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JENSEN–MARSHALL–KY FAN–TYPE INEQUALITIES AND THEIR APPLICATIONS IN BUSINESS PROFIT MANAGEMENT MODEL

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Abstract. This paper will introduce the theory of ϕ-Jensen coefficient. By means of the functional analysis, linear algebra, discrete mathematics and inequality theories with proper hypotheses, the Jensen-type inequality, Marshall-type inequality and the Ky Fan-type inequality are obtained as follows:

$$\widehat{f}|_{\phi} \geqslant \widehat{g}|_{\phi},\ \widetilde{f}_{\gamma} \geqslant \widetilde{g}_{\gamma} \text{ and } \widehat{\varphi}_{\gamma} \geqslant \widehat{(1-\varphi)}_{\gamma},$$

respectively, as well as we also displayed the applications of our main results in business profit management model, and some conditions such that $\mathbf{p} \prec_{\gamma} \mathbf{e}$ or $\mathbf{p} \backsim_{\gamma} \mathbf{e}$ hold are obtained, where $\mathbf{p}$ is the profit function and $\mathbf{e}$ is the cost function.

1. Introduction

Stability is an essential attribute of any random variable [7–10, 12, 13, 16, 17, 20, 24, 26, 30, 32]. The variance [8, 9, 12, 13, 16–19, 30, 32] and the coefficient of variation [24, 26] with coefficient of stable are important stability features of a random variable, their research and applications are important topics in mathematics.

As pointed out in [2], the theory of inequalities plays an important role in all the fields of mathematics, and the concept of mean value [7, 11, 21] is the most prominent in the theory, and the p-power mean [30] is the crucial one. The research of the Jensen-Marshall-Ky Fan-type inequalities [3–6, 15, 22, 23, 27–29] are important in the mean value, analysis of variance and nonlinear analysis theories. Unfortunately, it is very difficult to establish new Jensen-Marshall-Ky Fan-type inequalities. Therefore, it is of theoretical significance that to establish new Jensen-Marshall-Ky Fan-type inequalities. Since the Ky Fan-type inequality [15, 29] can be used to study business profit management model, it is of application value that to establish a new Ky Fan-type inequality.

In this paper, we will introduce the theory of ϕ-Jensen coefficient, this theory is based on our previous works, see [30–33]. In Sections 3–5, we will establish several Jensen-Marshall-Ky Fan-type inequalities. In particular, we will weaken the conditions for the Ky Fan-type inequality $\widehat{(1-\varphi)}_{\gamma} \leqslant \widehat{\varphi}_{\gamma}$ since the traditional conditions for the

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inequality are very strong. In Section 6, we will display the applications of our main results in business profit management model.

The research tools of the paper include the functional analysis, linear algebra, discrete mathematics, probability, statistics, mean value and inequality theories with the Mathematica software, especially the functional analysis.

2. Basic theory and preliminary results

Let $X \in \Omega$ be a random variable and its *probability density function* [32, 33] be $p : \Omega \to (0,\infty) \wedge \int_\Omega p = 1$, where $\Omega \subseteq \mathbb{R}^m$ be a measurable set and its measure $|\Omega| > 0$ with $\mathbb{R} \triangleq (-\infty,\infty)$. Then, for any function $\varphi : \Omega \to \mathbb{R}$, we say that the functionals

$$\mathrm{E}\varphi \triangleq \int_\Omega p\varphi,\ \mathrm{Var}\varphi \triangleq \mathrm{E}\varphi^2 - \mathrm{E}^2\varphi,\quad \widehat{\varphi} \triangleq \frac{\mathrm{Var}\varphi}{\mathrm{E}^2\varphi}\quad \text{and}\quad \widetilde{\varphi} \triangleq \sqrt{\widehat{\varphi}} \tag{1}$$

are the the *mathematical expectation*,*variance* [8, 9, 12, 13, 16–19, 30, 32], *coefficient of stable* and the *coefficient of variation* [24, 26] of the random variable $\varphi(X)$, respectively.

In the above definitions, the mathematical expectation is the crucial one, which is a *mean value* [7, 11, 21] of the function φ.

We remark here that, if $\varphi : \mathbb{N}_T \to \mathbb{R}$ is a discrete function, then we define

$$\mathrm{E}\varphi \triangleq \int_{\mathbb{N}_T} p\varphi \triangleq \sum_{t=0}^{T} p(t)\varphi(t), \tag{2}$$

where $\mathbb{N}_T \triangleq \{0,1,2,\ldots,T\}$, $1 \leqslant T \leqslant \infty$ and $p : \mathbb{N}_T \to (0,\infty) \wedge \sum_{t=0}^{T} p(t) = 1$ is the probability density function of the random variable $X \in \mathbb{N}_T$.

We also remark here that, if $\mathrm{E}\varphi = 0$, then we define

$$\widehat{\varphi} \triangleq \begin{cases} \lim_{\mathrm{E}\varphi\to 0} \widehat{\varphi} = 0 & , \quad \mathrm{Var}\varphi = 0 \\ \infty & , \quad \mathrm{Var}\varphi \neq 0 \end{cases}. \tag{3}$$

In [31], the authors applied the coefficient of variation to space science and established the stability inequalities involving gravity norm and temperature as follows:

$$\sqrt{\frac{4\pi}{15}} \times \frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}} \leqslant \widetilde{\|\mathbf{F}\|} \leqslant \sqrt{2} \times \frac{\mathrm{e}}{\sqrt{1-\mathrm{e}^2}}. \tag{4}$$

Let $\gamma \in \mathbb{R}$. Then, for any function $\varphi : \Omega \to (0,\infty)$, we define the *$\gamma$-order variance*,*$\gamma$-mean variance* [30, 32, 33] and *γ-coefficient of stable* of the random variable $\varphi(X)$ as follows:

$$\mathrm{Var}^{[\gamma]}\varphi \triangleq \begin{cases} \frac{2}{\gamma(\gamma-1)}\left(\mathrm{E}\varphi^\gamma - \mathrm{E}^\gamma\varphi\right) & , \quad \gamma \neq 0,1 \\ \lim_{\gamma\to 0} \mathrm{Var}^{[\gamma]}\varphi = 2\left(\log \mathrm{E}\varphi - \mathrm{E}\log\varphi\right) & , \quad \gamma = 0 \\ \lim_{\gamma\to 1} \mathrm{Var}^{[\gamma]}\varphi = 2\left(\mathrm{E}\varphi\log\varphi - \mathrm{E}\varphi\log\mathrm{E}\varphi\right) & , \quad \gamma = 1 \end{cases}, \tag{5}$$

$$\overline{\mathrm{Var}}^{[\gamma]}\varphi \triangleq \left(\mathrm{Var}^{[\gamma]}\varphi\right)^{1/\gamma} \text{ when } \gamma \neq 0, \tag{6}$$

and

$$\widehat{\varphi}_\gamma \triangleq \frac{\mathrm{Var}^{[\gamma]}\varphi}{\mathrm{E}^\gamma\varphi}, \tag{7}$$

respectively, where

$$\widehat{\varphi}_0 \triangleq \lim_{\gamma\to 0}\widehat{\varphi}_\gamma = \mathrm{Var}^{[0]}\varphi = 2\left(\log \mathrm{E}\varphi - \mathrm{E}\log\varphi\right), \tag{8}$$

$$\widehat{\varphi}_1 \triangleq \lim_{\gamma\to 1}\widehat{\varphi}_\gamma = \frac{2\left(\mathrm{E}\varphi\log\varphi - \mathrm{E}\varphi\log\mathrm{E}\varphi\right)}{\mathrm{E}\varphi} \tag{9}$$

and, we say that

$$\widetilde{\varphi}_\gamma \triangleq \left(\widehat{\varphi}_\gamma\right)^{1/\gamma} \text{ when } \gamma \neq 0, \tag{10}$$

is a γ*-coefficient of variation* of the random variable $\varphi(X)$.

The γ-coefficient of variation has the following property: Let $\gamma \in [1,2]$. Then, for any functions $f : \Omega \to (0,\infty)$ and $g : \Omega \to (0,\infty)$, we have

$$\widetilde{(f+g)}_\gamma \leqslant \max\left\{\widetilde{f}_\gamma,\ \widetilde{g}_\gamma\right\}. \tag{11}$$

Indeed, by Theorem 1.1 in [33], we have

$$\overline{\mathrm{Var}}^{[\gamma]}(f+g) \leqslant \overline{\mathrm{Var}}^{[\gamma]}f + \overline{\mathrm{Var}}^{[\gamma]}g. \tag{12}$$

So, by (12), we have

$$\begin{aligned}\widetilde{(f+g)}_\gamma &= \frac{\overline{\mathrm{Var}}^{[\gamma]}(f+g)}{\mathrm{E}(f+g)} \leqslant \frac{\overline{\mathrm{Var}}^{[\gamma]}f + \overline{\mathrm{Var}}^{[\gamma]}g}{\mathrm{E}(f+g)} = \frac{\overline{\mathrm{Var}}^{[\gamma]}f + \overline{\mathrm{Var}}^{[\gamma]}g}{\mathrm{E}f + \mathrm{E}g} = \frac{\mathrm{E}f\widetilde{f}_\gamma + \mathrm{E}g\widetilde{g}_\gamma}{\mathrm{E}f + \mathrm{E}g} \\ &\leqslant \frac{\mathrm{E}f\max\left\{\widetilde{f}_\gamma,\widetilde{g}_\gamma\right\} + \mathrm{E}g\max\left\{\widetilde{f}_\gamma,\widetilde{g}_\gamma\right\}}{\mathrm{E}f + \mathrm{E}g} = \max\left\{\widetilde{f}_\gamma,\ \widetilde{g}_\gamma\right\} \Rightarrow (11).\end{aligned}$$

In [30], for any function $\varphi : \Omega \to (0,\infty)$, the authors defined the *Dresher variance mean* $\mathrm{V}_{\gamma,\delta}(\varphi)$ of the random variable $\varphi(X)$ as follows:

$$\mathrm{V}_{\gamma,\delta}(\varphi) = \begin{cases} \left(\frac{\mathrm{Var}^{[\gamma]}\varphi}{\mathrm{Var}^{[\delta]}\varphi}\right)^{1/(\gamma-\delta)} & , \quad \gamma \neq \delta \\ \exp\left[\frac{\mathrm{E}\varphi^\gamma\log\varphi - \mathrm{E}^\gamma\varphi\log\mathrm{E}\varphi}{\mathrm{E}\varphi^\gamma - \mathrm{E}^\gamma\varphi} - \left(\frac{1}{\gamma} + \frac{1}{\gamma-1}\right)\right] & , \quad \gamma = \delta \neq 0,1 \\ \exp\left[\frac{\mathrm{E}\ln^2\varphi - \ln^2\mathrm{E}\varphi}{2(\mathrm{E}\log\varphi - \ln\mathrm{E}\varphi)} + 1\right] & , \quad \gamma = \delta = 0 \\ \exp\left[\frac{\mathrm{E}\varphi\ln^2\varphi - \mathrm{E}\varphi\ln^2\mathrm{E}\varphi}{2(\mathrm{E}\varphi\log\varphi - \mathrm{E}\varphi\log\mathrm{E}\varphi)} - 1\right] & , \quad \gamma = \delta = 1 \end{cases}, \tag{13}$$

and obtained the the following *Dresher variance mean inequality* (see Theorem 1 in [30])

$$\varphi_{\inf} \triangleq \inf_{t\in\Omega}\{\varphi(t)\} \leqslant \mathrm{V}_{\gamma,\delta}(\varphi) \leqslant \varphi_{\sup} \triangleq \sup_{t\in\Omega}\{\varphi(t)\},\ \forall\gamma\in\mathbb{R} \wedge \forall\delta\in\mathbb{R}, \tag{14}$$

Dresher-type inequality (see Theorem 2 in [30])

$$\max\{\gamma,\delta\} \geqslant \max\{\gamma^*,\delta^*\} \ \wedge \ \min\{\gamma,\delta\} \geqslant \min\{\gamma^*,\delta^*\} \Rightarrow \mathrm{V}_{\gamma,\delta}(\varphi) \geqslant \mathrm{V}_{\gamma^*,\delta^*}(\varphi) \tag{15}$$

with the *V-E inequality* (see Theorem 3 in [30])

$$\mathrm{V}_{\gamma,\delta}(\varphi) \geqslant \left(\frac{\delta}{\gamma}\right)^{1/(\gamma-\delta)} \mathrm{E}\varphi, \ \forall \gamma,\delta : \gamma > \delta \geqslant 1, \tag{16}$$

where the coefficient $(\delta/\gamma)^{1/(\gamma-\delta)}$ is the best constant, as well as the authors displayed the applications of these results in space science.

We remark here that, by (14) and (16), we have

$$0 < \max\left\{\frac{2}{3}\mathrm{E}\varphi, \varphi_{\inf}\right\} \leqslant \mathrm{V}_{3,2}(\varphi) \leqslant \varphi_{\sup}. \tag{17}$$

In Section 5, we will display the applications of the inequalities (17), see the proof of the inequalities (73).

Let the function $\phi : J \to \mathbb{R}$ be continuous, nonconstant and convex [3, 14], where J is an interval and its measure $|J| > 0$, and let $1 \in J$ with $\phi(1) \neq 0$. Then we say that the function $\phi : J \to \mathbb{R}$ is a Φ*-function*.

There are a large number of the Φ-functions. For example, the function [32]

$$\phi_\gamma : (0,\infty) \to (0,\infty), \ \ \phi_\gamma(t) \triangleq \frac{2}{\gamma(\gamma-1)} t^\gamma, \ \ \gamma \neq 0, 1, \tag{18}$$

is a Φ-function.

Let the function $\phi : J \to \mathbb{R}$ be a Φ-function. Then, for any function $\varphi : \Omega \to J$, we define ϕ*-Jensen variance* [32] of the random variable $\varphi(X)$ as

$$\mathrm{JVar}_\phi \varphi = \mathrm{E}\phi(\varphi) - \phi(\mathrm{E}\varphi) \tag{19}$$

and, for any function $\varphi : \Omega \to J$, we define the ϕ*-Jensen coefficient* of the random variable $\varphi(X)$ as

$$\widehat{\varphi}|_\phi \triangleq \begin{cases} \frac{\mathrm{JVar}_\phi \varphi}{|\phi(1)|^{-1}|\phi\mathrm{E}\varphi|} & , \quad \phi\mathrm{E}\varphi \neq 0 \\ \lim_{\phi\mathrm{E}\varphi \to 0} \widehat{\varphi}|_\phi = 0 & , \quad \phi\mathrm{E}\varphi = 0, \ \mathrm{JVar}_\Phi \varphi = 0 \\ \infty & , \quad \phi\mathrm{E}\varphi = 0, \ \mathrm{JVar}_\phi \varphi \neq 0 \end{cases}, \tag{20}$$

where $|\cdot|$ is the absolute value function [32].

In [32], the authors generalized classical covariance and variance of random variables, and defined ϕ-covariance, ϕ-variance, ϕ-Jensen variance, ϕ-Jensen covariance, integral variance and the γ-order variance, and studied the relationships among these variances, and proved the quasi-log concavity conjecture, as well as studied the monotonicity of the interval function $\mathrm{JVar}_\phi \varphi\left(X_{[a,b]}\right)$. They also displayed the effective applications of these results in higher education and show that the hierarchical teaching model is normally better than the traditional teaching model under the hypotheses that $X_I \subset X \sim N_k(\mu, \sigma)$, where $k \in \mathbb{R}$ and $k > 1$.

