(z)$ its derivative. The study of polynomial inequalities that relate the norms of the polynomial on different circles in a disk in the complex plane and generalizing the classical polynomial inequalities is a fertile area in mathematical analysis for researchers which is important especially for its wide range of applications in various fields of science and engineering. Here, we study some of the new inequalities centered around Rivlin's inequality that relate the uniform norms of the polynomial on different circles in a disk. These inequalities play a vital role in the literature for its various applications in the geometric function theory and of course have their own intrinsic appeals. These approximate bounds are quite accurate when computed effectively for the demands of investigators and scientists. As a result, there is a constant need for updates and more precise bounds that are superior to those described in the literature. We begin with the well-known Bernstein's inequality [2] for the uniform norm on the unit disk in the plane: Namely, if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)| \leqslant n\max_{|z|=1}|p(z)| \tag{1.1}$$

and

$$\max_{|z|=R}|p(z)| \leqslant R^n\max_{|z|=1}|p(z)|, \quad \text{whenever } R\geqslant 1. \tag{1.2}$$

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Inequality (1.1) is a direct consequence of Bernstein's theorem on the derivative of a trigonometric polynomial [18], and inequality (1.2) follows from the maximum modulus theorem (see [16, Problem 269]). The reverse analogue of the inequality (1.2) whenever $R \leqslant 1$ is given by Varga [19], and he proved that if $p(z)$ is a polynomial of degree n, then for $0 \leqslant r \leqslant 1$

$$\max_{|z|=r} |p(z)| \geqslant r^n \max_{|z|=1} |p(z)|. \tag{1.3}$$

Equality in (1.3) holds whenever $p(z) = az^n$.

For the class of polynomials having no zero inside the unit circle, it was Rivlin [17] who proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 \leqslant r \leqslant 1$

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |p(z)|. \tag{1.4}$$

Equality holds in (1.4) if $p(z) = (z+a)^n$ whenever $|a| = 1$.

The above inequalities are the starting point of a rich literature concerning their extensions, generalizations and improvements in several directions, see the papers ([1, 3, 4, 5, 7, 10, 11, 13, 15]) to mention only a few. For a deeper understanding about this kind of inequalities and their applications, we refer to the monographs [14, 8].

It was Jain [10] who generalized Rivlin's inequality (1.4) by studying the relative growth of a polynomial $p(z)$ having no zero in the open disk $|z| < k$, with respect to two circles $|z| = r$ and $|z| = R$ whenever $0 \leqslant r < R \leqslant k$. He proved that if $p(z)$ has no zero in $|z| < k$, $k > 0$, then for $0 \leqslant r < R \leqslant k$

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{k+r}{k+R}\right)^n \max_{|z|=R} |p(z)|. \tag{1.5}$$

Dewan et al. [7] further improved inequality (1.5) by proving the following result which also involves $\min_{|z|=k} |p(z)|$. If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 \leqslant r < R \leqslant k$

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{k+r}{k+R}\right)^n \max_{|z|=R} |p(z)| + \left\{1 - \left(\frac{k+r}{k+R}\right)^n\right\} m^*, \tag{1.6}$$

where $m^* = \min_{|z|=k} |p(z)|$ throughout the paper.

Although, the above inequalities (1.4), (1.5) and (1.6) are best possible with equality holding for polynomials $p(z) = (z+a)^n$ satisfying $|a| = 1$ for (1.4) and $|a| = k$ for (1.5) and (1.6), definitely the bounds given by these inequalities do not address the issue of how far the zeros of the polynomial of the respective inequalities (1.4) or (1.5) and (1.6) lie outside the circle $|z| = 1$ or $|z| = k$. Now, naturally a question arises: Is there any way to refine the inequalities (1.4), (1.5) and (1.6) for the class of polynomials satisfying the same hypotheses of these inequalities, by capturing some information on the moduli of the zeros? Can we obtain a bound via two extreme coefficients of $p(z)$ which are informative about the distances of these zeros from the origin? In view of the example for the equality case in (1.4) which holds the property $|a_0| = |a_n|$, it should be possible to improve upon the bounds for polynomials $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ having no zero in $|z| < 1$, satisfying $|a_0| \neq |a_n|$. In this direction, Kumar [12] recently proved the following result which sharpens inequality (1.4) significantly. In fact, he proved that if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $0 < r \leqslant 1$

$$\max_{|z|=r} |p(z)| \geqslant \left[\left(\frac{1+r}{2} \right)^n + \left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|} \right) \left(\frac{1-r}{2} \right)^n \right] \max_{|z|=1} |p(z)|. \tag{1.7}$$

Further, Kumar and Milovanović [13] generalized inequality (1.7) by considering a zero free open disk $|z| < k$, $k \geqslant 1$ that if $p(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z| < k$, $k \geqslant 1$, then for $0 < r \leqslant 1$

$$\max_{|z|=r} |p(z)| \geqslant \left[\left(\frac{k+r}{k+1} \right)^n + \frac{1}{k^{n-1}} \left(\frac{|a_0| - |a_n| k^n}{|a_0| + |a_n|} \right) \left(\frac{1-r}{k+1} \right)^n \right] \max_{|z|=1} |p(z)|. \tag{1.8}$$

In this paper, we approach this side of the inequality and obtain a bound which further extends inequality (1.8) and generalizes as well as sharpens the inequalities (1.4), (1.5) and (1.6) significantly.

2. Main results

THEOREM 2.1. *If $p(z) = \sum_{\upsilon=s}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, except zero of multiplicity s at the origin $0 \leqslant s < n$, then for $0 \leqslant l < 1$ and $0 < r \leqslant R \leqslant \rho$, $\rho \leqslant k$*

$$\begin{aligned} \max_{|z|=r} |p(z)| \geqslant & \left[\left(\frac{k+r}{k+R} \right)^{n-s} + \left(\frac{R}{k} \right)^{n-s-1} \left(\frac{|a_s| - l\frac{m^*}{k^s} - |a_n| k^{n-s}}{|a_s| - l\frac{m^*}{k^s} + |a_n| R^{n-s}} \right) \right. \\ & \left. \times \left(\frac{R-r}{k+R} \right)^{n-s} \right] \left(\frac{r}{R} \right)^s \max_{|z|=R} |p(z)| \\ & + \left[1 - \left\{ \left(\frac{k+r}{k+R} \right)^{n-s} + \left(\frac{R}{k} \right)^{n-s-1} \left(\frac{|a_s| - l\frac{m^*}{k^s} - |a_n| k^{n-s}}{|a_s| - l\frac{m^*}{k^s} + |a_n| R^{n-s}} \right) \right. \right. \\ & \left. \left. \times \left(\frac{R-r}{k+R} \right)^{n-s} \right\} \right] \left(\frac{r}{k} \right)^s l m^*. \end{aligned} \tag{2.1}$$

REMARK 2.2. When $s = 0$, Theorem 2.1 reduces to the following extension as well as generalization of inequality (1.8) due to Kumar and Milovanović [13].

COROLLARY 2.3. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having no zero in* $|z|<k$, $k>0$, *then for* $0\leqslant l<1$ *and* $0<r\leqslant R\leqslant\rho$, $\rho\leqslant k$

$$\max_{|z|=r}|p(z)|\geqslant\left[\left(\frac{k+r}{k+R}\right)^{n}+\left(\frac{R}{k}\right)^{n-1}\left(\frac{|a_0|-lm^*-|a_n|k^n}{|a_0|-lm^*+|a_n|R^n}\right)\left(\frac{R-r}{k+R}\right)^{n}\right]\max_{|z|=R}|p(z)|$$
$$+\left[1-\left\{\left(\frac{k+r}{k+R}\right)^{n}+\left(\frac{R}{k}\right)^{n-1}\left(\frac{|a_0|-lm^*-|a_n|k^n}{|a_0|-lm^*+|a_n|R^n}\right)\left(\frac{R-r}{k+R}\right)^{n}\right\}\right]lm^*. \tag{2.2}$$

REMARK 2.4. For $s=0$ and further letting $l\to1$ in Theorem 2.1, we have under the same hypotheses the following result which gives an improved bound over inequality (1.6) due to Dewan et al. [7].

COROLLARY 2.5. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having no zero in* $|z|<k$, $k>0$, *then for* $0<r\leqslant R\leqslant\rho$, $\rho\leqslant k$

$$\max_{|z|=r}|p(z)|\geqslant\left[\left(\frac{k+r}{k+R}\right)^{n}+\left(\frac{R}{k}\right)^{n-1}\left(\frac{|a_0|-m^*-|a_n|k^n}{|a_0|-m^*+|a_n|R^n}\right)\left(\frac{R-r}{k+R}\right)^{n}\right]\max_{|z|=R}|p(z)|$$
$$+\left[1-\left\{\left(\frac{k+r}{k+R}\right)^{n}+\left(\frac{R}{k}\right)^{n-1}\left(\frac{|a_0|-m^*-|a_n|k^n}{|a_0|-m^*+|a_n|R^n}\right)\left(\frac{R-r}{k+R}\right)^{n}\right\}\right]m^*. \tag{2.3}$$

REMARK 2.6. Inequality (2.3) can be rewritten as

$$\max_{|z|=r}|p(z)|\geqslant\left(\frac{k+r}{k+R}\right)^{n}\max_{|z|=R}|p(z)|+\left[1-\left(\frac{k+r}{k+R}\right)^{n}\right]m^*$$
$$+\left(\frac{R}{k}\right)^{n-1}\left(\frac{|a_0|-m^*-|a_n|k^n}{|a_0|-m^*+|a_n|R^n}\right)\left(\frac{R-r}{k+R}\right)^{n}\left[\max_{|z|=R}|p(z)|-m^*\right]. \tag{2.4}$$

By Lemma 4.4, we have
$$\frac{|a_0|-m^*-|a_n|k^n}{|a_0|-m^*+|a_n|R^n}\geqslant0$$
and by minimum modulus principle, we have

$$m^*=\min_{|z|=k}|p(z)|\leqslant\min_{|z|=R}|p(z)|\leqslant\max_{|z|=R}|p(z)|,\quad \text{for } R\leqslant k,$$

which verifies our claim that Corollary 2.5 gives an improved bound over inequality (1.6) due to Dewan et al. [7].

REMARK 2.7. Setting $s=0$, $R=\rho=1$ and letting $l \to 1$, and using the similar argument as in Remark 2.6 for $R=1$, Theorem 2.1 provides the following improvement of a result proved by Aziz [1].

COROLLARY 2.8. *If $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<k$, $k \geqslant 1$, then for $0<r \leqslant 1$*

$$\max_{|z|=r}|p(z)| \geqslant \left[\left(\frac{k+r}{k+1}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0|-m^*-|a_n|k^n}{|a_0|-m^*+|a_n|}\right)\left(\frac{1-r}{k+1}\right)^n\right]\max_{|z|=1}|p(z)|$$
$$+\left[1-\left\{\left(\frac{k+r}{k+1}\right)^n + \frac{1}{k^{n-1}}\left(\frac{|a_0|-m^*-|a_n|k^n}{|a_s|-m^*+|a_n|}\right)\left(\frac{1-r}{k+1}\right)^n\right\}\right]m^*. \tag{2.5}$$

REMARK 2.9. Putting $s=0$, $R=k=1$ and letting $l \to 1$ in Theorem 2.1, we obtain the following improvement of the famous result due to Rivlin [17] by following the similar arguments as in Remark 2.6 for $k=R=1$.

COROLLARY 2.10. *If $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<1$, then for $0<r \leqslant 1$*

$$\max_{|z|=r}|p(z)| \geqslant \left[\left(\frac{1+r}{2}\right)^n + \left(\frac{|a_0|-m_1-|a_n|}{|a_0|-m_1+|a_n|}\right)\left(\frac{1-r}{2}\right)^n\right]\max_{|z|=1}|p(z)|$$
$$+\left[1-\left\{\left(\frac{1+r}{2}\right)^n + \left(\frac{|a_0|-m_1-|a_n|}{|a_s|-m_1+|a_n|}\right)\left(\frac{1-r}{2}\right)^n\right\}\right]m_1, \tag{2.6}$$

where $m_1=\min_{|z|=1}|p(z)|$ throughout the paper.

REMARK 2.11. By taking $s=l=0$, Theorem 2.1 provides an improvement of inequality (1.5) due to Jain [10].

COROLLARY 2.12. *If $p(z)=\sum_{\upsilon=s}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for $0<r \leqslant R \leqslant \rho$, $\rho \leqslant k$*

$$\max_{|z|=r}|p(z)| \geqslant \left[\left(\frac{k+r}{k+R}\right)^n + \left(\frac{R}{k}\right)^{n-1}\left(\frac{|a_0|-|a_n|k^n}{|a_0|+|a_n|R^n}\right)\left(\frac{R-r}{k+R}\right)^n\right]\max_{|z|=R}|p(z)|. \tag{2.7}$$

REMARK 2.13. When $s=l=0$, $\rho=1$, Theorem 2.1 yields the following result which gives an improved bound over the result proved by Dewan [6].

COROLLARY 2.14. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having no zero in* $|z|<k,\ k\geqslant 1$, *then for* $0<r\leqslant R\leqslant 1$

$$\max_{|z|=r}|p(z)|\geqslant\left[\left(\frac{k+r}{k+R}\right)^{n}+\left(\frac{R}{k}\right)^{n-1}\left(\frac{|a_0|-|a_n|k^n}{|a_0|+|a_n|R^n}\right)\left(\frac{R-r}{k+R}\right)^{n}\right]\max_{|z|=R}|p(z)|. \tag{2.8}$$

REMARK 2.15. Moreover, if we consider $s=l=0,\ R=\rho=1$ in Theorem 2.1, we have inequality (1.8) recently proved by Kumar and Milovanović [13].

REMARK 2.16. Futher, if we assign $s=0,\ k=1$ and letting $l\to 1$ in Theorem 2.1, we obtain the following result which generalizes as well as improves an inequality proved by Govil [9].

COROLLARY 2.17. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having no zero in* $|z|<1$, *then for* $0<r\leqslant R<1$

$$\begin{aligned}\max_{|z|=r}|p(z)|\geqslant&\left[\left(\frac{1+r}{1+R}\right)^{n}+R^{n-1}\left(\frac{|a_0|-m_1-|a_n|}{|a_0|-m_1+|a_n|}\right)\left(\frac{R-r}{1+R}\right)^{n}\right]\max_{|z|=R}|p(z)|\\&+\left[1-\left\{\left(\frac{1+r}{1+R}\right)^{n}+R^{n-1}\left(\frac{|a_0|-m_1-|a_n|}{|a_s|-m_1+|a_n|}\right)\left(\frac{R-r}{1+R}\right)^{n}\right\}\right]m_1.\end{aligned} \tag{2.9}$$

REMARK 2.18. Lastly, if $s=l=0,\ k=1$, Theorem 2.1 becomes the following recent result of Kumar and Milovanović [13].

COROLLARY 2.19. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having no zero in* $|z|<1$, *then for* $0<r\leqslant R\leqslant 1$

$$\max_{|z|=r}|p(z)|\geqslant\left[\left(\frac{1+r}{1+R}\right)^{n}+R^{n-1}\left(\frac{|a_0|-|a_n|}{|a_0|+|a_n|R^n}\right)\left(\frac{R-r}{1+R}\right)^{n}\right]\max_{|z|=R}|p(z)|. \tag{2.10}$$

REMARK 2.20. Inequalities (1.8) and (2.10) due to Kumar and Milovanović [13] have a limitation in the sense that for k in (0,1), we do not have analogous bound of inequalities (1.8) and (2.10) for $0<r\leqslant R\leqslant k$. It is easily seen that this loophole is compensated by Theorem 2.1. Moreover, for $k>1$, the limits of r and R extend from $(0,1]$ to $(0,k]$.