According to the *Jensen inequality* [3–6, 23, 27, 28, 32], we have

$$0 \leqslant \mathrm{JVar}_{\phi}\varphi \leqslant \infty \wedge 0 \leqslant \widehat{\varphi}|_{\phi} \leqslant \infty. \tag{21}$$

In this paper, we assume that

$$0 < \mathrm{JVar}_{\phi}\varphi < \infty \wedge 0 < \widehat{\varphi}|_{\phi} < \infty \wedge 0 < |\phi \mathrm{E}\varphi| < \infty. \tag{22}$$

For the above definitions, we have

$$\mathrm{Var}^{[2]}\varphi = \mathrm{Var}\varphi \wedge \widehat{\varphi}_2 = \widehat{\varphi} \wedge \widetilde{\varphi}_2 = \widetilde{\varphi} \tag{23}$$

and

$$\widehat{\varphi}_\gamma \triangleq \begin{cases} \widehat{\varphi}|_{\phi_\gamma} & , \quad \gamma \neq 0,1 \\ \lim_{\gamma \to 0} \widehat{\varphi}_\gamma = \mathrm{Var}^{[0]}\varphi = 2\left[\log(\mathrm{E}\varphi) - \mathrm{E}(\log\varphi)\right] & , \quad \gamma = 0 \\ \lim_{\gamma \to 1} \widehat{\varphi}_\gamma = \widetilde{\varphi}_1 = \frac{2[\mathrm{E}(\varphi\log\varphi) - (\mathrm{E}\varphi)\log(\mathrm{E}\varphi)]}{\mathrm{E}\varphi} & , \quad \gamma = 1 \end{cases}. \tag{24}$$

Let $\phi''(x) \geqslant 0, \forall x \in \mathbb{R}$, where ϕ'' is the second derivative of the Φ-function ϕ. By Theorem 2 in [32], we have

$$\frac{|\phi(1)|\mathrm{E}^2\varphi}{2|\phi\mathrm{E}\varphi|} \times \inf_{t \in I}\left\{\phi''[\varphi(t)]\right\} \leqslant \frac{\widehat{\varphi}|_{\phi}}{\widehat{\varphi}} \leqslant \frac{|\phi(1)|\mathrm{E}^2\varphi}{2|\phi\mathrm{E}\varphi|} \times \sup_{t \in I}\left\{\phi''[\varphi(t)]\right\}. \tag{25}$$

According to the inequalities (25), we know that the ϕ-Jensen coefficient $\widehat{\varphi}|_{\phi}$ is an important stability feature of the random variable $\varphi(X)$.

Further, if $\mathrm{JVar}_{\phi}\varphi$ is very small and $|\phi\mathrm{E}\varphi|$ very large, then $\widehat{\varphi}|_{\phi}$ is very small. Conversely, if $\widehat{\varphi}|_{\phi}$ is very small, then $\mathrm{JVar}_{\phi}\varphi$ is very small or $|\phi\mathrm{E}\varphi|$ is very large. Suppose that the function $|\phi|$ is strictly increasing. Then, based on the above analysis, we have

$$0 < \mathrm{JVar}_{\phi} f < \mathrm{JVar}_{\phi} g \wedge \mathrm{E}f > \mathrm{E}g > 0 \Rightarrow 0 < \widehat{f}|_{\phi} < \widehat{g}|_{\phi} \tag{26}$$

and

$$0 < \widehat{f}|_{\phi} < \widehat{g}|_{\phi} \Rightarrow 0 < \mathrm{JVar}_{\phi} f < \mathrm{JVar}_{\phi} g \vee \mathrm{E}f > \mathrm{E}g > 0, \tag{27}$$

where $J = (0, \infty)$, $f : \Omega \to J$ and $g : \Omega \to J$ are two functions. This is the significance of the ϕ-Jensen coefficient $\widehat{\varphi}|_{\phi}$ in the analysis of variance.

For any function $\varphi : \Omega \to (0, \infty)$, we say that $\mathrm{E}^{1/\gamma}\varphi^\gamma \triangleq (\mathrm{E}\varphi^\gamma)^{1/\gamma}$ is a γ-*power mean* [30] of the function φ, where $\gamma \neq 0, -\infty, \infty$, and

$$\mathrm{E}^{1/\gamma}\varphi^\gamma \triangleq \begin{cases} \exp(\mathrm{E}\log\varphi) & , \quad \gamma = 0 \\ \varphi_{\inf} & , \quad \gamma = -\infty \\ \varphi_{\sup} & , \quad \gamma = \infty \end{cases}. \tag{28}$$

For the γ-power mean, we have the following γ-power mean inequality [30]:

$$\mathrm{E}^{1/\gamma_1}\varphi^{\gamma_1} \leqslant \mathrm{E}^{1/\gamma_1}\varphi^{\gamma_2}, \ \forall \gamma_1, \gamma_2 \in \mathbb{R} : \gamma_1 < \gamma_2. \tag{29}$$

The equality in (29) holds if and only if φ is a constant function.

For any discrete function $\varphi : \mathbb{N}_T \to \mathbb{R}$, we define the derivative of the function φ as

$$\varphi'(t) \triangleq \varphi(t) - \varphi(t-1),\ \forall t \in \mathbb{N}_T \setminus \{0\}, \tag{30}$$

which is called as the *left derivative* of the function φ in Economic Mathematics, and for any function $\varphi : J \to \mathbb{R}$, we define $\varphi'(t) \triangleq \mathrm{d}\varphi(t)/\mathrm{d}t$, which is the derivative of the function φ, where $J \subseteq \mathbb{R}$ is an interval and its measure $|J| > 0$.

Two differentiable functions $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}$ is an interval or $\Omega = \mathbb{N}_T$, are said to be *similarly ordered*, written as $f \uparrow g$, if and only if $f'(t) \times g'(t) \geqslant 0,\ \forall t \in \Omega$. If this inequality is reversed, then f and g are said to be *oppositely ordered*, written as $f \downarrow g$ [1,22].

A well-known *Marshall inequality* [22] can be described as: Let $f : \Omega \to (0,\infty)$ and $g : \Omega \to (0,\infty)$ be two differentiable functions, where Ω is an interval and its measure $|\Omega| > 0$ or $\Omega = \mathbb{N}_T$, and let $g \uparrow (f/g)$. Then we have

$$\left(\frac{\mathrm{E} f^{\gamma_1}}{\mathrm{E} g^{\gamma_1}}\right)^{1/\gamma_1} \leqslant \left(\frac{\mathrm{E} f^{\gamma_2}}{\mathrm{E} g^{\gamma_2}}\right)^{1/\gamma_2},\ \forall \gamma_1, \gamma_2 \in \mathbb{R} :\ \gamma_1 < \gamma_2. \tag{31}$$

The equality in (31) holds if and only if f/g is a constant function. Obviously, inequality (31) is an extension of the inequality (29).

Let $\gamma_1 = 1$ and $\gamma_2 = \gamma$. Then the Marshall inequality (31) can be rewritten as

$$\widetilde{f}_\gamma \geqslant \widetilde{g}_\gamma, \tag{32}$$

which is called as a *Marshall-type inequality*, where $g \uparrow (f/g), \gamma \in \mathbb{R}$. The equality in (32) holds if and only if f/g is a constant function.

A well-known *Ky Fan inequality* [15,29] can be described as: For any continuous function $\varphi : \Omega \to (0, 1/2]$, we have

$$\frac{\mathrm{E}^{1/0}\varphi^0}{\mathrm{E}^{1/0}(1-\varphi)^0} \leqslant \frac{\mathrm{E}\varphi}{\mathrm{E}(1-\varphi)}. \tag{33}$$

The equality in (33) holds if and only if φ is a constant function.

In [15,29], the authors obtained several interesting generalizations and extensions of the inequality (33). In particular, in [15], the author introduced the following Ky Fan-type inequalities: Let $-1 \leqslant \gamma_1 < \gamma_2 < 1 < \gamma_3 \leqslant 2$, and let $0 < \varphi(t) \leqslant 1/2,\ \forall t \in \Omega$. Then we have

$$\frac{\mathrm{E}^{1/\gamma_1}\varphi^{\gamma_1}}{\mathrm{E}^{1/\gamma_1}(1-\varphi)^{\gamma_1}} \leqslant \frac{\mathrm{E}^{1/\gamma_2}\varphi^{\gamma_2}}{\mathrm{E}^{1/\gamma_2}(1-\varphi)^{\gamma_2}} \leqslant \frac{\mathrm{E}\varphi}{\mathrm{E}(1-\varphi)} \leqslant \frac{\mathrm{E}^{1/\gamma_3}\varphi^{\gamma_3}}{\mathrm{E}^{1/\gamma_3}(1-\varphi)^{\gamma_3}}, \tag{34}$$

which are also the extensions of the inequality (29). The equalities in (34) hold if and only if φ is a constant function.

By inequalities (34), we have the following *Ky Fan-type inequality*:

$$\widehat{\varphi}_\gamma \geqslant \widehat{(1-\varphi)}_\gamma, \tag{35}$$

where $0<\varphi(t)\leqslant 1/2$, $\forall t\in\Omega$, and $\gamma\in[-1,2]$. The equalities in (35) holds if and only if φ is a constant function.

In this paper, we will further extend the inequalities (32) and (35). That is to say, we will study the conditions for the following *Jensen-type inequality* [4–6, 23, 27, 28]:

$$\widehat{f}|_{\phi}\geqslant\widehat{g}|_{\phi}. \tag{36}$$

3. Jensen-type inequalities

In this section, we will extend the inequalities (32) and (35) to the inequality (36). Here we assume that $\Omega\subseteq\mathbb{R}^m$ is a bounded and closed region, and its measure $|\Omega|>0$.

THEOREM 3.1. (Jensen-type inequality) *Let $\phi:(0,\infty)\to\mathbb{R}$ be a twice continuously differentiable Φ-function, and let the functions $f:\Omega\to(0,\infty)$ and $g:\Omega\to(0,\infty)$ be continuous. If*

(i) $|\phi|$ is strictly increasing and ϕ'' is strictly decreasing;

(ii) $|f(s_1)-f(s_2)|\geqslant|g(s_1)-g(s_2)|$ and $f(s)\leqslant g(s)$, $\forall s_1,s_2,s\in\Omega$,

then the Jensen-type inequality (36) holds. The equality in (36) holds if and only if $f(s)=g(s)$, $\forall s\in\Omega$.

In order to prove Theorem 3.1, we need several notations [30] as follows.

$$\mathbf{x}\triangleq(x_1,\ldots,x_n),\ \phi(\mathbf{x})\triangleq(\phi(x_1),\ldots,,\phi(x_n)),\ \mathbf{p}\triangleq(p_1,\ldots,p_n),$$

$$S^n\triangleq\left\{\mathbf{p}\in(0,\infty)^n:\sum_{i=1}^n p_i=1\right\},\ S\triangleq\left\{(t_1,t_2)\in[0,\infty)^2:t_1+t_2\leqslant 1\right\},$$

$$A(\mathbf{x},\mathbf{p})\triangleq\sum_{i=1}^n p_ix_i,\ w_{i,j}(\mathbf{x},\mathbf{p},t_1,t_2)\triangleq t_1x_i+t_2x_j+(1-t_1-t_2)A(\mathbf{x},\mathbf{p}),$$

where S^n and S are two simplices.

In order to prove Theorem 3.1, we also need the following Lemma 3.1.

LEMMA 3.1. (see Lemma 1 in [30]) *Let the function $\phi:J\to\mathbb{R}$ be twice continuously differentiable, where $J\subset\mathbb{R}$ is an interval and its measure $|J|>0$, and let $\mathbf{x}\in J^n$ with $\mathbf{p}\in S^n$. Then we have the following identity:*

$$A(\phi(\mathbf{x}),\mathbf{p})-\phi(A(\mathbf{x},\mathbf{p}))\equiv\sum_{1\leqslant i<j\leqslant n}p_ip_j\left\{\iint_S\phi''[w_{i,j}(\mathbf{x},\mathbf{p},t_1,t_2)]\,\mathrm{d}t_1\mathrm{d}t_2\right\}(x_i-x_j)^2. \tag{37}$$

We remark here that, the proof of Lemma 3.1 is based on the results in linear algebra, see the proof of Lemma 1 in [30].

Now let us start to prove Theorem 3.1.

Proof. Let $T\triangleq\{\Delta\Omega_1,\ldots,\Delta\Omega_n\}$ be a partition of the set Ω, and let

$$\|T\|\triangleq\max_{x,y\in\Delta\Omega_i,1\leqslant i\leqslant n}\{|x-y|\}$$

be the diameter of the partition T. Pick any $\xi_i \in \Delta\Omega_i$ for each $i = 1,2,\ldots,n$. Set

$$\xi \triangleq (\xi_1, \xi_2, \ldots, \xi_n) \ \wedge \ f(\xi) \triangleq (f(\xi_1), f(\xi_2), \ldots, f(\xi_n))$$

and

$$\overline{\mathbf{p}}(\xi) = (\overline{p}_1(\xi), \overline{p}_2(\xi), \ldots, \overline{p}_n(\xi)) \triangleq \frac{(p(\xi_1)|\Delta\Omega_1|, p(\xi_2)|\Delta\Omega_2|, \ldots, p(\xi_n)|\Delta\Omega_n|)}{\sum_{i=1}^n p(\xi_i)|\Delta\Omega_i|}.$$

Then

$$\overline{\mathbf{p}}(\xi) \in S^n \ \wedge \ \lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i)|\Delta\Omega_i| = \int_\Omega p = 1, \tag{38}$$

where $|\Delta\Omega_i| > 0$ is the measure of $\Delta\Omega_i$, $i = 1,2,\ldots,n$.

By (38) and the definition of the Riemann integral, we have

$$\begin{aligned}
\mathrm{JVar}_\phi f &= \mathrm{E}\phi(\varphi) - \phi(\mathrm{E}\varphi) \\
&= \lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i)\phi(f(\xi_i))|\Delta\Omega_i| - \phi\left(\lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i) f(\xi_i)|\Delta\Omega_i|\right) \\
&= \left(\lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i)|\Delta\Omega_i|\right)\left(\lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi)\phi(f(\xi_i))\right) \\
&\quad -\phi\left[\left(\lim_{\|T\|\to 0} \sum_{i=1}^n p(\xi_i)|\Delta\Omega_i|\right)\left(\lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi) f(\xi_i)\right)\right] \\
&= \lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi)\phi(f(\xi_i)) - \phi\left(\lim_{\|T\|\to 0} \sum_{i=1}^n \overline{p}_i(\xi) f(\xi_i)\right) \\
&= \lim_{\|T\|\to 0} \left[\sum_{i=1}^n \overline{p}_i(\xi)\phi(f(\xi_i)) - \phi\left(\sum_{i=1}^n \overline{p}_i(\xi) f(\xi_i)\right)\right] \\
&= \lim_{\|T\|\to 0} \left[A(\phi(f(\xi)), \mathbf{p}) - \phi(A(f(\xi), \mathbf{p}))\right],
\end{aligned}$$

i.e.

$$\mathrm{JVar}_\phi f = \lim_{\|T\|\to 0} \left[A(\phi(f(\xi)), \mathbf{p}) - \phi(A(f(\xi), \mathbf{p}))\right]. \tag{39}$$

By the conditions (i) and (ii) in Theorem 3.1, for any $i, j : 1 \leqslant i, j \leqslant n$, we have

$$(f(\xi_i) - f(\xi_j))^2 \geqslant (g(\xi_i) - g(\xi_j))^2 \ \wedge \ f(\xi_i) \leqslant g(\xi_i) \ \wedge \ f(\xi_j) \leqslant g(\xi_j) \tag{40}$$

and

$$\begin{aligned}
&w_{i,j}(f(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2) \leqslant w_{i,j}(g(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2) \ \wedge \\
&\quad |\phi(A(f(\xi), \mathbf{p}))| \leqslant |\phi(A(g(\xi), \mathbf{p}))|.
\end{aligned} \tag{41}$$

Since the Φ-function $\phi : J \to \mathbb{R}$ is twice continuously differentiable and convex, we have

$$\phi''\left[w_{i,j}\left(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right)\right] \geqslant 0 \ \wedge \ \phi''\left[w_{i,j}\left(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right)\right] \geqslant 0. \tag{42}$$

According to the Lemma 3.1, conditions (i)–(ii), and (39)–(42), we have

$$\begin{aligned}
&\widehat{f}|_{\phi} \\
&= \frac{\mathrm{JVar}_{\phi} f}{|\phi(1)|^{-1}|\phi \mathrm{E} f|} \\
&= \frac{\lim\limits_{\|T\|\to 0}\left[A\left(\phi\left(f(\xi)\right),\mathbf{p}\right)-\phi\left(A\left(f(\xi),\mathbf{p}\right)\right)\right]}{|\phi(1)|^{-1}|\phi \mathrm{E} f|} \\
&= \frac{\lim\limits_{\|T\|\to 0}\left[\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S \phi''\left[w_{i,j}\left(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right)\right]\mathrm{d}t_1\mathrm{d}t_2\right\}\left(f(\xi_i)-f(\xi_j)\right)^2\right]}{|\phi(1)|^{-1}|\phi \mathrm{E} f|} \\
&\geqslant \frac{\lim\limits_{\|T\|\to 0}\left[\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S \phi''\left[w_{i,j}\left(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right)\right]\mathrm{d}t_1\mathrm{d}t_2\right\}\left(g(\xi_i)-g(\xi_j)\right)^2\right]}{|\phi(1)|^{-1}|\phi \mathrm{E} f|} \\
&\geqslant \frac{\lim\limits_{\|T\|\to 0}\left[\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S \phi''\left[w_{i,j}\left(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right)\right]\mathrm{d}t_1\mathrm{d}t_2\right\}\left(g(\xi_i)-g(\xi_j)\right)^2\right]}{|\phi(1)|^{-1}|\phi \mathrm{E} f|} \\
&\geqslant \frac{\lim\limits_{\|T\|\to 0}\left[\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S \phi''\left[w_{i,j}\left(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right)\right]\mathrm{d}t_1\mathrm{d}t_2\right\}\left(g(\xi_i)-g(\xi_j)\right)^2\right]}{|\phi(1)|^{-1}|\phi \mathrm{E} g|} \\
&= \frac{\mathrm{JVar}_{\phi} g}{|\phi(1)|^{-1}|\phi \mathrm{E} g|} \\
&= \widehat{g}|_{\phi} \\
&\Rightarrow (36).
\end{aligned}$$

Hence the inequality (36) is proved.