For the class of polynomials having all their zeros in $|z|\leqslant k,\ k\leqslant 1$, Aziz [1] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant k,\ k\leqslant 1$, then

$$\max_{|z|=r}|p(z)|\geqslant\left(\frac{r+k}{1+k}\right)\max_{|z|=1}|p(z)|,\quad \text{for } r\geqslant 1. \tag{2.11}$$

As an application of Theorem 2.1, we obtain the following generalization as well as improvement of (2.11) due to Aziz [1].

THEOREM 2.21. *If* $p(z) = \sum_{\upsilon=s}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k > 0$ *with zero of multiplicity* s *at the origin* $0 \leqslant s < n$, *then for* $0 \leqslant l < 1$ *and* $0 < k \leqslant \rho \leqslant R \leqslant r$

$$\begin{aligned}\max_{|z|=r}|p(z)| \geqslant & \left[\left(\frac{k+r}{k+R}\right)^{n-s} \cdot \left(\frac{R}{r}\right)^{n-s} + \left(\frac{k}{R}\right)^{n-s-1} \left(\frac{|a_n| - \frac{lm^*}{k^n} - |a_s|\frac{1}{k^{n-s}}}{|a_n| - \frac{lm^*}{k^n} + |a_s|\frac{1}{R^{n-s}}}\right)\right. \\ & \left.\times \left(\frac{r-R}{k+R}\right)^{n-s} \left(\frac{k}{r}\right)^{n-s}\right] \left(\frac{r}{R}\right)^n \max_{|z|=R}|p(z)| \\ & + \left[1 - \left\{ \left(\frac{k+r}{k+R}\right)^{n-s} \cdot \left(\frac{R}{r}\right)^{n-s} + \left(\frac{k}{R}\right)^{n-s-1} \left(\frac{|a_n| - \frac{lm^*}{k^n} - |a_s|\frac{1}{k^{n-s}}}{|a_n| - \frac{lm^*}{k^n} + |a_s|\frac{1}{R^{n-s}}}\right)\right.\right. \\ & \left.\left.\times \left(\frac{r-R}{k+R}\right)^{n-s} \left(\frac{k}{r}\right)^{n-s}\right\}\right] \left(\frac{r}{k}\right)^n lm^*. \end{aligned} \tag{2.12}$$

REMARK 2.22. Putting $s = 0$ and further letting $l \to 1$ in Theorem 2.21, we have the following interesting result.

COROLLARY 2.23. *If* $p(z) = \sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k > 0$, *then for* $0 < k \leqslant \rho \leqslant R \leqslant r$

$$\begin{aligned}\max_{|z|=r}|p(z)| \geqslant & \left[\left(\frac{k+r}{k+R}\right)^{n} \cdot \left(\frac{R}{r}\right)^{n} + \left(\frac{k}{R}\right)^{n-1} \left(\frac{|a_n| - \frac{m^*}{k^n} - |a_0|\frac{1}{k^{n}}}{|a_n| - \frac{m^*}{k^n} + |a_0|\frac{1}{R^{n}}}\right)\right. \\ & \left.\times \left(\frac{r-R}{k+R}\right)^{n} \left(\frac{k}{r}\right)^{n}\right] \left(\frac{r}{R}\right)^n \max_{|z|=R}|p(z)| \\ & + \left[1 - \left\{ \left(\frac{k+r}{k+R}\right)^{n} \cdot \left(\frac{R}{r}\right)^{n} + \left(\frac{k}{R}\right)^{n-1} \left(\frac{|a_n| - \frac{m^*}{k^n} - |a_0|\frac{1}{k^{n}}}{|a_n| - \frac{m^*}{k^n} + |a_0|\frac{1}{R^{n}}}\right)\right.\right. \\ & \left.\left.\times \left(\frac{r-R}{k+R}\right)^{n} \left(\frac{k}{r}\right)^{n}\right\}\right] \left(\frac{r}{k}\right)^n m^*. \end{aligned} \tag{2.13}$$

REMARK 2.24. Setting $s = 0$, $R = \rho = 1$ and further letting $l \to 1$ in Theorem 2.21, we get under the same hypotheses, the following improvement of (2.11) due to Aziz [1].

COROLLARY 2.25. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k,\ k\leqslant 1$, *then for* $1\leqslant r$

$$\begin{aligned}\max_{|z|=r}|p(z)| \geqslant & \left[\left(\frac{k+r}{k+1}\right)^n + k^{2n-1}\left(\frac{|a_n|k^n-m^*-|a_0|}{|a_n|k^n-m^*+|a_0|k^n}\right)\left(\frac{r-1}{k+1}\right)^n\right]\max_{|z|=1}|p(z)| \\ & +\left[1-\left\{\left(\frac{k+r}{k+1}\right)^n\cdot\frac{1}{r^n}+k^{n-1}\left(\frac{|a_n|k^n-m^*-|a_0|}{|a_n|k^n-m^*+|a_0|k^n}\right)\right.\right. \\ & \left.\left.\times\left(\frac{r-1}{k+1}\right)^n\left(\frac{k}{r}\right)^n\right\}\right]\left(\frac{r}{k}\right)^n m^*. \qquad (2.14)\end{aligned}$$

REMARK 2.26. Inequality (2.14) can be rewritten as

$$\begin{aligned}\max_{|z|=r}|p(z)| \geqslant & \left(\frac{k+r}{k+1}\right)^n\max_{|z|=1}|p(z)| + \left[1-\left(\frac{\frac{k}{r}+1}{k+1}\right)\right]\left(\frac{r}{k}\right)^n m^* \\ & + k^{n-1}\left(\frac{|a_n|k^n-m^*-|a_0|}{|a_n|k^n-m^*+|a_0|k^n}\right)\left(\frac{r-1}{k+1}\right)^n\left[k^n\max_{|z|=1}|p(z)|-m^*\right].\end{aligned}$$

By Lemma 4.5, we have

$$\frac{|a_n|k^n-m^*-|a_0|}{|a_n|k^n-m^*+|a_0|k^n}\geqslant 0. \qquad (2.15)$$

Suppose $\underline{p(z)}$ is a polynomial of degree n having all its zeros in $|z|\leqslant k,\ k\leqslant 1$. Then $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$ has all it zeros in $|z|\geqslant\frac{1}{k},\ \frac{1}{k}\geqslant 1$, i.e., has no zero in $|z|<\frac{1}{k},\ \frac{1}{k}\geqslant 1$, therefore applying minimum modulus principle to $q(z)$, we get $|q(z)|\geqslant\min_{|z|=\frac{1}{k}}|q(z)| = \frac{1}{k^n}\min_{|z|=k}|p(z)|$, for $|z|\leqslant\frac{1}{k},\frac{1}{k}\geqslant 1$. Hence, in particular, for $|z|=1$

$$|q(z)|\geqslant\frac{1}{k^n}\min_{|z|=k}|p(z)|. \qquad (2.16)$$

Also, for $|z|=1$, we know

$$|p(z)|=|q(z)|. \qquad (2.17)$$

From (2.16) and (2.17), we have

$$\max_{|z|=1}|p(z)|\geqslant\frac{1}{k^n}\min_{|z|=k}|p(z)|. \qquad (2.18)$$

The above two facts (2.15) and (2.18) verify our claim that under the same hypotheses, Corollary 2.25 gives improved bound over (2.11) due to Aziz [1].

REMARK 2.27. When $s=l=0$. Theorem 2.21 reduces to the following interesting result.

COROLLARY 2.28. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k,\ k>0$*, then for* $0<k\leqslant\rho\leqslant R\leqslant r$

$$\max_{|z|=r}|p(z)|\geqslant\left[\left(\frac{k+r}{k+R}\right)^n+\left(\frac{k}{R}\right)^{n-1}\left(\frac{|a_n|k^n-|a_0|}{|a_n|R^n+|a_0|}\right)\left(\frac{r-R}{k+R}\right)^n\right]\max_{|z|=R}|p(z)|. \quad (2.19)$$

REMARK 2.29. Further, if we assign $\rho=R=1$ and $s=l=0$ in Theorem 2.21, the following result recently proved by Kumar and Milovanović [13] is recovered.

COROLLARY 2.30. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k,\ k\leqslant 1$*, then for* $r\geqslant 1$

$$\max_{|z|=r}|p(z)|\geqslant\left[\left(\frac{k+r}{k+1}\right)^n+k^{n-1}\left(\frac{|a_n|k^n-|a_0|}{|a_n|+|a_0|}\right)\left(\frac{r-1}{k+1}\right)^n\right]\max_{|z|=1}|p(z)|. \quad (2.20)$$

REMARK 2.31. Lastly, if $k=1$ and $s=l=0$, Theorem 2.21 yeilds in particular the following result recently proved by Kumar and Milovanović [13].

COROLLARY 2.32. *If* $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant 1$*, then for* $1\leqslant R<r$

$$\max_{|z|=r}|p(z)|\geqslant\left[\left(\frac{1+r}{1+R}\right)^n+\frac{1}{R^{n-1}}\left(\frac{|a_n|-|a_0|}{|a_n|R^n+|a_0|}\right)\left(\frac{r-R}{1+R}\right)^n\right]\max_{|z|=R}|p(z)|. \quad (2.21)$$

REMARK 2.33. Corollaries 2.30 and 2.32 due to Kumar and Milovanović [13] have a limitation in the sense that for $k\geqslant 1$, we do not have analogous bounds of inequalities (2.20) and (2.21) for $1\leqslant k\leqslant R\leqslant r$. It is easily seen that this deficiency is compensated by Theorem 2.21. Moreover, for $k\leqslant 1$ the limits of r and R extend from $[1,\infty)$ to $[k,\infty)$.

3. Numerical examples and graphical illustration

EXAMPLE 3.1. Consider the polynomial $p(z)=z^3-18z^2+101z-168$ of degree 3 having all its zeros in $|z|\geqslant 3$. We take $k=2$. On the circle $|z|=R$, we have

$$|p(Re^{i\theta})|=\sqrt{(R^2+9-6R\cos\theta)(R^2+49-14R\cos\theta)(R^2+64-16R\cos\theta)}$$

and the level graphs for $R=0.4,1,1.5$ in $0\leqslant\theta<2\pi$ are presented in Fig. 1.

And also,

$$M_R=\max_{|z|=R}|p(z)|=R^3+18R^2+101R+168,$$

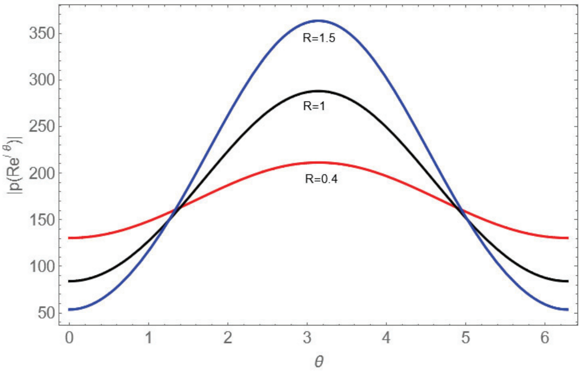

Figure 1: *Level graphs of the function* $\theta \longmapsto |p(Re^{i\theta})|$ *for* $R = 0.4, 1, 1.5$ *in* $0 \leqslant \theta \leqslant 2\pi$.

$$M_{0.4} = \max_{|z|=0.4} |p(z)| = 211.344$$

as well as $m^* = \min\limits_{|z|=2} |p(z)| = 30$. We consider the difference between $M_{0.4}$ and the right hand side of inequality (2.2),

$$\begin{aligned} \phi(l,R) = 211.344 &- \left(\frac{2.4}{2+R}\right)^3 \left(R^3 + 18R^2 + 101R + 168\right) - 30l \left[1 - \left(\frac{2.4}{2+R}\right)^3\right] \\ &- \left(\frac{R}{2}\right)^2 \cdot \frac{160 - 30l}{168 - 30l + R^3} \left(\frac{R - 0.4}{2+R}\right)^3 \left(R^3 + 18R^2 + 101R + 168 - 30l\right). \end{aligned}$$

In Fig. 2, we present the difference $\Delta(R)$, $0.2 \leqslant R \leqslant 2$ between the left $(M_{0.4})$ and right hand sides of inequalities (1.5), (2.3) and (2.7). i.e.,

$$\Delta(R) = \begin{cases} 211.344 - \left(\frac{2.4}{2+R}\right)^3 \left(R^3 + 18R^2 + 101R + 168\right), & \text{inequality (1.5)}, \\ \phi(1,R), & \text{inequality (2.3)}, \\ \phi(0,R), & \text{inequality(2.7)}. \end{cases}$$

REMARK 3.2. At any point on the $R-$axis, the inequality whose $\Delta(R)$ graph is nearer to the $R-$axis gives improved bound over the others. From Fig. 2, it is clear that inequality (2.3) gives the most improved bound for all values of R, $0.4 \leqslant R \leqslant 2$ for this particular example. When $R = 1$ inequalities (1.5), (2.3) and (2.7) reduce respectively to inequalities (1.4), (2.5) and (1.8) and hence at the point $R = 1$, $\Delta(R)$ corresponds to the difference between the left and right hand sides of inequalities (1.4), (2.5) and (1.8). In general, it is not possible to compare the bounds of inequalities (2.5) and (1.8) with regard to sharpness. However, from the graph in Fig. 2, it is evident that the bound (2.5) of Corollary 2.8 gives a significant improvement over the bound (1.8) due to Kumar and Milovanović [13] for this particular example 3.1.

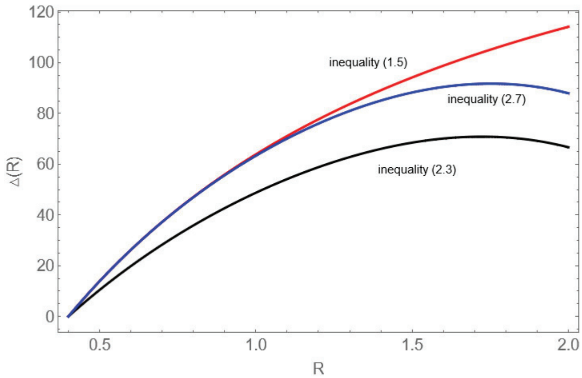

Figure 2: *Comparison of the difference* $R \longmapsto \Delta(R)$ *for* $0.4 \leqslant R \leqslant 2$ *in the inequalities* (1.5), (2.3) *and* (2.7).

EXAMPLE 3.3. Consider the polynomial $p(z) = 6z^2 - 5z + 1$ of degree 2 having all its zeros in $|z| \leqslant 0.5$. We take $k = 0.6$. On the circle $|z| = R$, we have

$$|p(Re^{i\theta})| = \sqrt{(4R^2 + 1 - 4R\cos\theta)\,(9R^2 + 1 - 6R\cos\theta)}$$

and their level graphs for $R = 1, 1.5, 2$ in $0 \leqslant \theta < 2\pi$ are presented in Fig. 3.