Based on the above proof, we see that the equality in (36) holds if and only if $f(s) = g(s)$, $\forall s \in \Omega$. This completes the proof of Theorem 3.1. □

There are a large number of the Φ-functions satisfying the condition (i) in Theorem 3.1. For example, let the Φ-function ϕ_{γ} be defined by (18), where $0<\gamma<2$, $\gamma \neq 1$. Then $|\phi_{\gamma}|$ is strictly increasing and ϕ''_{γ} is strictly decreasing. Another example is that the Φ-function

$$\phi : (0,\infty) \to (0,\infty) \ \wedge \ \phi(t) \triangleq (t+e)\left[\log(t+e)-1\right]$$

also satisfies the condition (i) in Theorem 3.1. Indeed,

$$\phi(t) > 0 \wedge \phi'(t) = \log(t+e) > 0 \wedge \phi''(t) = (t+e)^{-1} > 0 \\ \wedge \phi'''(t) = -(t+e)^{-2} < 0, \ \forall t \in (0,\infty).$$

The connotation of Theorem 3.1 is very rich.

EXAMPLE 3.1. Let the Φ-function ϕ_γ be defined by (18), where $0 < \gamma < 2$, $\gamma \neq 1$, and let

$$\Omega \triangleq (0, \pi/4) \wedge f(t) \triangleq \sin t \wedge g(t) \triangleq \cos t.$$

Then the condition (i) in Theorem 3.1 holds and, for any $s_1, s_2, s \in \Omega$, we have

$$\begin{aligned} |f(s_1) - f(s_2)| &= \left| 2\sin\frac{s_1 - s_2}{2} \cos\frac{s_1 + s_2}{2} \right| \\ &\geqslant \left| -2\sin\frac{s_1 - s_2}{2} \sin\frac{s_1 + s_2}{2} \right| = |g(s_1) - g(s_2)| \end{aligned}$$

and

$$0 < f(s) = \sin s \leqslant \cos s = g(s).$$

Hence the condition (ii) in Theorem 3.1 also holds. So, by Theorem 3.1, we have

$$\widehat{\sin t}_\gamma = \widehat{\sin t}|_{\phi_\gamma} > \widehat{\cos t}|_{\phi_\gamma} = \widehat{\cos t}_\gamma, \ \forall \gamma \in (0,2) \setminus \{1\}. \tag{43}$$

REMARK 3.1. In the proof of Theorem 3.1, we first transform a continuous mathematical problem into a discrete mathematical problem, and then use the numerical analysis theory to solve this problem. Based on the proof of Theorem 3.1, we know that Theorem 3.1 is also true when $\Omega = \mathbb{N}_T$, where $f : \Omega \to (0,\infty)$ and $g : \Omega \to (0,\infty)$ are two discrete functions, and $1 \leqslant T \leqslant \infty$.

4. Marshall-type inequalities

In this section, we will weaken the condition $g \uparrow (f/g)$ for the Marshall-type inequality (32) and, we assume that $J \subset \mathbb{R}$ is an interval and its measure $|J| > 0$.

THEOREM 4.1. (Marshall-type inequality) *Let the functions $f : J \to (0,\infty)$ and $g : J \to (0,\infty)$ be differentiable. If*

$$2\left[1 - \frac{\log(\lambda_3/\lambda_1)}{\log(\lambda_2/\lambda_1)}\right] \leqslant \gamma \leqslant 2\left[1 + \frac{\log(\lambda_3/\lambda_2)}{\log(\lambda_2/\lambda_1)}\right] \tag{44}$$

and

$$0 \leqslant \lambda_1 \triangleq \left(\frac{f}{g}\right)_{\inf} \leqslant \lambda_2 \triangleq \left(\frac{f}{g}\right)_{\sup} \leqslant \lambda_3 \triangleq \left|\frac{f'}{g'}\right|_{\inf}, \tag{45}$$

then the Marshall-type inequality (32) *holds. The equality in* (32) *holds if and only if f/g is a constant function.*

We remark here that, if $\lambda_1 = 0,$ then we define

$$2\left[1-\frac{\log(\lambda_3/\lambda_1)}{\log(\lambda_2/\lambda_1)}\right] \triangleq \lim_{\lambda_1\to 0^+} 2\left[1-\frac{\log(\lambda_3/\lambda_1)}{\log(\lambda_2/\lambda_1)}\right] = 0$$

and

$$2\left[1+\frac{\log(\lambda_3/\lambda_2)}{\log(\lambda_2/\lambda_1)}\right] \triangleq \lim_{\lambda_1\to 0^+} 2\left[1+\frac{\log(\lambda_3/\lambda_2)}{\log(\lambda_2/\lambda_1)}\right] = 2.$$

Now let us start the proof.

Proof. In the following proof, we continue to use the proof of Theorem 3.1.

Based on the continuity, we may assume that $\gamma \neq 0, 1, \lambda_1 < \lambda_2$ and $\lambda_1 > 0.$ In the proof of Theorem 3.1, set $\phi = \phi_\gamma,$ where ϕ_γ is defined by (18).

Case 1: $0 < \gamma \leqslant 2.$ According to the Cauchy mean value theorem, there exists a real $\theta_{i,j} \in (0,1)$ such that

$$\left|\frac{f(\xi_i)-f(\xi_j)}{g(\xi_i)-g(\xi_j)}\right| = \left|\frac{f'((1-\theta_{i,j})\xi_i+\theta_{i,j}\xi_j)}{g'((1-\theta_{i,j})\xi_i+\theta_{i,j}\xi_j)}\right| \geqslant \lambda_3,\ \forall i,j : 0 \leqslant i < j \leqslant n. \tag{46}$$

Since

$$\begin{aligned} w_{i,j}(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2) &= t_1 f(\xi_i) + t_2 f(\xi_j) + (1-t_1-t_2)A(f(\xi),\mathbf{p}) \\ &\geqslant t_1\lambda_1 g(\xi_i) + t_2\lambda_1 g(\xi_j) + (1-t_1-t_2)A(\lambda_1 g(\xi),\mathbf{p}) \\ &= \lambda_1\left[t_1 g(\xi_i) + t_2 g(\xi_j) + (1-t_1-t_2)A(g(\xi),\mathbf{p})\right] \end{aligned}$$

and

$$\begin{aligned} w_{i,j}(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2) &= t_1 f(\xi_i) + t_2 f(\xi_j) + (1-t_1-t_2)A(f(\xi),\mathbf{p}) \\ &\leqslant t_1\lambda_2 g(\xi_i) + t_2\lambda_2 g(\xi_j) + (1-t_1-t_2)A(\lambda_2 g(\xi),\mathbf{p}) \\ &= \lambda_2\left[t_1 g(\xi_i) + t_2 g(\xi_j) + (1-t_1-t_2)A(g(\xi),\mathbf{p})\right], \end{aligned}$$

we have

$$\lambda_1 w_{i,j}(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2) \leqslant w_{i,j}(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2) \leqslant \lambda_2 w_{i,j}(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2). \tag{47}$$

Similarly, we have

$$\lambda_1 \mathrm{E}g = \mathrm{E}(\lambda_1 g) \leqslant \mathrm{E}f \leqslant \mathrm{E}(\lambda_2 g) = \lambda_2 \mathrm{E}g. \tag{48}$$

According to the Lemma 3.1 and (45)–(48), we have

$$\begin{aligned} &A(\phi(f(\xi)),\mathbf{p}) - \phi(A(f(\xi),\mathbf{p})) \\ &= \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S \phi''\left[w_{i,j}(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)\right]\mathrm{d}t_1\mathrm{d}t_2\right\}(f(\xi_i)-f(\xi_j))^2 \\ &= 2\sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S \left[w_{i,j}(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)\right]^{\gamma-2}\mathrm{d}t_1\mathrm{d}t_2\right\}(f(\xi_i)-f(\xi_j))^2 \end{aligned}$$

$$\geqslant 2\sum_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S[\lambda_2 w_{i,j}(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)]^{\gamma-2}\,\mathrm{d}t_1\mathrm{d}t_2\right\}\lambda_3^2(g(\xi_i)-g(\xi_j))^2$$

$$\geqslant 2\sum_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S[\lambda_2 w_{i,j}(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)]^{\gamma-2}\,\mathrm{d}t_1\mathrm{d}t_2\right\}\lambda_2^2(g(\xi_i)-g(\xi_j))^2$$

$$=2\lambda_2^{\gamma}\sum_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S[w_{i,j}(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)]^{\gamma-2}\,\mathrm{d}t_1\mathrm{d}t_2\right\}(g(\xi_i)-g(\xi_j))^2$$

$$=\lambda_2^{\gamma}[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))].$$

Hence

$$A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))\geqslant\lambda_2^{\gamma}[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]. \tag{49}$$

By (24), (39), (48) and (49), we have

$$\begin{aligned}
\widehat{f}_\gamma &= \widehat{f}|_\phi\\
&=\frac{\mathrm{JVar}_\phi f}{|\phi(1)|^{-1}|\phi\mathrm{E}f|}\\
&=\frac{\lim_{\|T\|\to 0}[A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))]}{\mathrm{E}^\gamma f}\\
&=\lim_{\|T\|\to 0}\frac{A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))}{\mathrm{E}^\gamma f}\\
&\geqslant\lim_{\|T\|\to 0}\frac{\lambda_2^{\gamma}[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]}{\mathrm{E}^\gamma f}\\
&\geqslant\lim_{\|T\|\to 0}\frac{\lambda_2^{\gamma}[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]}{(\lambda_2\mathrm{E}g)^\gamma}\\
&=\frac{\lim_{\|T\|\to 0}[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]}{\mathrm{E}^\gamma g}\\
&=\widehat{g}|_\phi\\
&=\widehat{g}_\gamma\\
&\Rightarrow(32).
\end{aligned}$$

Consequently, the inequality (32) holds for the case where $0<\gamma\leqslant 2$.

Case 2: $\gamma>2$. According to the proof of Case 1, we have

$$\begin{aligned}
&A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))\\
&=2\sum_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S[w_{i,j}(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)]^{\gamma-2}\,\mathrm{d}t_1\mathrm{d}t_2\right\}(f(\xi_i)-f(\xi_j))^2\\
&\geqslant 2\sum_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S[\lambda_1 w_{i,j}(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)]^{\gamma-2}\,\mathrm{d}t_1\mathrm{d}t_2\right\}\lambda_3^2(g(\xi_i)-g(\xi_j))^2\\
&=\lambda_1^{\gamma-2}\lambda_3^2[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))],
\end{aligned}$$

that is,

$$A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))\geqslant \lambda_1^{\gamma-2}\lambda_3^2[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]. \quad (50)$$

Since

$$\frac{\lambda_1^{\gamma-2}\lambda_3^2}{\lambda_2^\gamma}\geqslant 1 \Leftrightarrow 1\leqslant\left(\frac{\lambda_2}{\lambda_1}\right)^{\gamma-2}\leqslant\left(\frac{\lambda_3}{\lambda_2}\right)^2 \Leftrightarrow 2<\gamma\leqslant 2\left[1+\frac{\log(\lambda_3/\lambda_2)}{\log(\lambda_2/\lambda_1)}\right],$$

by the conditions of Theorem 4.1, (24), (39), (48) and (50), we have

$$\begin{aligned}
\widehat{f}_\gamma &= \lim_{\|T\|\to 0}\frac{A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))}{\mathrm{E}^\gamma f}\\
&\geqslant \lim_{\|T\|\to 0}\frac{\lambda_1^{\gamma-2}\lambda_3^2[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]}{\mathrm{E}^\gamma f}\\
&\geqslant \lim_{\|T\|\to 0}\frac{\lambda_1^{\gamma-2}\lambda_3^2[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]}{(\lambda_2\mathrm{E}g)^\gamma}\\
&= \frac{\lambda_1^{\gamma-2}\lambda_3^2}{\lambda_2^\gamma}\widehat{g}_\gamma\\
&\geqslant \widehat{g}_\gamma\\
&\Rightarrow (32).
\end{aligned}$$

Hence the inequality (32) also holds for the case where

$$2<\gamma\leqslant 2\left[1+\frac{\log(\lambda_3/\lambda_2)}{\log(\lambda_2/\lambda_1)}\right].$$

Case 3: $\gamma<0$. According to the proof of Case 1, we have

$$\begin{aligned}
&A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))\\
&=2\sum_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S[w_{i,j}(f(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)]^{\gamma-2}\,\mathrm{d}t_1\mathrm{d}t_2\right\}(f(\xi_i)-f(\xi_j))^2\\
&\geqslant 2\sum_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left\{\iint_S[\lambda_2 w_{i,j}(g(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2)]^{\gamma-2}\,\mathrm{d}t_1\mathrm{d}t_2\right\}\lambda_3^2(g(\xi_i)-g(\xi_j))^2\\
&=\lambda_2^{\gamma-2}\lambda_3^2[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))],
\end{aligned}$$

that is,

$$A(\phi(f(\xi)),\mathbf{p})-\phi(A(f(\xi),\mathbf{p}))\geqslant \lambda_2^{\gamma-2}\lambda_3^2[A(\phi(g(\xi)),\mathbf{p})-\phi(A(g(\xi),\mathbf{p}))]. \quad (51)$$

Since $\gamma<0$ and

$$\frac{\lambda_2^{\gamma-2}\lambda_3^2}{\lambda_1^\gamma}\geqslant 1 \Leftrightarrow 1\geqslant\left(\frac{\lambda_2}{\lambda_1}\right)^{\gamma-2}\geqslant\left(\frac{\lambda_3}{\lambda_1}\right)^{-2} \Leftrightarrow 2\left[1-\frac{\log(\lambda_3/\lambda_1)}{\log(\lambda_2/\lambda_1)}\right]\leqslant\gamma<0,$$

by the conditions of Theorem 4.1, (24), (39), (48) and (51), we have

$$\begin{aligned}
\widehat{f}_\gamma &= \lim_{\|T\|\to 0} \frac{A(\phi(f(\xi)),\mathbf{p}) - \phi(A(f(\xi),\mathbf{p}))}{\mathrm{E}^\gamma f} \\
&\geqslant \lim_{\|T\|\to 0} \frac{\lambda_1^{\gamma-2}\lambda_3^2 [A(\phi(g(\xi)),\mathbf{p}) - \phi(A(g(\xi),\mathbf{p}))]}{\mathrm{E}^\gamma f} \\
&\geqslant \lim_{\|T\|\to 0} \frac{\lambda_1^{\gamma-2}\lambda_3^2 [A(\phi(g(\xi)),\mathbf{p}) - \phi(A(g(\xi),\mathbf{p}))]}{(\lambda_1 \mathrm{E} g)^\gamma} \\
&= \frac{\lambda_1^{\gamma-2}\lambda_3^2}{\lambda_1^\gamma} \widehat{g}_\gamma \\
&\geqslant \widehat{g}_\gamma \\
&\Rightarrow (32).
\end{aligned}$$

Therefore, the inequality (32) still holds for the case where

$$2\left[1 - \frac{\log(\lambda_3/\lambda_1)}{\log(\lambda_2/\lambda_1)}\right] \leqslant \gamma < 0.$$

Based on the above proof, we see that the equality in (32) holds if and only if f/g is a constant function. This completes the proof of Theorem 4.1. □

THEOREM 4.2. (Marshall-type inequality) *Let the functions $f : J \to (0,\infty)$ and $g : J \to (0,\infty)$ be differentiable. If*

$$2\left[1 - \frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma \leqslant 2\left[1 + \frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right] \tag{52}$$

and

$$0 \leqslant \lambda_1 \triangleq \left(\frac{f}{g}\right)_{\inf} \leqslant \lambda_4 \triangleq \frac{\mathrm{E}f}{\mathrm{E}g} \leqslant \lambda_2 \triangleq \left(\frac{f}{g}\right)_{\sup} \leqslant \lambda_3 \triangleq \left|\frac{f'}{g'}\right|_{\inf}, \tag{53}$$

then the Marshall-type inequality (32) holds. The equality in (32) holds if and only if f/g is a constant function.