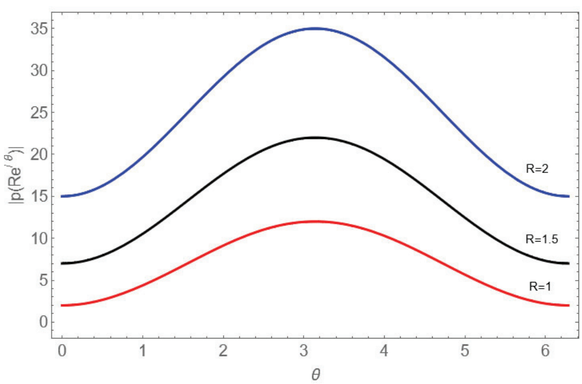

Figure 3: *Level graphs of the function* $\theta \longmapsto |p(Re^{i\theta})|$ *for* $R = 1, 1.5, 2$ *in* $0 \leqslant \theta \leqslant 2\pi$.

And also,

$$M_R = \max_{|z|=R} |p(z)| = 6R^2 + 5R + 1,$$

$$M_2 = \max_{|z|=2} |p(z)| = 35$$

as well as $m^* = \min\limits_{|z|=0.6} |p(z)| = 0.16$.

In Fig. 4, we present the difference $\Delta(R)$, $0.6 \leqslant R \leqslant 2$ between the left (M_2) and right hand sides of inequalities (2.13) and (2.19). i.e.,

$$\Delta(R) = \begin{cases} 35 - \left[\left(\frac{2.6}{0.6+R}\right)^2 + \left(\frac{0.6}{R}\right)^3 \cdot \frac{2.7778}{5.5556+\frac{1}{R^2}} \cdot \left(\frac{2-R}{0.6+R}\right)^2\right](6R^2+5R+1) \\ \quad - \left[1 - \left\{\left(\frac{2.6}{0.6+R}\right)^2 \cdot \frac{R^2}{4} + \frac{0.054}{R} \cdot \frac{2.7778}{5.5556+\frac{1}{R^2}} \cdot \left(\frac{2-R}{0.6+R}\right)^2\right\}\right] 1.7778, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{inequality (2.13)}, \\ 35 - \left[\left(\frac{2.6}{0.6+R}\right)^2 + \left(\frac{0.6}{R}\right)^3 \cdot \frac{3.222}{6+\frac{1}{R^2}} \cdot \left(\frac{2-R}{0.6+R}\right)^2\right](6R^2+5R+1), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{inequality (2.19)}. \end{cases}$$

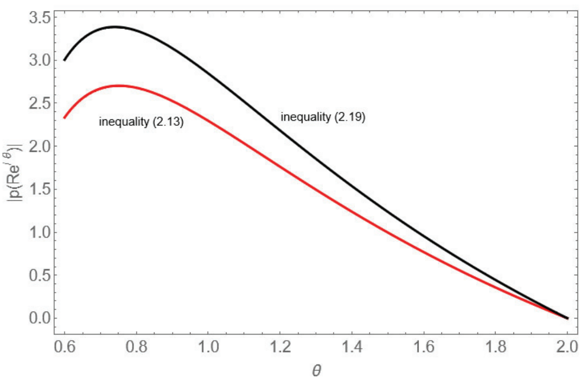

Figure 4: *Comparison of the difference* $R \longmapsto \Delta(R)$ *for* $0.6 \leqslant R \leqslant 2$ *in the inequalities* (2.13) *and* (2.19).

REMARK 3.4. At any point on $R-$axis, the inequality whose $\Delta(R)$ graph is nearer to the $R-$axis gives improved bound over the others. From Fig. 4, it is clear that inequality (2.13) provides the most improved bound for all values of R, $0.6 \leqslant R \leqslant 2$ for this particular example. When $R = 1$ inequalities (2.13) and (2.19) reduce to inequalities (2.14) and (2.20) and hence, at the point $R = 1$, $\Delta(R)$ corresponds to the difference between the left and right hand sides of inequalities (2.14) and (2.20). Generally, it is not possible to compare the bounds given by inequalities (2.14) and (2.20) in regard to sharpness. However, from the graph in Fig. 4, it is evident that the bound (2.14) of Corollary 2.23 gives significant improvement over the bound (2.20) due to Kumar and Milovanović [13] for this particular example 3.3.

4. Lemmas

We shall need the following lemmas in order to prove the above theorems and verify the claims. For a polynomial $p(z)$ of degree n, we will use $q(z)=z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

LEMMA 4.1. *For any $a\geqslant k^m$, $k\geqslant R>0$, and m is any positive integer, the function*

$$\begin{aligned}f(x)&=kR^{m-1}(a-k^m)(x-k)+kR^{m-1}(a-k^m)(x+R)\\&\quad+k^m(x-k)(a+R^m)-2R^m(ax-k^{m+1})\geqslant 0\end{aligned}\tag{4.1}$$

for all values of x in $[k,\infty)$.

Proof. Considering the first derivative of f with respect to x, we have

$$\begin{aligned}f'(x)&=kR^{m-1}(a-k^m)+kR^{m-1}(a-k^m)+k^m(a+R^m)-2aR^m\\&=kR^{m-1}(a-k^m)+R^{m-1}(k-R)(a-k^m)+a(k^m-R^m)\geqslant 0\end{aligned}$$

which is a non-decreasing function of x by derivative test, and

$$\begin{aligned}f(k)&=kR^{m-1}(a-k^m)(k+R)-2kR^m(a-k^m)\\&=kR^{m-1}(k-R)(a-k^m)\geqslant 0,\end{aligned}$$

which verifies the claim. $\square$

LEMMA 4.2. *If $a\geqslant k^m$, $b\geqslant k$, $k\geqslant R>0$, and m is any positive integer, then*

$$\left(\frac{R}{k}\right)^{m-1}\frac{a-k^m}{a+R^m}\cdot\frac{b-k}{b+R}+\left(\frac{R}{k}\right)^{m-1}\frac{a-k^m}{a+R^m}+\frac{b-k}{b+R}\geqslant\left(\frac{R}{k}\right)^{m}\frac{ab-k^{m+1}}{ab+R^{m+1}}\cdot\tag{4.2}$$

Proof. We need to show

$$\left(\frac{R}{k}\right)^{m-1}\frac{a-k^m}{a+R^m}\cdot\frac{b-k}{b+R}+\left(\frac{R}{k}\right)^{m-1}\frac{a-k^m}{a+R^m}+\frac{b-k}{b+R}-\left(\frac{R}{k}\right)^{m}\frac{ab-k^{m+1}}{ab+R^{m+1}}\geqslant 0.$$

Equivalently, it suffices to show that

$$\begin{aligned}&kR^{m-1}\left(ab+R^{m+1}\right)(a-k^m)(b-k)+kR^{m-1}\left(ab+R^{m+1}\right)(a-k^m)(b+R)\\&+k^m(b-k)\left(ab+R^{m+1}\right)(a+R^m)-R^m\left(ab-k^{m+1}\right)(a+R^m)(b+R)\geqslant 0.\end{aligned}$$

Since

$$(a+R^m)(b+R)=ab+R^{m+1}+aR+bR^m$$

and

$$ab+R^{m+1}-aR-bR^m=(a-R^m)(b-R)\geqslant 0,$$

we must have $2(ab+R^{m+1}) \geqslant (a+R^m)(b+R)$ and therefore it is sufficient to prove that

$$kR^{m-1}(a-k^m)(b-k) + kR^{m-1}(a-k^m)(b+R)+k^m(b-k)(a+R^m) \\ -2R^m(ab-k^{m+1}) \geqslant 0,$$

which is always true by Lemma 4.1 and hence the proof is complete. $\square$

LEMMA 4.3. *For any $0<r\leqslant R\leqslant k$, and $R_l \geqslant k>0$, $1\leqslant l\leqslant n$, we have*

$$\prod_{l=1}^{n}\frac{r+R_l}{R+R_l} \geqslant \left(\frac{k+r}{k+R}\right)^n + \left(\frac{R}{k}\right)^{n-1}\left[\frac{R_1R_2\cdots R_n-k^n}{R_1R_2\cdots R_n+R^n}\right]\left(\frac{R-r}{k+R}\right)^n. \tag{4.3}$$

Proof. We prove the result by induction on n. The identity

$$\frac{r+R_1}{R+R_1} = \left(\frac{k+r}{k+R}\right)+\left[\frac{R_1-k}{R_1+R}\right]\left(\frac{R-r}{k+R}\right), \tag{4.4}$$

justifies the validity of (4.3) for $n=1$. Let us assume that (4.3) is true for $n=m$. Then using the result for m and with the help of (4.4), we will have

$$\begin{aligned}
\prod_{l=1}^{m+1}\frac{r+R_l}{R+R_l} &= \left(\prod_{l=1}^{m}\frac{r+R_l}{R+R_l}\right)\left(\frac{r+R_{m+1}}{R+R_{m+1}}\right)\\
&\geqslant \left[\left(\frac{k+r}{k+R}\right)^m + \left(\frac{R}{k}\right)^{m-1}\left[\frac{R_1R_2\cdots R_m-k^m}{R_1R_2\cdots R_m+R^m}\right]\left(\frac{R-r}{k+R}\right)^m\right]\\
&\quad\times\left[\left(\frac{k+r}{k+R}\right)+\left[\frac{R_{m+1}-k}{R_{m+1}+R}\right]\left(\frac{R-r}{k+R}\right)\right]\\
&= \left(\frac{k+r}{k+R}\right)^m + \left(\frac{R}{k}\right)^{m-1}\left[\frac{R_1R_2\cdots R_m-k^m}{R_1R_2\cdots R_m+R^m}\right]\left[\frac{R_{m+1}-k}{R_{m+1}+R}\right]\left(\frac{R-r}{k+R}\right)^{m+1}\\
&\quad+\left(\frac{k+r}{k+R}\right)^m\left(\frac{R-r}{k+R}\right)\left[\frac{R_{m+1}-k}{R_{m+1}+R}\right]\\
&\quad+\left(\frac{R}{k}\right)^{m-1}\left(\frac{R-r}{k+R}\right)^m\left(\frac{k+r}{k+R}\right)\left[\frac{R_1R_2\cdots R_m-k^m}{R_1R_2\cdots R_m+R^m}\right].
\end{aligned}$$

Therefor, we will have

$$\begin{aligned}
\prod_{l=1}^{m+1}\frac{r+R_l}{R+R_l} \geqslant \left(\frac{k+r}{k+R}\right)^{m+1} &+ \left(\frac{R-r}{k+R}\right)^{m+1}\\
&\times\left[\left(\frac{R}{k}\right)^{m-1}\left[\frac{R_1R_2\cdots R_m-k^m}{R_1R_2\cdots R_m+R^m}\right]\left[\frac{R_{m+1}-k}{R_{m+1}+R}\right]\right.\\
&\left.+\frac{R_{m+1}-k}{R_{m+1}+R}+\left(\frac{R}{k}\right)^{m-1}\frac{R_1R_2\cdots R_m-k^m}{R_1R_2\cdots R_m+R^m}\right].
\end{aligned}$$

Applying Lemma 4.2 to the second term in the right hand side of the above inequality, we obtain

$$\prod_{l=1}^{m+1}\frac{r+R_l}{R+R_l} \geqslant \left(\frac{k+r}{k+R}\right)^{m+1} + \left(\frac{R}{k}\right)^{m}\left[\frac{R_1R_2\cdots R_{m+1}-k^{m+1}}{R_1R_2\cdots R_{m+1}+R^{m+1}}\right]\left(\frac{R-r}{k+R}\right)^{m+1},$$

by which the method of induction is completed. □

LEMMA 4.4. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z|<k,\ k>0$*, then for any complex number* α *with* $|\alpha|<1$ *and* $m^*=\min_{|z|=k}|p(z)|$

$$|a_0|-|\alpha|m^*-k^n|a_n|\geqslant 0. \tag{4.5}$$

Proof. By hypothesis, $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\geqslant k,\ k>0$. Then, the polynomial $P(z)=e^{-i\arg a_0}p(z)$ has the same zeros as $p(z)$. Now,

$$\begin{aligned}P(z)&=e^{-i\arg a_0}\left\{|a_0|e^{i\arg a_0}+a_1z+\cdots+a_{n-1}z^{n-1}+a_nz^n\right\}\\&=|a_0|+e^{-i\arg a_0}\left\{a_1z+\cdots+a_{n-1}z^{n-1}+a_nz^n\right\}.\end{aligned} \tag{4.6}$$

Now, on $|z|=k$ for any complex number α with $|\alpha|<1$ and $m^*=\min_{|z|=k}p(z)\neq 0$, we have

$$|\alpha|m^*<m^*\leqslant|P(z)|.$$

Then by Rouche's theorem, $R(z)=P(z)-|\alpha|m^*$ has all its zeros in $|z|>k$ and in case $m^*=0$, $R(z)=P(z)$. Thus, in any case $R(z)$ has all its zeros in $|z|\geqslant k$. Now, applying Vieta's formula to $R(z)$, we get

$$\frac{|a_0|-|\alpha|m^*}{|a_n|}\geqslant k^n,$$

i.e.,

$$|a_0|-|\alpha|m^*-k^n|a_n|\geqslant 0. \quad \square$$

LEMMA 4.5. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant k,\ k>0$*, then for any complex number* α *with* $|\alpha|<1$ *and* $m^*=\min_{|z|=k}|p(z)|$

$$|a_n|k^n-|\alpha|m^*-|a_0|\geqslant 0. \tag{4.7}$$

Proof. Following the same argument as in the beginning of the proof of Lemma 4.4, it follows that, $R(z) = p(z) + |\alpha|m^*$ has all its zeros in $|z| \leqslant k$. Now, applying Viet's formula to $R(z)$, we get

$$\frac{|a_0| + |\alpha|m^*}{|a_n|} \leqslant k^n,$$

i.e.,

$$k^n|a_n| - |\alpha|m^* - |a_0| \geqslant 0. \quad \square$$

5. Proofs of the theorems

Proof of Theorem 2.1. Without loss of generality, we assume that $p(z) = z^s h(z)$, where $h(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z| \geqslant k$, $k > 0$.

Now, $\min\limits_{|z|=k} |h(z)| = \min\limits_{|z|=k} \left|\frac{p(z)}{z^s}\right| = \frac{1}{k^s} \min\limits_{|z|=k} |p(z)| = \frac{m^*}{k^s}$, where $m^* = \min\limits_{|z|=k} |p(z)|$.

By Rouche's Theorem, for any complex number α with $|\alpha| < 1$, the polynomial

$$F(z) = h(z) + \frac{m^*\alpha}{k^s} = \frac{p(z)}{z^s} + \frac{m^*\alpha}{k^s} \tag{5.1}$$

has all its zeros in $|z| \geqslant k$.