Proof. In the following proof, we continue to use the proof of Theorem 4.1. Notice that

$$\frac{\lambda_1^{\gamma-2}\lambda_3^2}{\lambda_4^\gamma} \geqslant 1 \Leftrightarrow \left(\frac{\lambda_4}{\lambda_1}\right)^{\gamma-2} \leqslant \left(\frac{\lambda_3}{\lambda_4}\right)^2 \Leftrightarrow -\infty < \gamma \leqslant 2\left[1 + \frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right]$$

and

$$\frac{\lambda_2^{\gamma-2}\lambda_3^2}{\lambda_4^\gamma} \geqslant 1 \Leftrightarrow \left(\frac{\lambda_2}{\lambda_4}\right)^{\gamma-2} \geqslant \left(\frac{\lambda_3}{\lambda_4}\right)^{-2} \Leftrightarrow 2\left[1 - \frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma < \infty.$$

Case 1: $2 \leqslant \gamma < \infty$. By the proof of Theorem 4.1, we have

$$\begin{aligned}
\widehat{f}_\gamma &= \lim_{\|T\|\to 0} \frac{A(\phi(f(\xi)),\mathbf{p}) - \phi(A(f(\xi),\mathbf{p}))}{\mathrm{E}^\gamma f} \\
&\geqslant \lim_{\|T\|\to 0} \frac{\lambda_1^{\gamma-2}\lambda_3^2 [A(\phi(g(\xi)),\mathbf{p}) - \phi(A(g(\xi),\mathbf{p}))]}{(\mathrm{E}f)^\gamma} \\
&= \frac{\lambda_1^{\gamma-2}\lambda_3^2}{\lambda_4^\gamma}\widehat{g}_\gamma \\
&\geqslant \widehat{g}_\gamma \\
&\Rightarrow (32).
\end{aligned}$$

Hence the inequality (32) holds for the case where

$$2 \leqslant \gamma \leqslant 2\left[1 + \frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right].$$

Case 2: $-\infty < \gamma < 2$. By the proof of Theorem 4.1, we have

$$\begin{aligned}
\widehat{f}_\gamma &= \lim_{\|T\|\to 0} \frac{A(\phi(f(\xi)),\mathbf{p}) - \phi(A(f(\xi),\mathbf{p}))}{\mathrm{E}^\gamma f} \\
&\geqslant \lim_{\|T\|\to 0} \frac{\lambda_2^{\gamma-2}\lambda_3^2 [A(\phi(g(\xi)),\mathbf{p}) - \phi(A(g(\xi),\mathbf{p}))]}{(\mathrm{E}f)^\gamma} \\
&= \frac{\lambda_2^{\gamma-2}\lambda_3^2}{\lambda_4^\gamma}\widehat{g}_\gamma \\
&\geqslant \widehat{g}_\gamma \\
&\Rightarrow (32).
\end{aligned}$$

Therefore, inequality (32) also holds for the case where

$$2\left[1 - \frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma < 2.$$

Based on the above proof, we see that the equality in (32) holds if and only if f/g is a constant function. The proof of Theorem 4.2 is completed. □

The connotation of Theorem 4.2 is also very rich.

EXAMPLE 4.1. Let

$$p : \left(\frac{\pi}{6}, \frac{\pi}{4}\right) \to \mathbb{R} \wedge p(t) \triangleq \frac{12}{\pi} \wedge f(t) \triangleq \sin t \wedge g(t) \triangleq \cos t.$$

Since

$$\mathrm{E}f = \int_{\pi/6}^{\pi/4} p(t)\sin t \mathrm{d}t = -\frac{12}{\pi}\cot t \,|_{\pi/6}^{\pi/4} = \frac{6}{\pi}\left(\sqrt{3} - \sqrt{2}\right)$$

and

$$\mathrm{E}g=\int_{\pi/6}^{\pi/4}p(t)\cos t\,\mathrm{d}t=\frac{12}{\pi}\sin t\mid_{\pi/6}^{\pi/4}=\frac{6}{\pi}\left(\sqrt{2}-1\right),$$

we have

$$0\leqslant\lambda_1\triangleq\left(\frac{f}{g}\right)_{\inf}=\frac{1}{\sqrt{3}}<\lambda_4\triangleq\frac{\mathrm{E}f}{\mathrm{E}g}=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}<\lambda_2\triangleq\left(\frac{f}{g}\right)_{\sup}=1=\lambda_3\triangleq\left|\frac{f'}{g'}\right|_{\inf}.$$

So, by Theorem 4.2, we know that the inequality

$$\widetilde{\sin t_\gamma}\geqslant\widetilde{\cos t_\gamma} \tag{54}$$

holds when

$$0=2\left[1-\frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right]\leqslant\gamma\leqslant2\left[1+\frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right]=3.8620447160546743\cdots. \tag{55}$$

REMARK 4.1. The advantage of Theorem 4.1 is that the conditions of the theorem do not depend on the probability density function p, and the defect of Theorem 4.2 is that the conditions of the theorem depend on the probability density function p, but Theorem 4.2 is an improvement of Theorem 4.1. Indeed,

$$\begin{aligned}2\left[1-\frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right]&\leqslant2\left[1-\frac{\log(\lambda_3/\lambda_1)}{\log(\lambda_2/\lambda_1)}\right]\ \wedge\ 2\left[1+\frac{\log(\lambda_3/\lambda_2)}{\log(\lambda_2/\lambda_1)}\right]\\&\leqslant2\left[1+\frac{\log(\lambda_3/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right].\end{aligned}$$

5. Ky Fan-type inequalities

In this section, we will establish several new Ky Fan-type inequalities.

We say that the function $\varphi:(0,T)\to\mathbb{R}$ is *symmetric* for the interval $(0,T)$, where $0<T<\infty$, if and only if $\varphi(T-t)=\varphi(t)$, $\forall t\in(0,T)$.

We remark here that there are a large number of symmetric functions for the interval $(0,T)$. For example, the functions p_1,p_2, here

$$p_1:(0,T)\to(0,\infty)\wedge p_1(t)\triangleq\frac{1}{T}\wedge p_2:(0,T)\to(0,\infty)\wedge p_2(t)\triangleq\frac{p\left(t;T/2,\sigma,k\right)}{\int_0^T p\left(t;T/2,\sigma,k\right)\mathrm{d}t},$$

are symmetric for the interval $(0,T)$, which are the probability density functions of random variables, where

$$p\left(t;\mu,\sigma,k\right)\triangleq\frac{k^{1-k^{-1}}}{2\Gamma\left(k^{-1}\right)\sigma}\exp\left(-\frac{|t-\mu|^k}{k\sigma^k}\right),\ k>1,\ \sigma>0,\ \mu,t\in\mathbb{R},$$

$\Gamma(s)$ is the gamma function and $p\left(t;\mu,\sigma,k\right)$ is the probability density function of the $k-$ normal distribution [24, 25, 32].

Marshall inequality (32) implies the following Corollary 5.1.

COROLLARY 5.1. (Ky Fan-type inequality) *Let the probability density function $p:(0,T)\to(0,\infty)$ of the random variable $X\in(0,T)$ be symmetric for the interval $(0,T)$, and let the function $\varphi:(0,T)\to(0,1)$ be differentiable. If*

$$\frac{\varphi'(t)}{\varphi'(T-t)}-\frac{\varphi(t)}{1-\varphi(T-t)}\geqslant 0,\ \forall t\in(0,T), \tag{56}$$

then, for any real γ, we have the Ky Fan-type inequality (35)*. Equality in* (35) *holds if and only if the equality in* (56) *holds for any $t\in(0,T)$.*

Proof. We first prove that

$$[1-\widehat{\varphi(T-X)}]_\gamma\equiv[1-\widehat{\varphi(X)}]_\gamma,\ \forall\gamma\in\mathbb{R}. \tag{57}$$

Indeed, without losing of generality, we may assume that $\gamma\neq 0,1$. Set $s\triangleq T-t$. Then we have

$$\begin{aligned}
[1-\widehat{\varphi(T-X)}]_\gamma &= \frac{\mathrm{Var}^{[\gamma]}[1-\varphi(T-X)]}{\mathrm{E}^\gamma[1-\varphi(T-X)]}\\
&= \frac{2}{\gamma(\gamma-1)}\frac{\mathrm{E}[1-\varphi(T-t)]^\gamma-\mathrm{E}^\gamma[1-\varphi(T-X)]}{\mathrm{E}^\gamma[1-\varphi(T-X)]}\\
&= \frac{2}{\gamma(\gamma-1)}\left\{\frac{\mathrm{E}[1-\varphi(T-X)]^\gamma}{\mathrm{E}^\gamma[1-\varphi(T-X)]}-1\right\}\\
&= \frac{2}{\gamma(\gamma-1)}\left\{\frac{\int_0^T p(t)[1-\varphi(T-t)]^\gamma\mathrm{d}t}{\left[\int_0^T p(t)(1-\varphi(T-t))\mathrm{d}t\right]^\gamma}-1\right\}\\
&= \frac{2}{\gamma(\gamma-1)}\left\{\frac{\int_T^0 p(T-s)[1-\varphi(s)]^\gamma\mathrm{d}(T-s)}{\left[\int_T^0 p(T-s)(1-\varphi(s))\mathrm{d}(T-s)\right]^\gamma}-1\right\}\\
&= \frac{2}{\gamma(\gamma-1)}\left\{\frac{\int_0^T p(T-s)[1-\varphi(s)]^\gamma\mathrm{d}s}{\left[\int_0^T p(T-s)(1-\varphi(s))\mathrm{d}s\right]^\gamma}-1\right\}\\
&= \frac{2}{\gamma(\gamma-1)}\left\{\frac{\int_0^T p(s)[1-\varphi(s)]^\gamma\mathrm{d}s}{\left[\int_0^T p(s)(1-\varphi(s))\mathrm{d}s\right]^\gamma}-1\right\}\\
&= \frac{2}{\gamma(\gamma-1)}\left\{\frac{\int_0^T p(t)[1-\varphi(t)]^\gamma\mathrm{d}t}{\left[\int_0^T p(t)(1-\varphi(t))\mathrm{d}t\right]^\gamma}-1\right\}\\
&= [1-\widehat{\varphi(X)}]_\gamma,\ \forall\gamma\in\mathbb{R}\\
&\Rightarrow (57).
\end{aligned}$$

That is, the identity (57) holds.

Next, we prove that

$$[1-\varphi(T-t)]\uparrow\frac{\varphi(t)}{1-\varphi(T-t)}\Leftrightarrow\frac{\varphi'(t)}{\varphi'(T-t)}-\frac{\varphi(t)}{1-\varphi(T-t)}\geqslant 0,\ \forall t\in(0,T). \tag{58}$$

Indeed,

$$\begin{aligned}[1-\varphi(T-t)]\uparrow\frac{\varphi(t)}{1-\varphi(T-t)}&\Leftrightarrow[1-\varphi(T-t)]\uparrow\log\left[\frac{\varphi(t)}{1-\varphi(T-t)}\right]\\&\Leftrightarrow[1-\varphi(T-t)]'\times\left\{\log\left[\frac{\varphi(t)}{1-\varphi(T-t)}\right]\right\}'\geqslant 0\\&\Leftrightarrow\varphi'(T-t)\times\{\log\varphi(t)-\log[1-\varphi(T-t)]\}'\geqslant 0\\&\Leftrightarrow\varphi'(T-t)\times\left\{\frac{\varphi'(t)}{\varphi(t)}-\frac{\varphi'(T-t)}{1-\varphi(T-t)}\right\}\geqslant 0\\&\Leftrightarrow\frac{[\varphi'(T-t)]^2}{\varphi(t)}\times\left\{\frac{\varphi'(t)}{\varphi'(T-t)}-\frac{\varphi(t)}{1-\varphi(T-t)}\right\}\geqslant 0\\&\Leftrightarrow\frac{\varphi'(t)}{\varphi'(T-t)}-\frac{\varphi(t)}{1-\varphi(T-t)}\geqslant 0,\ \forall t\in(0,T)\\&\Rightarrow(58).\end{aligned}$$

Hence (58) holds.

Finally, we prove the inequality (35) as follows.

By (56) and (58), we have

$$[1-\varphi(T-t)]\uparrow\frac{\varphi(t)}{1-\varphi(T-t)}. \tag{59}$$

By (57), (59) and the Marshall inequality (32), we get

$$\widehat{\varphi}_\gamma=\widehat{\varphi(X)}_\gamma\geqslant[1-\widehat{\varphi(T-X)}]_\gamma=[1-\widehat{\varphi(X)}]_\gamma=(\widehat{1-\varphi})_\gamma,\ \forall\gamma\in\mathbb{R}\Rightarrow(35).$$

Hence inequality (35) is proved.

Based on the above proof, we see that the equality in (35) holds if and only if

$$\left[\frac{\varphi(t)}{1-\varphi(T-t)}\right]'=0,\ \forall t\in(0,T)\Leftrightarrow\frac{\varphi'(t)}{\varphi'(T-t)}-\frac{\varphi(t)}{1-\varphi(T-t)}=0,\ \forall t\in(0,T),$$

that is, the equality in (35) holds if and only if the equality in (56) holds for any $t\in(0,T)$. This ends the proof of Corollary 5.1. □

The connotation of Corollary 5.1 is still very rich.