If $R_1e^{i\theta_1}, R_2e^{i\theta_2}, \cdots, R_{n-s}e^{i\theta_{n-s}}$, are the zeros of $F(z)$, then for any $0 \leqslant r \leqslant R$, and $0 \leqslant \theta \leqslant 2\pi$

$$\left|\frac{F(re^{i\theta})}{F(Re^{i\theta})}\right| = \left|\frac{\frac{p(re^{i\theta})}{(re^{i\theta})^s} + \frac{m^*\alpha}{k^s}}{\frac{p(Re^{i\theta})}{(Re^{i\theta})^s} + \frac{m^*\alpha}{k^s}}\right|. \tag{5.2}$$

Also, we have

$$\begin{aligned}
\frac{|F(re^{i\theta})|}{|F(Re^{i\theta})|} &= \prod_{l=1}^{n-s} \frac{|re^{i\theta} - R_le^{i\theta_l}|}{|Re^{i\theta} - R_le^{i\theta_l}|} \\
&= \prod_{l=1}^{n-s} \frac{|re^{i(\theta-\theta_l)} - R_l|}{|Re^{i(\theta-\theta_l)} - R_l|} \\
&= \prod_{l=1}^{n-s} \left(\frac{r^2 + R_l^2 - 2rR_l\cos(\theta-\theta_l)}{R^2 + R_l^2 - 2RR_l\cos(\theta-\theta_l)}\right)^{\frac{1}{2}} \\
&\geqslant \prod_{l=1}^{n-s} \frac{r + R_l}{R + R_l}.
\end{aligned}$$

Therefore we have

$$|F(re^{i\theta})| \geqslant \prod_{l=1}^{n-s} \frac{r + R_l}{R + R_l} |F(Re^{i\theta})|. \tag{5.3}$$

Now applying Lemma 4.3 to the right hand side of the inequality (5.3) and using the fact that

$$R_1, R_2, \cdots, R_{n-s} = \frac{|a_s + \alpha m^*|}{|a_n|},$$

we have the following inequality

$$|F(re^{i\theta})| \geqslant \left[\left(\frac{k+r}{k+R}\right)^{n-s} + \left(\frac{R}{k}\right)^{n-s}\left(\frac{|a_s+\frac{m^*\alpha}{k^s}|-|a_n|k^{n-s}}{|a_s+\frac{m^*\alpha}{k^s}|+|a_n|R^{n-s}}\right)\right.$$
$$\left.\times\left(\frac{R-r}{k+R}\right)^{n-s}\right]|F(Re^{i\theta})|.$$

Now, using the fact that the function $\frac{x-a}{x+b}$ is a non-decreasing function of $x \neq -b$ for $a \geqslant 0, b \geqslant 0$, and $|a_s + \frac{m^*\alpha}{k^s}| \geqslant |a_s| - |\frac{m^*\alpha}{k^s}|$, we have

$$|F(re^{i\theta})| \geqslant \left[\left(\frac{k+r}{k+R}\right)^{n-s} + \left(\frac{R}{k}\right)^{n-s}\left(\frac{|a_s|-\frac{m^*|\alpha|}{k^s}-|a_n|k^{n-s}}{|a_s|-\frac{m^*|\alpha|}{k^s}+|a_n|R^{n-s}}\right)\right.$$
$$\left.\times\left(\frac{R-r}{k+R}\right)^{n-s}\right]|F(Re^{i\theta})|. \tag{5.4}$$

From (5.2) and (5.4), we have

$$\max_{\theta\in[0,2\pi)}\left|\frac{p(re^{i\theta})}{(re^{i\theta})^s}+\frac{m^*\alpha}{k^s}\right| \geqslant \left[\left(\frac{k+r}{k+R}\right)^{n-s} + \left(\frac{R}{k}\right)^{n-s}\left(\frac{|a_s|-\frac{m^*|\alpha|}{k^s}-|a_n|k^{n-s}}{|a_s|-\frac{m^*|\alpha|}{k^s}+|a_n|R^{n-s}}\right)\right.$$
$$\left.\times\left(\frac{R-r}{k+R}\right)^{n-s}\right]\max_{\theta\in[0,2\pi)}\left|\frac{p(Re^{i\theta})}{(Re^{i\theta})^s}+\frac{m^*\alpha}{k^s}\right|. \tag{5.5}$$

Let $\theta_0 \in [0,2\pi)$ be such that

$$\max_{\theta\in[0,2\pi)}\left|\frac{p(re^{i\theta})}{(re^{i\theta})^s}+\frac{m^*\alpha}{k^s}\right| = \left|\frac{p(re^{i\theta_0})}{(re^{i\theta_0})^s}+\frac{m^*\alpha}{k^s}\right|. \tag{5.6}$$

We choose the argument of α in the right-hand side of (5.6) such that

$$\left|\frac{p(re^{i\theta_0})}{(re^{i\theta_0})^s}+\frac{m^*\alpha}{k^s}\right| = \left|\frac{p(re^{i\theta_0})}{(re^{i\theta_0})^s}\right| - \frac{m^*|\alpha|}{k^s}$$
$$\leqslant \max_{\theta\in[0,2\pi)}\left|\frac{p(re^{i\theta_0})}{(re^{i\theta_0})^s}\right| - \frac{m^*|\alpha|}{k^s}. \tag{5.7}$$

Also,

$$\left|\frac{p(Re^{i\theta})}{(Re^{i\theta})^s}\right| - \frac{m^*|\alpha|}{k^s} \leqslant \left|\frac{p(Re^{i\theta})}{(Re^{i\theta})^s}+\frac{m^*\alpha}{k^s}\right|. \tag{5.8}$$

Using (5.7) and (5.8) in (5.5), we get

$$\max_{\theta\in[0,2\pi)}\left|\frac{p(re^{i\theta_0})}{(re^{i\theta_0})^s}\right| - \frac{m^*|\alpha|}{k^s} \geqslant \left[\left(\frac{k+r}{k+R}\right)^{n-s} + \left(\frac{R}{k}\right)^{n-s}\left(\frac{|a_s|-\frac{m^*|\alpha|}{k^s}-|a_n|k^{n-s}}{|a_s|-\frac{m^*|\alpha|}{k^s}+|a_n|R^{n-s}}\right)\right.$$
$$\left.\times\left(\frac{R-r}{k+R}\right)^{n-s}\right]\max_{\theta\in[0,2\pi)}\left\{\left|\frac{p(Re^{i\theta})}{(Re^{i\theta})^s}\right| - \frac{m^*|\alpha|}{k^s}\right\}.$$

i.e.,

$$\frac{\max\limits_{|z|=r}|p(z)|}{r^s}-\frac{m^*|\alpha|}{k^s}\geqslant\left[\left(\frac{k+r}{k+R}\right)^{n-s}+\left(\frac{R}{k}\right)^{n-s}\left(\frac{|a_s|-\frac{m^*|\alpha|}{k^s}-|a_n|k^{n-s}}{|a_s|-\frac{m^*|\alpha|}{k^s}+|a_n|R^{n-s}}\right)\right.$$
$$\left.\times\left(\frac{R-r}{k+R}\right)^{n-s}\right]\left\{\frac{\max\limits_{|z|=R}|p(z)|}{R^s}-\frac{m^*|\alpha|}{k^s}\right\}. \tag{5.9}$$

Setting $|\alpha|=l$, $0\leqslant l<1$ in (5.9) gives

$$\max_{|z|=r}|p(z)|\geqslant\frac{lm^*r^s}{k^s}+\left[\left(\frac{k+r}{k+R}\right)^{n-s}+\left(\frac{R}{k}\right)^{n-s}\left(\frac{|a_s|-\frac{lm^*}{k^s}-|a_n|k^{n-s}}{|a_s|-\frac{lm^*}{k^s}+|a_n|R^{n-s}}\right)\right.$$
$$\left.\times\left(\frac{R-r}{k+R}\right)^{n-s}\right]\left(\frac{r}{R}\right)^s\left\{\max_{|z|=R}|p(z)|-\frac{lm^*R^s}{k^s}\right\},$$

which on simplification gives the required result. □

Proof of Theorem 2.21. If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k>0$ with zero of multiplicity s at the origin $0\leqslant s<n$, then $q(z)=z^n\overline{p\left(\frac{1}{\bar z}\right)}$, is a polynomial of degree at most $n-s$ having no zero in $|z|<\frac{1}{k}$, $\frac{1}{k}>0$. Now, if $k\leqslant R\leqslant r$, then $\frac{1}{r}\leqslant\frac{1}{R}\leqslant\frac{1}{k}$. Therefore applying Corollary 2.3 to the polynomial $q(z)$, we get

$$\max_{|z|=\frac{1}{r}}|q(z)|\geqslant\left[\left(\frac{\frac{1}{k}+\frac{1}{r}}{\frac{1}{k}+\frac{1}{R}}\right)^{n-s}+\left(\frac{k}{R}\right)^{n-s-1}\left(\frac{|a_n|-lm'-|a_s|\frac{1}{k^{n-s}}}{|a_n|-lm'+|a_s|\frac{1}{R^{n-s}}}\right)\right.$$
$$\left.\times\left(\frac{\frac{1}{R}-\frac{1}{r}}{\frac{1}{k}+\frac{1}{R}}\right)^{n-s}\right]\max_{|z|=\frac{1}{R}}|q(z)|$$
$$+\left[1-\left\{\left(\frac{\frac{1}{k}+\frac{1}{r}}{\frac{1}{k}+\frac{1}{R}}\right)^{n-s}+\left(\frac{k}{R}\right)^{n-s-1}\left(\frac{|a_n|-lm'-|a_s|\frac{1}{k^{n-s}}}{|a_n|-lm'+|a_s|\frac{1}{R^{n-s}}}\right)\right.\right.$$
$$\left.\left.\times\left(\frac{\frac{1}{R}-\frac{1}{r}}{\frac{1}{k}+\frac{1}{R}}\right)^{n-s}\right\}\right]m'. \tag{5.10}$$

where $m'=\min\limits_{|z|=\frac{1}{k}}|q(z)|$.

Substituting the following results

$$\max_{|z|=\frac{1}{r}}|q(z)|=\frac{1}{r^n}\max_{|z|=r}|p(z)|,\quad\max_{|z|=\frac{1}{R}}|q(z)|=\frac{1}{R^n}\max_{|z|=R}|p(z)|$$

and $\min\limits_{|z|=\frac{1}{k}}|q(z)| = \dfrac{1}{k^n}\min\limits_{|z|=k}|p(z)|$, in (5.10) and simplifying, we finally have

$$\begin{aligned}
\max_{|z|=r}|p(z)| \geqslant & \Bigg[\left(\frac{k+r}{k+R}\right)^{n-s}\cdot\left(\frac{R}{r}\right)^{n-s}+\left(\frac{k}{R}\right)^{n-s-1}\left(\frac{|a_n|-\frac{lm^*}{k^n}-|a_s|\frac{1}{k^{n-s}}}{|a_n|-\frac{lm^*}{k^n}+|a_s|\frac{1}{R^{n-s}}}\right) \\
&\times\left(\frac{r-R}{k+R}\right)^{n-s}\left(\frac{k}{r}\right)^{n-s}\Bigg]\left(\frac{r}{R}\right)^n\max_{|z|=R}|p(z)| \\
&+\Bigg[1-\Bigg\{\left(\frac{k+r}{k+R}\right)^{n-s}\cdot\left(\frac{R}{r}\right)^{n-s}+\left(\frac{k}{R}\right)^{n-s-1}\left(\frac{|a_n|-\frac{lm^*}{k^n}-|a_s|\frac{1}{k^{n-s}}}{|a_n|-\frac{lm^*}{k^n}+|a_s|\frac{1}{R^{n-s}}}\right) \\
&\times\left(\frac{r-R}{k+R}\right)^{n-s}\left(\frac{k}{r}\right)^{n-s}\Bigg\}\Bigg]\left(\frac{r}{k}\right)^n lm^*, \qquad (5.11)
\end{aligned}$$

where $m^* = \min\limits_{|z|=k}|p(z)|$.

This completes the proof of Theorem 2.21. $\square$

6. Conclusion

In the past few years, a series of papers related to the Rivlin-type inequalities has been published and significant advances have been achieved in different directions. In this paper, we continue the study of this type of inequality for polynomials, following up on a study started by various authors in the recent past. Generally, under similar hypotheses, Rivlin's inequality has been improved and generalized by adopting two different approaches (see the papers [3, 4, 5, 7, 11, 13, 15]) to mention only a few. More precisely, we adopt one of these approaches and established some new inequalities while taking into account the placement of the zeros of the underlying polynomial. Our results generalize as well as refine some well-known polynomial inequalities. The techniques we have used in this paper could implicate further work in polynomial inequalities. Two numerical examples are given in order to graphically illustrate and compare the obtained inequalities with some recent results.

Declarations

Conflict of interest. The authors declare that there is no conflict of interest.

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REFERENCES

[1] A. AZIZ, *Growth of polynomials whose zeros are within or outside a circle*, Bull. Aust. Math. Soc., **35** (2), 1987, 247–256.

[2] S. BERNSTEIN, *Lecons Sur Les Propriétés extrémales et la meilleure approximation desfunctions analytiques d'une fonctions reele*, Gauthier-Villars (Paris, 1926).

[3] B. CHANAM, K. B. DEVI, K. KRISHNADAS AND M. T. DEVI, *On maximum modulus of polynomials with restricted zeros*, Bull. Iranian Math. Soc., **48**, 2022, 1325–1338, https://doi.org/10.1007/s41980-021-00575-x.

[4] B. CHANAM AND K. K. DEWAN, *Inequalities for a polynomial and its derivative*, J. Math. Anal. Appl., **336**, 2007, 171–179.

[5] M. T. DEVI, T. B. SINGH AND B. CHANAM, *Improvement and generalization of polynomial inequality of T. J. Rivlin*, Sao Paulo J. Math. Sci., **17**, 2023, 1006–1016, https://doi.org/10.1007/s40863-022-00300-4.

[6] K. K. DEWAN, *Inequalities for polynomial and its derivative*, J. Math. Anal. Appl., **190**, 1995, 625–629.

[7] K. K. DEWAN, R. S. YADAV AND M. S. PUKHTA, *Inequalities for a polynomial and its derivative*, Math. Inequal. and Appl., **2** (2), 1999, 203–205.

[8] R. B. GARDNER, N. K. GOVIL AND G. V. MILOVANOVIĆ, *Extremal Problems and Inequalities of Markov-Bernstein Type for Algebraic Polynomials*, Mathematical Analysis and its Application, Elsevier/Academic Press, London, 2022.

[9] N. K. GOVIL, *On the maximum modulus of polynomials*, J. Math. Anal. Appl., **112** (1), 1985, 253–258.

[10] V. K. JAIN, *On maximum modulus of polynomials with zeros outside a circle*, Glasnik Matematicki, **29**, 1994, 267–274.

[11] KRISHNADAS K., NGAMCHUI R. AND CHANAM B., *Generalized and extended versions of Ankeny-Rivlin and improved, Generalized, and extended versions of Rivlin type inequalities for the s^{th} derivative of a polynomial*, Mathematics, **9**, 887, 2021, https://doi.org/10.3390/math9080887.

[12] P. KUMAR, *A remark on a theorem of Rivlin*, C. R. Acad. Bulg. Sci., **74** (12), 2021, 1723–1728.

[13] P. KUMAR AND G. V. MILOVANOVIĆ, *On sharpening and generalization of Rivlin's inequality*, Turk. J. Math., **46**, 2022, 1436–1445.

[14] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ AND D. S. RASSIAS, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World Scientific Publishing Co. Singapore, 1994.

[15] A. MIR, A. WANI, I. HUSSIAN, *A note on a theorem of T. J. Rivlin*, Anal. Theory Appl. **34** (4), 2018, 293–296.

[16] G. PÓLYA AND G. SZEGÖ, *Aufgaben and Lehratze ous der Aalysis*, Springer, Barlin, (1925).

[17] T. J. RIVLIN, *On the maximum modulus of polynomials*, Amer. Math. Monthly., **67** (3), 1960, 251–253.

[18] A. C. SCHAEFFER, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc., **47** (8), 1941, 565–579.

[19] R. S. VARGA, *A comparison of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomial*, J. Soc. Indust. Appl. Math., **5** (2), 1957, 39–46.

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