EXAMPLE 5.1. Let

$$p:\left(0,\frac{\pi}{2}\right)\to(0,\infty)\ \wedge\ p(t)\triangleq\frac{2}{\pi}\ \wedge\ \varphi(t)\triangleq 1-\cos t.$$

Then the probability density function $p:(0,\pi/2)\to(0,\infty)$ of the random variable $X\in(0,\pi/2)$ is symmetric for the interval $(0,\pi/2)$. Since,

$$\frac{\varphi'(t)}{\varphi'(\frac{2}{\pi}-t)}-\frac{\varphi(t)}{1-\varphi(\frac{2}{\pi}-t)}=\frac{\sin t}{\sin\left(\frac{\pi}{2}-t\right)}-\frac{1-\cos t}{\cos\left(\frac{\pi}{2}-t\right)}$$
$$=\frac{1-\cos t}{\sin t\cos t}>0,\ \forall t\in\left(0,\frac{\pi}{2}\right),$$

according to Corollary 5.1, we have

$$\widehat{(1-\cos t)}_{\gamma}>\widehat{(\cos t)}_{\gamma},\ \forall\gamma\in\mathbb{R}. \tag{60}$$

Corollary 5.1 implies the following Corollary 5.2.

COROLLARY 5.2. (Ky Fan-type inequality) *Let the probability density function $p:(0,1)\to(0,\infty)$ of the random variable $X\in(0,1)$ be symmetric for the interval $(0,1)$, and let $0<a\leqslant 1$ with $1\leqslant\alpha<\infty$. Then, for any $\gamma\in\mathbb{R}$, we have the following Ky Fan-type inequality:*

$$\widehat{(aX^{\alpha})}_{\gamma}\geqslant\widehat{(1-aX^{\alpha})}_{\gamma}. \tag{61}$$

The equality in (61) holds if and only if $a=1$ and $\alpha=1$.

Proof. Let $\varphi(t)\triangleq at^{\alpha}\in(0,1)$, $t\in(0,1)$. Since

$$\frac{\varphi'(t)}{\varphi'(1-t)}-\frac{\varphi(t)}{1-\varphi(1-t)}\geqslant 0\Leftrightarrow\frac{t^{\alpha-1}}{(1-t)^{\alpha-1}}-\frac{at^{\alpha}}{1-a(1-t)^{\alpha}}\geqslant 0$$
$$\Leftrightarrow t^{\alpha-1}[1-a(1-t)^{\alpha}]-at^{\alpha}(1-t)^{\alpha-1}\geqslant 0$$
$$\Leftrightarrow t^{\alpha-1}-at^{\alpha-1}(1-t)^{\alpha-1}\geqslant 0$$
$$\Leftrightarrow a(1-t)^{\alpha-1}\leqslant 1$$
$$\Leftarrow 0<a\leqslant 1\ \wedge\ 1\leqslant\alpha<\infty\ \wedge\ 0<t<1,$$

we have

$$\frac{\varphi'(t)}{\varphi'(1-t)}-\frac{\varphi(t)}{1-\varphi(1-t)}\geqslant 0,\ \forall t\in(0,1). \tag{62}$$

According to Corollary 5.1, we have

$$\widehat{(aX^{\alpha})}_{\gamma}=\widehat{\varphi}_{\gamma}\geqslant\widehat{(1-\varphi)}_{\gamma}=\widehat{(1-aX^{\alpha})}_{\gamma},\ \forall\gamma\in\mathbb{R}\Rightarrow(61).$$

That is, inequality (61) is proved.

Based on the above proof, we see that the equality in (61) holds if and only if $a=1$ and $\alpha=1$. Corollary 5.2 is proved. □

Corollary 5.1 also implies the following Corollary 5.3.

COROLLARY 5.3. (Ky Fan-type inequality) *Let the probability density function* $p:(0,T)\to(0,\infty)$ *of the random variable* $X\in(0,T)$ *be symmetric for the interval* $(0,T)$, *and let the function* $\psi:(0,T)\to(0,1)$ *be differentiable and strictly monotonic. Then, for any* $\gamma\in\mathbb{R}$, *the Ky Fan-type inequality* (35) *holds, where*

$$\varphi:(0,T)\to\mathbb{R}\ \wedge\ \varphi(t)\triangleq\frac{\psi(t)\left[1-\psi(T-t)\right]}{1-\psi(t)\psi(T-t)}. \tag{63}$$

Equality in (35) *holds if and only if* ψ *is a constant function.*

Proof. First, we assume that the function $\psi:(0,T)\to(0,1)$ is strictly increasing. By $0<\psi(t)<1$, $0<\psi(T-t)<1$ and (63), we see that

$$0<\varphi(t)<1,\ \forall t\in(0,T). \tag{64}$$

Since the function $\psi:(0,T)\to(0,1)$ is differentiable and strictly increasing, we have

$$0<\psi(t)<1\ \wedge\ 0<\psi(T-t)<1\ \wedge\ \psi'(t)\geqslant 0\ \wedge\ \psi'(T-t)\geqslant 0,\ \forall t\in(0,T), \tag{65}$$

and

$$\psi'(t)\neq 0,\ \exists t\in(0,T). \tag{66}$$

By (63), we have

$$\frac{\varphi(t)}{1-\varphi(T-t)}=\frac{\frac{\psi(t)[1-\psi(T-t)]}{1-\psi(t)\psi(T-t)}}{1-\frac{\psi(T-t)[1-\psi(t)]}{1-\psi(T-t)\psi(t)}}=\psi(t)\Rightarrow\left[\frac{\varphi(t)}{1-\varphi(T-t)}\right]'=\psi'(t)\geqslant 0$$

and

$$\begin{aligned}
&[1-\varphi(T-t)]'\\
=&\left\{1-\frac{\psi(T-t)\left[1-\psi(t)\right]}{1-\psi(T-t)\psi(t)}\right\}'\\
=&\left[\frac{1-\psi(T-t)}{1-\psi(t)\psi(T-t)}\right]'\\
=&\frac{[1-\psi(T-t)]'\left[1-\psi(t)\psi(T-t)\right]-\left[1-\psi(T-t)\right]\left[1-\psi(t)\psi(T-t)\right]'}{\left[1-\psi(t)\psi(T-t)\right]^2}\\
=&\frac{\psi'(T-t)\left[1-\psi(t)\psi(T-t)\right]+\left[1-\psi(T-t)\right]\left[\psi'(t)\psi(T-t)-\psi(t)\psi'(T-t)\right]}{\left[1-\psi(t)\psi(T-t)\right]^2}\\
=&\frac{\psi'(T-t)-\psi(t)\psi'(T-t)+\psi'(t)\psi(T-t)-\psi'(t)\psi^2(T-t)}{\left[1-\psi(t)\psi(T-t)\right]^2}\\
=&\frac{\psi'(T-t)\left[1-\psi(t)\right]+\psi'(t)\psi(T-t)\left[1-\psi(T-t)\right]}{\left[1-\psi(t)\psi(T-t)\right]^2}\\
\geqslant&\ 0.
\end{aligned}$$

By (58), we have

$$[1-\varphi(T-t)]'\left[\frac{\varphi(t)}{1-\varphi(T-t)}\right]' \geqslant 0,\ \forall t\in(0,T)$$
$$\Rightarrow [1-\varphi(T-t)]\uparrow \frac{\varphi(t)}{1-\varphi(T-t)}$$
$$\Rightarrow \frac{\varphi'(t)}{\varphi'(T-t)}-\frac{\varphi(t)}{1-\varphi(T-t)}$$
$$\geqslant 0,\ \forall t\in(0,T).$$

According to Corollary 5.1, we see that the Ky Fan-type inequality (35) holds for any $\gamma\in\mathbb{R}$.

Next, we assume that the function $\psi:(0,T)\to(0,1)$ is strictly decreasing. Then, we can similarly prove that

$$\left[\frac{\varphi(t)}{1-\varphi(T-t)}\right]' \leqslant 0 \wedge [1-\varphi(T-t)]'$$
$$\leqslant 0 \wedge \left[\frac{\varphi(t)}{1-\varphi(T-t)}\right]'[1-\varphi(T-t)]' \geqslant 0,\ \forall t\in(0,T),$$

and

$$\frac{\varphi'(t)}{\varphi'(T-t)}-\frac{\varphi(t)}{1-\varphi(T-t)} \geqslant 0,\ \forall t\in(0,T).$$

By Corollary 5.1, we see that the Ky Fan-type inequality (35) also holds for any $\gamma\in\mathbb{R}$.

Based on the above proof, we see that the equality in (35) holds if and only if ψ is a constant function. This ends the proof of Corollary 5.3. □

We remark here that, the solution of the function equation

$$\frac{\varphi(t)}{1-\varphi(T-t)} = \psi(t),\ \forall t\in(0,T), \tag{67}$$

is (63). Indeed,

$$\frac{\varphi(t)}{1-\varphi(T-t)} = \psi(t),\ \forall t\in(0,T) \Rightarrow \frac{\varphi(T-t)}{1-\varphi(t)} = \psi(T-t),\ \forall t\in(0,T)$$
$$\Rightarrow \varphi(T-t) = \psi(T-t)[1-\varphi(t)],\ \forall t\in(0,T),$$

that is,

$$\varphi(T-t) = \psi(T-t)[1-\varphi(t)],\ \forall t\in(0,T). \tag{68}$$

Substituting the value of (68) into (67), we get

$$\frac{\varphi(t)}{1-\psi(T-t)[1-\varphi(t)]} = \psi(t) \Rightarrow \varphi(t) \triangleq \frac{\psi(t)[1-\psi(T-t)]}{1-\psi(t)\psi(T-t)},\ \forall t\in(0,T).$$

Theorem 3.1 implies the following Corollary 5.4

COROLLARY 5.4. (Ky Fan-type inequality) *Let the* Φ*-function* $\phi:\mathbb{R}\to\mathbb{R}$ *be a twice continuously differentiable, and let the function* $\varphi:\Omega\to(0,1/2]$ *be continuous. If* $|\phi|$ *is strictly increasing and* ϕ'' *is strictly decreasing, then we have the following Ky Fan-type inequality:*

$$\widehat{\varphi}|_\phi \geqslant \widehat{(1-\varphi)}|_\phi. \tag{69}$$

The equality in (69) *holds if and only if* $\varphi(s)=1/2,\ \forall s\in\Omega$.

Proof. Let $f\triangleq\varphi$ and $g\triangleq 1-\varphi$. Then the functions $f:\Omega\to(0,1/2]$ and $g:\Omega\to[1/2,1)$ are continuous and,

(i) $|\phi|$ is increasing and ϕ'' is decreasing;

(ii) $|f(s_1)-f(s_2)|=|g(s_1)-g(s_2)|$ and $f(s)\leqslant 1/2\leqslant g(s),\ \forall s_1,s_2,s\in\Omega$.

According to Theorem 3.1, we see that (69) holds.

Based on the above proof, we see that the equality in (69) holds if and only if $\varphi(s)=1/2,\ \forall s\in\Omega$. This ends the proof of Corollary 5.4. □

We remark here that, if the continuous function $\varphi:\Omega\to\mathbb{R}$ is not a constant function, where $\Omega\subseteq\mathbb{R}^m$ is a measurable set and its measure $|\Omega|>0$, then

$$\widehat{\varphi}>\widehat{(1-\varphi)}\Leftrightarrow \mathrm{E}\varphi<\frac{1}{2}, \tag{70}$$

and

$$\widehat{\varphi}=\widehat{(1-\varphi)}\Leftrightarrow \mathrm{E}\varphi=\frac{1}{2}. \tag{71}$$

Indeed, since the continuous function $\varphi:\Omega\to\mathbb{R}$ is not a constant, by the γ-power mean inequality (29), we have

$$\mathrm{E}^2\varphi<\mathrm{E}|\varphi|^2=\mathrm{E}\varphi^2.$$

Hence

$$\begin{aligned}
\widehat{\varphi}>\widehat{(1-\varphi)} &\Leftrightarrow \widehat{(1-\varphi)}<\widehat{\varphi}\\
&\Leftrightarrow \frac{\mathrm{E}(1-\varphi)^2}{\mathrm{E}^2(1-\varphi)}-1<\frac{\mathrm{E}\varphi^2}{\mathrm{E}^2\varphi}-1\\
&\Leftrightarrow \frac{1-2\mathrm{E}\varphi+\mathrm{E}\varphi^2}{1-2\mathrm{E}\varphi+\mathrm{E}^2\varphi}<\frac{\mathrm{E}\varphi^2}{\mathrm{E}^2\varphi}\\
&\Leftrightarrow \mathrm{E}^2\varphi\left(1-2\mathrm{E}\varphi+\mathrm{E}\varphi^2\right)-\mathrm{E}\varphi^2\left(1-2\mathrm{E}\varphi+\mathrm{E}^2\varphi\right)<0\\
&\Leftrightarrow \mathrm{E}^2\varphi-\mathrm{E}\varphi^2-2\mathrm{E}^3\varphi+2\mathrm{E}\varphi\mathrm{E}\varphi^2<0\\
&\Leftrightarrow \left(\mathrm{E}^2\varphi-\mathrm{E}\varphi^2\right)(1-2\mathrm{E}\varphi)<0\\
&\Leftrightarrow 1-2\mathrm{E}\varphi>0\\
&\Leftrightarrow \mathrm{E}\varphi<1/2\\
&\Rightarrow (70).
\end{aligned}$$

Similarly, we can prove (71).

Unfortunately, the inequality (35) is inconvenient for applications since the condition $0 < \varphi_{\sup} \leqslant 1/2$ is very strong. In this section, we will weaken the condition $0 < \varphi_{\sup} \leqslant 1/2$ for the inequality (35).

In the following discussion, we assume that the function $\varphi : \Omega \to (0,1)$ is continuous, where $\Omega \subseteq \mathbb{R}^m$ is a bounded and closed region and its measure $|\Omega| > 0$, and define that

$$\lambda_0 \triangleq \frac{\mathrm{V}_{3,2}(\varphi)}{1-\mathrm{V}_{3,2}(\varphi)} \wedge \lambda_1 \triangleq \frac{\varphi_{\inf}}{1-\varphi_{\inf}} \wedge \lambda_2 \triangleq \frac{\varphi_{\sup}}{1-\varphi_{\sup}} \wedge \lambda_4 \triangleq \frac{\mathrm{E}\varphi}{1-\mathrm{E}\varphi}. \tag{72}$$

Then, by (17), we have

$$0 \leqslant \lambda_1 \leqslant \lambda_4 \leqslant \lambda_2 \wedge 0 \leqslant \lambda_1 \leqslant \lambda_0 \leqslant \lambda_2. \tag{73}$$

In this section, Our main result is the following Theorem 5.1.

THEOREM 5.1. (Ky Fan-type inequality) *Let the function $\varphi : \Omega \to (0,1)$ be continuous, and let $0 < \lambda_4 \leqslant 1$. If (i) or (ii) or (iii) hold, where*

(i) $\lambda_2\lambda_4 \leqslant 1$, $\lambda_0 > \lambda_4$, and

$$\max\left\{2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right], 4\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right]\right\} \leqslant \gamma \leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right], \tag{74}$$

(ii) $\lambda_2\lambda_4 \leqslant 1$, $0 < \lambda_0 \leqslant \lambda_4$, and

$$4\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma \leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right], \tag{75}$$

(iii) $\lambda_2\lambda_4 > 1$, and

$$2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma \leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right], \tag{76}$$

then we have the Ky Fan-type inequality (35). *The equality in* (35) *holds if and only if φ is a constant function.*

We remark here that

$$0 < \lambda_4 \leqslant 1 \Leftrightarrow 0 < \mathrm{E}\varphi \leqslant \frac{1}{2} \wedge \lambda_2\lambda_4 \leqslant 1 \Leftrightarrow \varphi_{\sup} + \mathrm{E}\varphi \leqslant 1. \tag{77}$$

In order to prove Theorem 5.1, we need the following Lemmas 5.1–5.5.

LEMMA 5.1. (Continuous Jensen inequality, see [15]) *Let $E \subset \mathbb{R}^m$ be a bounded and closed region, and let the functions $f : E \to \mathbb{R}$ and $\phi : f(E) \to \mathbb{R}$ be Riemann integrable, where $f(E)$ is an interval, which is the value field of the function f. If $\phi : f(E) \to \mathbb{R}$ is a convex function* [3,14]*, then we have the following continuous Jensen inequality:*

$$\frac{\int_E \phi(f)}{\int_E} \geqslant \phi\left(\frac{\int_E f}{\int_E}\right). \tag{78}$$

Inequality (78) *is reversed if $\phi : f(E) \to \mathbb{R}$ is a concave function.*

LEMMA 5.2. (Discrete Jensen inequality, see [4–6, 15, 23, 27, 28]) *Let* $f : J \to \mathbb{R}$ *be a convex function, where* $J \subset \mathbb{R}$ *is an interval and its measure* $|J| > 0$*, and let* $p_j \geqslant 0$, $x_j \in J$, $j = 1, 2, \ldots, n$*, where* $\sum_{j=1}^n p_j > 0$*, then we have the following discrete Jensen inequality:*

$$\frac{\sum_{j=1}^n p_j f(x_j)}{\sum_{j=1}^n p_j} \geqslant f\left(\frac{\sum_{j=1}^n p_j x_j}{\sum_{j=1}^n p_j}\right). \tag{79}$$

Inequality (79) *is reversed if* $f : J \to \mathbb{R}$ *is a concave function.*

LEMMA 5.3. *Let the function* $\varphi : \Omega \to (0,1)$ *be continuous, and let* $0 < \lambda_4 \leqslant 1$ *with* $\lambda_2\lambda_4 \leqslant 1$*. If* $\lambda_0 > \lambda_4$*, then we have*

$$2\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right] \leqslant 2. \tag{80}$$

Proof. By (73), we have

$$\log(1/\lambda_4) \geqslant 0 \wedge \log(\lambda_0/\lambda_4) > 0 \Rightarrow \frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)} \geqslant 0 \Rightarrow 2\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right] \leqslant 2.$$

This completes the proof. □

LEMMA 5.4. *Let the function* $\varphi : \Omega \to (0,1)$ *be continuous, and let* $0 < \lambda_4 \leqslant 1$ *with* $\lambda_2\lambda_4 \leqslant 1$*. Then we have*

$$4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant 2 \leqslant 2\left[1 + \frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right]. \tag{81}$$

Proof. By (73), we have

$$\begin{aligned} 4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant 2 &\Leftrightarrow 1 \leqslant 2\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)} \\ &\Leftrightarrow \log(\lambda_2/\lambda_4) \leqslant 2\log(1/\lambda_4) \\ &\Leftrightarrow \lambda_2/\lambda_4 \leqslant (1/\lambda_4)^2 \\ &\Leftrightarrow \lambda_2\lambda_4 \leqslant 1. \end{aligned}$$

Hence

$$4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant 2.$$

Since

$$\begin{aligned} 0 < \lambda_1 \leqslant \lambda_4 \leqslant 1 &\Rightarrow \log(1/\lambda_4) \geqslant 0 \wedge \log(\lambda_4/\lambda_1) \geqslant 0 \\ &\Rightarrow \frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)} \geqslant 0 \\ &\Rightarrow 2 \leqslant 2\left[1 + \frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right], \end{aligned}$$

we have

$$2 \leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right].$$

This proves Lemma 5.4. □

LEMMA 5.5. *Let the function* $\varphi:\Omega\to(0,1)$ *be continuous, and let* $0<\lambda_4\leqslant 1$ *with* $\lambda_2\lambda_4>1$. *Then we have*

$$2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right]\leqslant 2\leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right]. \tag{82}$$

Proof. Indeed, by (73), we have

$$\begin{aligned} 0<\lambda_4\leqslant\lambda_2 \ \wedge\ 0<\lambda_4\leqslant 1 &\Rightarrow \log(1/\lambda_4)\geqslant 0 \ \wedge\ \log(\lambda_2/\lambda_4)\geqslant 0\\ &\Rightarrow \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\geqslant 0\\ &\Rightarrow 2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right]\leqslant 2 \end{aligned}$$

and

$$\begin{aligned} 0\leqslant\lambda_1\leqslant\lambda_4\leqslant 1 &\Rightarrow \log(1/\lambda_4)\geqslant 0 \ \wedge\ \log(\lambda_4/\lambda_1)\geqslant 0\\ &\Rightarrow \frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\geqslant 0\\ &\Rightarrow 2\leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right]. \end{aligned}$$

This ends the proof of Lemma 5.5. □

Now let us start the proof.

Proof. By Lemmas 5.3–5.5, we know that there exist real γ such that (74), (75) and (76) hold.

In the following proof, we continue to use the proof of Theorem 3.1.

Without losing of generality, we may assume that

$$\gamma\in\mathbb{R}\setminus\{0,1\} \ \wedge\ 0<\lambda_1\leqslant\lambda_2<1.$$

We first consider the case (i) in Theorem 5.1.

Let the Φ-function ϕ_γ be defined by (18), and let $f\triangleq\varphi$ with $g\triangleq 1-\varphi$. Since

$$w_{i,j}\left(1-\varphi(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right)=1-w_{i,j}\left(\varphi(\xi),\overline{\mathbf{p}}(\xi),t_1,t_2\right), \tag{83}$$

by (39), (83) and Lemma 3.1, we have

$$
\begin{aligned}
& \widehat{\varphi}_\gamma \geqslant \widehat{(1-\varphi)}_\gamma \\
\Leftrightarrow\ & \widehat{\varphi}|_{\phi_\gamma} \geqslant \widehat{(1-\varphi)}|_{\phi_\gamma} \\
\Leftrightarrow\ & \frac{\mathrm{JVar}_{\phi_\gamma} f}{|\phi_\gamma(1)|^{-1}|\phi_\gamma \mathrm{E} f|} \geqslant \frac{\mathrm{JVar}_{\phi_\gamma} g}{|\phi_\gamma(1)|^{-1}|\phi_\gamma \mathrm{E} g|} \\
\Leftrightarrow\ & \mathrm{JVar}_{\phi_\gamma} f - \frac{|\phi_\gamma \mathrm{E} f|}{|\phi_\gamma \mathrm{E} g|} \mathrm{JVar}_{\phi_\gamma} g \geqslant 0 \\
\Leftrightarrow\ & \mathrm{JVar}_{\phi_\gamma} f - \lambda_4^\gamma \mathrm{JVar}_{\phi_\gamma} g \geqslant 0 \\
\Leftrightarrow\ & \lim_{\|T\|\to 0} \left[A\left(\phi_\gamma\left(f(\xi)\right), \mathbf{p}\right) - \phi_\gamma\left(A\left(f(\xi), \mathbf{p}\right)\right)\right] \\
& -\lambda_4^\gamma \lim_{\|T\|\to 0} \left[A\left(\phi_\gamma\left(g(\xi)\right), \mathbf{p}\right) - \phi_\gamma\left(A\left(g(\xi), \mathbf{p}\right)\right)\right] g \geqslant 0 \\
\Leftrightarrow\ & \lim_{\|T\|\to 0} \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left\{ \iint_S \phi_\gamma''\left[w_{i,j}\left(f(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right)\right] \mathrm{d}t_1 \mathrm{d}t_2 \right\} \left(f(\xi_i) - f(\xi_j)\right)^2 \\
& - \lim_{\|T\|\to 0} \lambda_4^\gamma \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left\{ \iint_S \phi_\gamma''\left[w_{i,j}\left(g(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right)\right] \mathrm{d}t_1 \mathrm{d}t_2 \right\} \\
& \times \left(g(\xi_i) - g(\xi_j)\right)^2 \geqslant 0 \\
\Leftrightarrow\ & \lim_{\|T\|\to 0} \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left\{ \iint_S \phi_\gamma''\left[w_{i,j}\left(\varphi(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right)\right] \mathrm{d}t_1 \mathrm{d}t_2 \right\} \\
& \times \left(\varphi(\xi_i) - \varphi(\xi_j)\right)^2 \\
& - \lim_{\|T\|\to 0} \lambda_4^\gamma \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left\{ \iint_S \phi_\gamma''\left[w_{i,j}\left(1-\varphi(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right)\right] \mathrm{d}t_1 \mathrm{d}t_2 \right\} \\
& \times \left(\varphi(\xi_j) - \varphi(\xi_i)\right)^2 \geqslant 0 \\
\Leftrightarrow\ & \lim_{\|T\|\to 0} \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left\{ \iint_S \phi_\gamma''\left[w_{i,j}\left(\varphi(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right)\right] \mathrm{d}t_1 \mathrm{d}t_2 \right\} \\
& \times \left(\varphi(\xi_i) - \varphi(\xi_j)\right)^2 \\
& - \lim_{\|T\|\to 0} \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left\{ \iint_S \lambda_4^\gamma \phi_\gamma''\left[1 - w_{i,j}\left(\varphi(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right)\right] \mathrm{d}t_1 \mathrm{d}t_2 \right\} \\
& \times \left(\varphi(\xi_i) - \varphi(\xi_j)\right)^2 \geqslant 0 \\
\Leftrightarrow\ & \lim_{\|T\|\to 0} \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left[\iint_S \varpi(w_{i,j}) \mathrm{d}t_1 \mathrm{d}t_2 \right] \left(\varphi(\xi_i) - \varphi(\xi_j)\right)^2 \geqslant 0,
\end{aligned}
$$

i.e.

$$
\widehat{\varphi}_\gamma \geqslant \widehat{(1-\varphi)}_\gamma \Leftrightarrow \lim_{\|T\|\to 0} \sum_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi) \left[\iint_S \varpi(w_{i,j}) \mathrm{d}t_1 \mathrm{d}t_2 \right] \left(\varphi(\xi_i) - \varphi(\xi_j)\right)^2 \geqslant 0, \tag{84}
$$

where

$$\varpi(x) \triangleq \phi_\gamma''(x) - \lambda_4^\gamma \phi_\gamma''(1-x) = 2\left[x^{\gamma-2} - \lambda_4^\gamma (1-x)^{\gamma-2}\right],\ x \in (0,1), \tag{85}$$

and

$$w_{i,j} \triangleq w_{i,j}\left(\varphi(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right) = t_1\varphi(\xi_i) + t_2\varphi(\xi_j) + (1-t_1-t_2)A\left(\varphi(\xi), \mathbf{p}\right). \tag{86}$$

Since

$$\begin{aligned} w_{i,j}\left(\varphi(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right) &= t_1\varphi(\xi_i) + t_2\varphi(\xi_j) + (1-t_1-t_2)A\left(\varphi(\xi), \mathbf{p}\right) \\ &\geqslant t_1\varphi_{\inf} + t_2\varphi_{\inf} + (1-t_1-t_2)A\left(\varphi_{\inf}, \mathbf{p}\right) \\ &= t_1\varphi_{\inf} + t_2\varphi_{\inf} + (1-t_1-t_2)\varphi_{\inf} \\ &= \varphi_{\inf} \end{aligned}$$

and

$$\begin{aligned} w_{i,j}\left(\varphi(\xi), \overline{\mathbf{p}}(\xi), t_1, t_2\right) &= t_1\varphi(\xi_i) + t_2\varphi(\xi_j) + (1-t_1-t_2)A\left(\varphi(\xi), \mathbf{p}\right) \\ &\leqslant t_1\varphi_{\sup} + t_2\varphi_{\sup} + (1-t_1-t_2)A\left(\varphi_{\sup}, \mathbf{p}\right) \\ &= t_1\varphi_{\sup} + t_2\varphi_{\sup} + (1-t_1-t_2)\varphi_{\sup} \\ &= \varphi_{\sup}, \end{aligned}$$

we have

$$0 < \varphi_{\inf} \leqslant w_{i,j} \leqslant \varphi_{\sup} < 1,\ \forall i,j : 1 \leqslant i < j \leqslant n. \tag{87}$$

Case (i).1: $\gamma \leqslant 2.$ we first prove that the function

$$\varpi : \left(\varphi_{\inf}, \varphi_{\sup}\right) \to \mathbb{R}\ \wedge\ \varpi(x) \triangleq 2\left[x^{\gamma-2} - \lambda_4^\gamma(1-x)^{\gamma-2}\right] \tag{88}$$

is convex.

Indeed, by $\lambda_2 \geqslant \lambda_4 > 0$, (73) and (74), we have

$$\begin{aligned} \gamma \geqslant 4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] &\Rightarrow (\gamma-4)\log(\lambda_2/\lambda_4) \geqslant -4\log(1/\lambda_4) \\ &\Rightarrow \left(\frac{\lambda_2}{\lambda_4}\right)^{\gamma-4} \geqslant \left(\frac{1}{\lambda_4}\right)^{-4} \\ &\Rightarrow \lambda_2^{\gamma-4} - \lambda_4^\gamma \geqslant 0 \\ &\Rightarrow \varphi_{\sup}^{\gamma-4} - \lambda_4^\gamma(1-\varphi_{\sup})^{\gamma-4} \\ &= (1-\varphi_{\sup})^{\gamma-4}\left(\lambda_2^{\gamma-4} - \lambda_4^\gamma\right) \geqslant 0. \end{aligned}$$

Hence

$$\varphi_{\sup}^{\gamma-4} - \lambda_4^\gamma(1-\varphi_{\sup})^{\gamma-4} \geqslant 0. \tag{89}$$

By (74), (88), (89), Lemma 5.3 and

$$4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma \leqslant 2,$$

we have

$$\begin{aligned}\varpi'''(x) &= 2(\gamma-2)(\gamma-3)(\gamma-4)\left[x^{\gamma-5}+\lambda_4^{\gamma}(1-x)^{\gamma-5}\right]\leqslant 0,\ \forall x\in(0,1)\\ &\Rightarrow \varpi''(x)=2(\gamma-2)(\gamma-3)\left[x^{\gamma-4}-\lambda_4^{\gamma}(1-x)^{\gamma-4}\right]\\ &\geqslant \varpi''(\varphi_{\sup})=2(\gamma-2)(\gamma-3)\left[\varphi_{\sup}^{\gamma-4}-\lambda_4^{\gamma}(1-\varphi_{\sup})^{\gamma-4}\right]\\ &\geqslant 0,\ \forall x\in\left(\varphi_{\inf},\varphi_{\sup}\right).\end{aligned}$$

Hence

$$\varpi''(x)\geqslant 0,\ \forall x\in\left(\varphi_{\inf},\varphi_{\sup}\right). \tag{90}$$

By (90), we see that the function $\varpi:\left(\varphi_{\inf},\varphi_{\sup}\right)\to\mathbb{R}$ is convex.

Since the function $\varpi:\left(\varphi_{\inf},\varphi_{\sup}\right)\to\mathbb{R}$ is convex, by Lemma 5.1, we have

$$\frac{\iint_S \varpi(w_{i,j})\mathrm{d}t_1\mathrm{d}t_2}{\iint_S \mathrm{d}t_1\mathrm{d}t_2}\geqslant \varpi\left(\frac{\iint_S w_{i,j}\mathrm{d}t_1\mathrm{d}t_2}{\iint_S \mathrm{d}t_1\mathrm{d}t_2}\right). \tag{91}$$

Since

$$\phi_\gamma(t)=\frac{2}{\gamma(\gamma-1)}t^{\gamma}\ \wedge\ \phi_\gamma''(t)=2t^{\gamma-2}\ \wedge\ w_{i,j}=\frac{1}{2}\phi_3''(w_{i,j})\ \wedge\ 1=\frac{1}{2}\phi_2''(w_{i,j}), \tag{92}$$

by (91), (92) and Lemma 5.2, we have

$$\begin{aligned}&\frac{\lim_{\|T\|\to 0}\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)\left[\iint_S\varpi(w_{i,j})\mathrm{d}t_1\mathrm{d}t_2\right](\varphi(\xi_i)-\varphi(\xi_j))^2}{\lim_{\|T\|\to 0}\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\mathrm{d}t_1\mathrm{d}t_2\right)}\\
&=\lim_{\|T\|\to 0}\frac{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left[\iint_S\varpi(w_{i,j})\mathrm{d}t_1\mathrm{d}t_2\right]}{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\mathrm{d}t_1\mathrm{d}t_2\right)}\\
&\geqslant\lim_{\|T\|\to 0}\frac{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\mathrm{d}t_1\mathrm{d}t_2\right)\varpi\left(\frac{\iint_S w_{i,j}\mathrm{d}t_1\mathrm{d}t_2}{\iint_S\mathrm{d}t_1\mathrm{d}t_2}\right)}{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\mathrm{d}t_1\mathrm{d}t_2\right)}\\
&\geqslant\lim_{\|T\|\to 0}\varpi\left[\frac{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\mathrm{d}t_1\mathrm{d}t_2\right)\left(\frac{\iint_S w_{i,j}\mathrm{d}t_1\mathrm{d}t_2}{\iint_S\mathrm{d}t_1\mathrm{d}t_2}\right)}{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\mathrm{d}t_1\mathrm{d}t_2\right)}\right]\\
&=\lim_{\|T\|\to 0}\varpi\left[\frac{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\iint_S w_{i,j}\mathrm{d}t_1\mathrm{d}t_2}{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\mathrm{d}t_1\mathrm{d}t_2\right)}\right]\\
&=\lim_{\|T\|\to 0}\varpi\left[\frac{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\iint_S\frac{1}{2}\phi_3''(w_{i,j})\mathrm{d}t_1\mathrm{d}t_2}{\sum\limits_{1\leqslant i<j\leqslant n}\overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2\left(\iint_S\frac{1}{2}\phi_2''(w_{i,j})\mathrm{d}t_1\mathrm{d}t_2\right)}\right]\end{aligned}$$

$$= \lim_{\|T\|\to 0} \varpi \left[\frac{\sum\limits_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2 \iint_S \phi_3''(w_{i,j})\,\mathrm{d}t_1\mathrm{d}t_2}{\sum\limits_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2 \left(\iint_S \phi_2''(w_{i,j})\,\mathrm{d}t_1\mathrm{d}t_2\right)} \right]$$

$$= \varpi \left[\frac{\lim_{\|T\|\to 0} \sum\limits_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2 \iint_S \phi_3''(w_{i,j})\,\mathrm{d}t_1\mathrm{d}t_2}{\lim_{\|T\|\to 0} \sum\limits_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2 \left(\iint_S \phi_2''(w_{i,j})\,\mathrm{d}t_1\mathrm{d}t_2\right)} \right]$$

$$= \varpi \left(\frac{\mathrm{Var}^{[3]}\varphi}{\mathrm{Var}^{[2]}\varphi} \right)$$

$$= \varpi\left(\mathrm{V}_{3,2}(\varphi)\right).$$

Hence

$$\frac{\lim_{\|T\|\to 0} \sum\limits_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)\left[\iint_S \varpi(w_{i,j})\mathrm{d}t_1\mathrm{d}t_2\right](\varphi(\xi_i)-\varphi(\xi_j))^2}{\lim_{\|T\|\to 0} \sum\limits_{1\leqslant i<j\leqslant n} \overline{p}_i(\xi)\overline{p}_j(\xi)(\varphi(\xi_i)-\varphi(\xi_j))^2 \left(\iint_S \mathrm{d}t_1\mathrm{d}t_2\right)} \geqslant \varpi\left(\mathrm{V}_{3,2}(\varphi)\right). \tag{93}$$

By $\lambda_0 > \lambda_4 > 0$, (73) and (74), we have

$$\begin{aligned} 2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right] \leqslant \gamma \leqslant 2 &\Rightarrow (\gamma-2)\log(\lambda_0/\lambda_4) \geqslant -2\log(1/\lambda_4) \\ &\Rightarrow \left(\frac{\lambda_0}{\lambda_4}\right)^{\gamma-2} \geqslant \left(\frac{1}{\lambda_4}\right)^{-2} \\ &\Rightarrow \lambda_0^{\gamma-2} - \lambda_4^{\gamma} \geqslant 0 \\ &\Rightarrow \varpi\left(\mathrm{V}_{3,2}(\varphi)\right) = 2\left[\mathrm{V}_{3,2}^{\gamma-2}(\varphi) - \lambda_4^{\gamma}(1-\mathrm{V}_{3,2}(\varphi))^{\gamma-2}\right] \\ &= 2(1-\mathrm{V}_{3,2}(\varphi))^{\gamma-2}\left(\lambda_0^{\gamma-2}-\lambda_4^{\gamma}\right) \geqslant 0. \end{aligned}$$

So we get

$$\varpi\left(\mathrm{V}_{3,2}(\varphi)\right) \geqslant 0. \tag{94}$$

Combining with (84), (93) and (94), we see that the Ky Fan-type inequality (35) holds when

$$\max\left\{2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right], 4\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right]\right\} \leqslant \gamma \leqslant 2.$$

Case (i).2: $\gamma \geqslant 2$. By $\lambda_4 \geqslant \lambda_1 > 0$, (73), (74) and (87), we have

$$\begin{aligned} 2 \leqslant \gamma \leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right] &\Rightarrow (\gamma-2)\log(\lambda_4/\lambda_1) \leqslant 2\log(1/\lambda_4) \\ &\Rightarrow \left(\frac{\lambda_4}{\lambda_1}\right)^{\gamma-2} \leqslant \left(\frac{1}{\lambda_4}\right)^{2} \\ &\Rightarrow \lambda_1^{\gamma-2} - \lambda_4^{\gamma} \geqslant 0 \end{aligned}$$

$$\begin{aligned}
\Rightarrow \varpi(w_{i,j}) &= 2\left[w_{i,j}^{\gamma-2}(\varphi) - \lambda_4^{\gamma}(1-w_{i,j})^{\gamma-2}\right] \\
&\geqslant 2\left[\varphi_{\inf}^{\gamma-2}(\varphi) - \lambda_4^{\gamma}(1-\varphi_{\inf})^{\gamma-2}\right] \\
&= 2(1-\varphi_{\inf})^{\gamma-2}\left(\lambda_1^{\gamma-2} - \lambda_4^{\gamma}\right) \\
&\geqslant 0.
\end{aligned}$$

Thus,

$$\varpi(w_{i,j}) \geqslant 0,\ \forall i,j : 1 \leqslant i,j \leqslant n. \tag{95}$$

Combining with (84) and (95), we see that the Ky Fan-type inequality (35) holds when

$$2 \leqslant \gamma \leqslant 2\left[1 + \frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right].$$

Based on the above analysis, we know that the Ky Fan-type inequality (35) holds under the hypotheses (74).

Next, we consider the case (ii) in Theorem 5.1.

Case (ii).1: $\gamma \leqslant 2$. Since

$$0 < \lambda_0 \leqslant \lambda_4 \leqslant 1 \Rightarrow \log(\lambda_0/\lambda_4) \leqslant 0 \ \wedge\ \log(1/\lambda_4) \geqslant 0,$$

we have

$$\begin{aligned}
\gamma \leqslant 2 \Rightarrow \gamma \leqslant 2 \leqslant 2\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right] \\
&\Rightarrow \gamma \leqslant 2\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right] \\
&\Rightarrow (\gamma-2)\log(\lambda_0/\lambda_4) \geqslant -2\log(1/\lambda_4) \\
&\Rightarrow \left(\frac{\lambda_0}{\lambda_4}\right)^{\gamma-2} \geqslant \left(\frac{1}{\lambda_4}\right)^{-2} \\
&\Rightarrow \lambda_0^{\gamma-2} - \lambda_4^{\gamma} \geqslant 0 \\
&\Rightarrow \varpi\left(\mathrm{V}_{3,2}(\varphi)\right) = 2\left[\mathrm{V}_{3,2}^{\gamma-2}(\varphi) - \lambda_4^{\gamma}(1-\mathrm{V}_{3,2}(\varphi))^{\gamma-2}\right] \\
&= 2(1-\mathrm{V}_{3,2}(\varphi))^{\gamma-2}\left(\lambda_0^{\gamma-2} - \lambda_4^{\gamma}\right) \geqslant 0.
\end{aligned}$$

Hence inequality (94) also holds.

Based on the proof of case (i).1, we see that (84), (93) and (94) also hold under the hypotheses that

$$4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma \leqslant 2.$$

So, by the proof of case (i).1, we see that the Ky Fan-type inequality (35) holds for this case.

Case (ii).2: $\gamma \geqslant 2$. By the proof of case (i).2, we see that the Ky Fan-type inequality (35) also holds when

$$2 \leqslant \gamma \leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right].$$

Based on the above analysis, we know that the Ky Fan-type inequality (35) also holds under the hypotheses (75).

Finally, we consider the case (iii) in Theorem 5.1.

Case (iii).1: $\gamma \leqslant 2$. By (73), (76) and (87), we have

$$\begin{aligned} 2 \geqslant \gamma \geqslant 2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] &\Rightarrow \left(\frac{\lambda_2}{\lambda_4}\right)^{\gamma-2} \geqslant \left(\frac{1}{\lambda_4}\right)^{-2} \\ &\Rightarrow \lambda_2^{\gamma-2} - \lambda_4^{\gamma} \geqslant 0 \\ &\Rightarrow \varpi(w_{i,j}) = 2\left[w_{i,j}^{\gamma-2}(\varphi) - \lambda_4^{\gamma}(1-w_{i,j})^{\gamma-2}\right] \\ &\geqslant 2\left[\varphi_{\sup}^{\gamma-2}(\varphi) - \lambda_4^{\gamma}(1-\varphi_{\sup})^{\gamma-2}\right] \\ &= 2(1-\varphi_{\sup})^{\gamma-2}\left(\lambda_2^{\gamma-2}-\lambda_4^{\gamma}\right) \geqslant 0. \end{aligned}$$

Hence

$$\varpi(w_{i,j}) \geqslant 0,\ \forall i,j : 1 \leqslant i,j \leqslant n. \tag{96}$$

Combining with (84) and (96), we see that the Ky Fan-type inequality (35) holds when

$$2\left[1-\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \leqslant \gamma \leqslant 2.$$

Case (iii).2: $\gamma \geqslant 2$. By (73), (76) and the proof of Case (i).2, we see that (96) also hold. Combining with (84) and (96), we see that the Ky Fan-type inequality (35) also holds when

$$2 \leqslant \gamma \leqslant 2\left[1+\frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right].$$

According to the above analysis, we know that the Ky Fan-type inequality (35) still holds under the hypotheses (76). Therefore, the Ky Fan-type inequality (35) is proved.

Based on the above proof, we see that the equality in (35) holds if and only if φ is a constant function. This completes the proof of Theorem 5.1. □

Theorem 5.1 implies the following Corollary.

COROLLARY 5.5. (Ky Fan-type inequality) *Let the function* $\varphi : \Omega \to (0,1)$ *be continuous, and let*

$$\varphi_{\sup} + \mathrm{E}\varphi \leqslant 1 \ \wedge\ \varphi_{\sup} \geqslant \frac{1}{2}. \tag{97}$$

If (75) *hold, then we have the Ky Fan-type inequality* (35)*. The equality in* (35) *holds if and only if* φ *is a constant function.*

Proof. Since

$$\left(\varphi_{\sup} \geqslant \mathrm{E}\varphi > 0 \wedge \varphi_{\sup} + \mathrm{E}\varphi \leqslant 1 \Rightarrow 0 < \mathrm{E}\varphi \leqslant \frac{1}{2}\right) \wedge 0 < \lambda_4 \leqslant 1 \wedge \lambda_2\lambda_4 \leqslant 1, \quad (98)$$

by Theorem 5.1, we just need to prove that

$$\lambda_0 > \lambda_4 > 0 \Rightarrow 2\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right] \leqslant 4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right]. \quad (99)$$

Indeed, since

$$\varphi_{\sup} \geqslant \frac{1}{2} \Rightarrow \lambda_2 \geqslant 1, \quad (100)$$

by (73), (98) and (100), we have

$$\begin{aligned} 0 < \lambda_4 < \lambda_0 \leqslant \lambda_2,\ \lambda_2 \geqslant 1 \Rightarrow\ & 2\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_0/\lambda_4)}\right] - 4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \\ \leqslant\ & 2\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] - 4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] \\ =\ & 2\left[\frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)} - 1\right] \\ =\ & 2\frac{\log(1/\lambda_4) - \log(\lambda_2/\lambda_4)}{\log(\lambda_2/\lambda_4)} \\ =\ & 2\frac{\log(1/\lambda_2)}{\log(\lambda_2/\lambda_4)} \leqslant 0 \\ \Rightarrow\ & (99). \end{aligned}$$

Hence (99) is proved.

Based on the above proof, we see that the equality in (35) holds if and only if φ is a constant function. This completes the proof. □

REMARK 5.1. Based on the proofs of Theorem 5.1 and Corollary 5.5, we know that Theorem 5.1 and Corollary 5.5 are also true when $\Omega = \mathbb{N}_T$, where $\varphi : \Omega \to (0,1)$ is a discrete function, and $1 \leqslant T \leqslant \infty$.

6. Applications in business profit management model

DEFINITION 6.1. Let $f : \Omega \to (0,\infty)$ and $g : \Omega \to (0,\infty)$ be two continuous functions, where Ω is an interval and its measure $|\Omega| > 0$, and let $\gamma \in \mathbb{R}$ with $s \triangleq f + g$. Then we define that

(i) f is more stable than g, write as $f \prec g$, if $\widehat{(f/s)} < \widehat{(g/s)}$;

(ii) The stability of f is no worse than g, write as $f \preccurlyeq g$, if $\widehat{(f/s)} \leqslant \widehat{(g/s)}$;

(iii) The stability of f and g are the same, write as $f \backsim g$, if $\widehat{(f/s)} = \widehat{(g/s)}$;

(iv) f is more γ-stable than g, write as $f \prec_\gamma g$, if $\widehat{(f/s)}_\gamma < \widehat{(g/s)}_\gamma$;

(v) The γ-stability of f is no worse than g, write as $f \preccurlyeq_\gamma g$, if $\widehat{(f/s)}_\gamma \leqslant \widehat{(g/s)}_\gamma$;
(vi) The γ-stability of f and g are the same, write as $f \backsim_\gamma g$, if $\widehat{(f/s)}_\gamma = \widehat{(g/s)}_\gamma$.

Let $\mathbf{C}$ be a business company which only sells a product $\mathbf{P}$, and let

$$\mathbf{p} \triangleq \mathbf{p}(t) > 0,\ \mathbf{e} \triangleq \mathbf{e}(t) > 0 \text{ and } \mathbf{i} \triangleq \mathbf{i}(t) \triangleq \mathbf{p}(t) + \mathbf{e}(t) \tag{101}$$

be the *profit function, cost function* and the *income function* respectively [33] of $\mathbf{C}$ in a day, where $t \in (0,1)$ is the sale volume of $\mathbf{C}$ in a day. Then we can think that the variable t is a continuous random variable, which follows a uniform distribution, that is to say, it's probability density function is

$$p : (0,1) \to (0,\infty) \ \wedge \ p(t) = 1.$$

For the business company $\mathbf{C}$, it is desirable that the variance $\mathrm{Var}(\mathbf{e}/\mathbf{i})$ to be very large (in other words, it is desirable that the company's funds do not spend frequently) and the mathematical expectation $\mathrm{E}(\mathbf{e}/\mathbf{i})$ to be very small, hence it is desirable that the coefficient of stable $\widehat{(\mathbf{e}/\mathbf{i})}$ to be very large. Similarly, it is desirable that the variance $\mathrm{Var}(\mathbf{p}/\mathbf{i})$ to be very small and the mathematical expectation $\mathrm{E}(\mathbf{p}/\mathbf{i})$ to be very large, hence it is also desirable that the coefficient of stable $\widehat{(\mathbf{p}/\mathbf{i})}$ to be very small. Therefore, the business company desires that $\mathbf{p} \prec \mathbf{e}$ or $\mathbf{p} \backsim \mathbf{e}$ hold. This is the significance of the coefficient of stable and Definition 6.1 in business sector.

PROBLEM 6.1. (Business profit management model) Let the sale price of the product $\mathbf{P}$ is 1, i.e.

$$\mathbf{p} + \mathbf{e} = \mathbf{i} = t,\ t \in (0,1), \tag{102}$$

and let

$$\mathbf{e} \triangleq at^\alpha + bt \ \wedge \ a > 0 \ \wedge \ b \geqslant 0 \ \wedge \ a+b \leqslant 1 \ \wedge \ \alpha > 1, \tag{103}$$

where at^α is an allowance function [32, 33] which is the allowance of a salesperson, and bt is the production cost function which is the cost of production of the product $\mathbf{P}$. Our problem is that: How to find the parameters a, b, α and γ such that $\mathbf{p} \prec_\gamma \mathbf{e}$ or $\mathbf{p} \backsim_\gamma \mathbf{e}$?

We remark here that, the above business profit management model is an ordinary case when $\alpha = 1$ and, by (102) and (103), we have

$$0 < \mathbf{p} < t \ \wedge \ 0 < \mathbf{e} < t,\ \forall t \in (0,1). \tag{104}$$

We also remark here that, if $t \in (0,T), 0 < T < \infty$, then $t_* \triangleq t/T \in (0,1)$. Therefore, we can replace t with t_*, where

$$p : (0,T) \to (0,\infty) \ \wedge \ p(t) = 1/T \ \wedge \ \mathbf{p} + \mathbf{e} = \mathbf{i} = Tt_* \ \wedge \ \mathbf{e} \triangleq T(at_*^\alpha + bt_*).$$

For Problem 6.1, we have the following Assertions 6.1–6.7.

First, we demonstrate the applications of assertions (70) and (71) in business profit management model.

ASSERTION 6.1. *For Problem 6.1, we have*

$$\mathbf{p} \prec \mathbf{e} \Leftrightarrow \frac{a}{\alpha} + b < \frac{1}{2} \tag{105}$$

and

$$\mathbf{p} \backsim \mathbf{e} \Leftrightarrow \frac{a}{\alpha} + b = \frac{1}{2}. \tag{106}$$

Proof. In the following discussion, we define a auxiliary function as follows:

$$\varphi : (0,1) \to (0,1) \ \wedge \ \varphi(t) \triangleq \frac{\mathbf{e}}{\mathbf{i}} = at^{\alpha-1} + b. \tag{107}$$

Since $\alpha > 1$, we see that the function φ is not a constant function. By Definition 6.1 and the assertion (70), we have

$$\mathbf{p} \prec \mathbf{e} \Leftrightarrow \widehat{\left(\frac{\mathbf{p}}{\mathbf{i}}\right)} < \widehat{\left(\frac{\mathbf{e}}{\mathbf{i}}\right)} \Leftrightarrow \widehat{\varphi} > \widehat{(1-\varphi)} \Leftrightarrow \mathrm{E}\varphi = \int_0^1 \left(at^{\alpha-1} + b\right) \mathrm{d}t = \frac{a}{\alpha} + b < \frac{1}{2}.$$

Hence (105) is proved. Similarly, we can prove (106). This proves Assertion 6.1. □

Next, we demonstrate the applications of Corollary 5.5 in business profit management model.

ASSERTION 6.2. *Let*

$$a(\alpha^{-1} + 1) + 2b < 1 \ \wedge \ 1/2 < a + b \leqslant 1. \tag{108}$$

If

$$4\left[1 - \frac{\log(1/\lambda_4)}{\log(\lambda_2/\lambda_4)}\right] < \gamma < 2\left[1 + \frac{\log(1/\lambda_4)}{\log(\lambda_4/\lambda_1)}\right], \tag{109}$$

where

$$\lambda_1 = \frac{b}{1-b} \ \wedge \ \lambda_2 = \frac{a+b}{1-(a+b)} \ \wedge \ \lambda_4 = \frac{a\alpha^{-1} + b}{1-(a\alpha^{-1} + b)}, \tag{110}$$

then $\mathbf{p} \prec_\gamma \mathbf{e}$.

Proof. Indeed, the inequality $\widehat{(\mathbf{p}/\mathbf{i})}_\gamma < \widehat{(\mathbf{e}/\mathbf{i})}_\gamma$ can be rewritten as (35). By the proof of Assertion 6.1, we have

$$\mathrm{E}\varphi = a\alpha^{-1} + b. \tag{111}$$

Since

$$\varphi_{\inf} = b \ \wedge \ \varphi_{\sup} = a + b, \tag{112}$$

by (72), (111) and (112), we see that (110) hold. Since

$$\varphi_{\sup} + \mathrm{E}\varphi = a(\alpha^{-1} + 1) + 2b < 1 \ \wedge \ \varphi_{\sup} = a + b > 1/2 \tag{113}$$

and the function φ is not a constant function, according to Corollary 5.5, we have

$$\widehat{\varphi}_\gamma > \widehat{(1-\varphi)}_\gamma \Leftrightarrow \widehat{(\mathbf{p}/\mathbf{i})}_\gamma < \widehat{(\mathbf{e}/\mathbf{i})}_\gamma \Leftrightarrow \mathbf{p} \prec_\gamma \mathbf{e} \tag{114}$$

under the hypotheses (109). This completes the proof. □

Next, we demonstrate the applications of Corollary 5.4 in business profit management model.

ASSERTION 6.3. *Let* $a+b<1/2$ *and* $\gamma \in (0,2)$. *Then* $\mathbf{p} \prec_\gamma \mathbf{e}$.

Proof. Let the Φ-function $\phi : (0,\infty) \to \mathbb{R}$ be a twice continuously differentiable, and let $|\phi|$ is strictly increasing and ϕ'' is strictly decreasing. Since $\varphi_{\sup} = a+b < 1/2$ and the function φ is not a constant function, by Corollary 5.4, we have

$$\widehat{\varphi}|_\phi > \widehat{(1-\varphi)}|_\phi. \tag{115}$$

In (115), set $\phi = \phi_\gamma$, $\gamma \in (0,2)$, where ϕ_γ is defined by (18), we know that (114) hold for any $\gamma \in (0,2)$. Hence $\mathbf{p} \prec_\gamma \mathbf{e}$. This ends the proof of Assertion 6.3. □

Next, we demonstrate the applications of Corollary 5.2 in business profit management model.

ASSERTION 6.4. *Let* $\alpha > 2$ *and* $b = 0$. *Then, for any* $\gamma \in \mathbb{R}$, *we have* $\mathbf{p} \prec_\gamma \mathbf{e}$.

Proof. Indeed, by (107), we see that

$$\varphi : (0,1) \to (0,1) \wedge \varphi(t) = at^{\alpha-1} \wedge 0 < a \leqslant 1 \wedge \alpha - 1 > 1 \tag{116}$$

and the function φ is not a constant function. According to Corollary 5.2, we know that (114) hold for any $\gamma \in \mathbb{R}$. This proves Assertion 6.4. □

ASSERTION 6.5. *For any* $\gamma \in \mathbb{R}$, *the formula*

$$\mathbf{p} \backsim_\gamma \mathbf{e} \tag{117}$$

hold if and only if

$$\alpha = 2 \wedge a + 2b = 1. \tag{118}$$

Proof. First, assume that (117) hold for any $\gamma \in \mathbb{R}$, now we prove that (118) hold.

Set $\gamma = 2$ in (117), by Definition 6.1, we have $\mathbf{p} \backsim \mathbf{e}$. By the assertion (71), we have

$$\mathrm{E}\varphi = \frac{a}{\alpha} + b = \frac{1}{2}. \tag{119}$$

Let $\gamma \neq 0,1$. By (119) and $\mathrm{E}(1-\varphi)=1-\mathrm{E}\varphi=1/2$, we have

$$\begin{aligned}
\mathbf{p} \backsim_\gamma \mathbf{e},\ \forall\gamma\in\mathbb{R} &\Leftrightarrow \widehat{(1-\varphi)}_\gamma=\widehat{\varphi}_\gamma,\ \forall\gamma\in\mathbb{R}\\
&\Leftrightarrow \frac{\mathrm{Var}^{[\gamma]}(1-\varphi)}{\mathrm{E}^\gamma(1-\varphi)}=\frac{\mathrm{Var}^{[\gamma]}\varphi}{\mathrm{E}^\gamma\varphi},\ \forall\gamma\in\mathbb{R}\\
&\Leftrightarrow \frac{2}{\gamma(\gamma-1)}\frac{\mathrm{E}(1-\varphi)^\gamma-\mathrm{E}^\gamma(1-\varphi)}{\mathrm{E}^\gamma(1-\varphi)}\\
&= \frac{2}{\gamma(\gamma-1)}\frac{\mathrm{E}\varphi^\gamma-\mathrm{E}^\gamma\varphi}{\mathrm{E}^\gamma\varphi},\ \forall\gamma\in\mathbb{R}\\
&\Leftrightarrow \frac{\mathrm{E}(1-\varphi)^\gamma}{\mathrm{E}^\gamma(1-\varphi)}=\frac{\mathrm{E}\varphi^\gamma}{\mathrm{E}^\gamma\varphi},\ \forall\gamma\in\mathbb{R}\\
&\Leftrightarrow \mathrm{E}(1-\varphi)^\gamma=\mathrm{E}\varphi^\gamma,\ \forall\gamma\in\mathbb{R}\\
&\Rightarrow [\mathrm{E}(1-\varphi)^\gamma]^{1/\gamma}=[\mathrm{E}\varphi^\gamma]^{1/\gamma},\ \forall\gamma\in\mathbb{R}\setminus\{0\}\\
&\Rightarrow \lim_{\gamma\to\infty}[\mathrm{E}(1-\varphi)^\gamma]^{1/\gamma}=\lim_{\gamma\to\infty}[\mathrm{E}\varphi^\gamma]^{1/\gamma}\\
&\Leftrightarrow (1-\varphi)_{\sup}=\varphi_{\sup}\\
&\Leftrightarrow 1-\varphi_{\inf}=\varphi_{\sup}\\
&\Leftrightarrow 1-b=a+b\\
&\Leftrightarrow a+2b=1.
\end{aligned}$$

Hence $a+2b=1$. Since $0<a\leqslant 1$, by (119) and $a+2b=1$, we have $\alpha=2$. Thus, (118) is proved.

Next, assume that (118) hold, now we prove (117) hold for any $\gamma\in\mathbb{R}$.

Indeed, by (118), we have $\mathrm{E}(1-\varphi)=1-\mathrm{E}\varphi=1/2$. So, by the above proof, we see that

$$\mathbf{p} \backsim_\gamma \mathbf{e},\ \forall\gamma\in\mathbb{R} \Leftrightarrow \mathrm{E}(1-\varphi)^\gamma=\mathrm{E}\varphi^\gamma,\ \forall\gamma\in\mathbb{R}. \tag{120}$$

By (118), we have

$$\varphi=at+\frac{1-a}{2}. \tag{121}$$

Let $x\triangleq(1-a)/2$. Then $(1+a)/2=1-x$. Set $\varphi^*\triangleq 1-\varphi$. Then $\varphi=1-\varphi^*$. By (121), we have

$$\begin{aligned}
\mathrm{E}(1-\varphi)^\gamma &= \int_0^1(1-\varphi)^\gamma\mathrm{d}t=a^{-1}\int_{(1-a)/2}^{(1+a)/2}(1-\varphi)^\gamma\mathrm{d}\varphi\\
&=a^{-1}\int_x^{1-x}(1-\varphi)^\gamma\mathrm{d}\varphi=a^{-1}\int_{1-x}^{x}(\varphi^*)^\gamma\mathrm{d}(1-\varphi^*)\\
&=a^{-1}\int_x^{1-x}(\varphi^*)^\gamma\mathrm{d}\varphi^*=a^{-1}\int_x^{1-x}\varphi^\gamma\mathrm{d}\varphi\\
&=\mathrm{E}\varphi^\gamma,\ \forall\gamma\in\mathbb{R},
\end{aligned}$$

that is,

$$\mathrm{E}(1-\varphi)^\gamma=\mathrm{E}\varphi^\gamma,\ \forall\gamma\in\mathbb{R}. \tag{122}$$

By (120) and (122), we see that (117) hold for any $\gamma \in \mathbb{R}$.

The proof of Assertion 6.5 is completed. □

ASSERTION 6.6. *Let* $\mathbf{p} \backsim \mathbf{e}$. *Then we have the following two assertions.*
(i) If $\alpha > 2$, *then there exists a real* $\gamma^* \geqslant 2$ *such that* $\mathbf{p} \prec_\gamma \mathbf{e}$, $\forall \gamma > \gamma^*$.
(ii) If $1 < \alpha < 2$, *then there exists a real* $\gamma_* \leqslant 1$ *such that* $\mathbf{p} \prec_\gamma \mathbf{e}$, $\forall \gamma < \gamma_*$.

Proof. Two real numbers x and y are said to have the same sign [32], written as $x \sim y$, if

$$x > 0 \Rightarrow y > 0 \ \wedge \ x = 0 \Rightarrow y = 0 \ \wedge \ x < 0 \Rightarrow y < 0. \tag{123}$$

Let $\gamma \in \mathbb{R} \setminus \{1\}$. Then, For any function $f : \Omega \to (0,\infty)$ and $g : \Omega \to (0,\infty)$, we have

$$\begin{aligned}
\widehat{f}_\gamma - \widehat{g}_\gamma &= \frac{2}{\gamma(\gamma-1)}\frac{\mathrm{E}f^\gamma - \mathrm{E}^\gamma f}{\mathrm{E}^\gamma f} - \frac{2}{\gamma(\gamma-1)}\frac{\mathrm{E}g^\gamma - \mathrm{E}^\gamma g}{\mathrm{E}^\gamma g} \\
&= \frac{2}{\gamma(\gamma-1)}\left(\frac{\mathrm{E}f^\gamma}{\mathrm{E}^\gamma f} - \frac{\mathrm{E}g^\gamma}{\mathrm{E}^\gamma g}\right) \\
&\sim \frac{2}{\gamma(\gamma-1)}\left(\frac{\mathrm{E}f^\gamma}{\mathrm{E}g^\gamma} - \frac{\mathrm{E}^\gamma f}{\mathrm{E}^\gamma g}\right) \\
&\sim \frac{2}{(\gamma-1)}\left[\frac{(\mathrm{E}f^\gamma)^{1/\gamma}}{(\mathrm{E}g^\gamma)^{1/\gamma}} - \frac{\mathrm{E}f}{\mathrm{E}g}\right] \\
&\sim \mathrm{sign}(\gamma-1)\left[\frac{(\mathrm{E}f^\gamma)^{1/\gamma}}{(\mathrm{E}g^\gamma)^{1/\gamma}} - \frac{\mathrm{E}f}{\mathrm{E}g}\right],
\end{aligned}$$

that is

$$\widehat{f}_\gamma - \widehat{g}_\gamma \sim \mathrm{F}_{f,g}(\gamma) \triangleq \mathrm{sign}(\gamma-1)\left[\frac{(\mathrm{E}f^\gamma)^{1/\gamma}}{(\mathrm{E}g^\gamma)^{1/\gamma}} - \frac{\mathrm{E}f}{\mathrm{E}g}\right], \ \forall \gamma \in \mathbb{R} \setminus \{1\}, \tag{124}$$

where $\mathrm{sign}(\gamma)$ is the sign function [32].

We say that the function $\mathrm{F}_{f,g}(\gamma)$ in (124) is a *feature function* of the function f for g.

By (124), we see that, $f \prec_\gamma g$, $\forall \gamma \in \mathbb{R} \setminus \{1\}$, if and only if

$$\mathrm{F}_{f/s,g/s}(\gamma) < 0, \ \forall \gamma \in \mathbb{R} \setminus \{1\}, \tag{125}$$

where $s \triangleq f + g$.

By (28), (107), (119) and (124), we have

$$\mathrm{F}_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\infty) = \frac{(1-\varphi)_{\sup}}{\varphi_{\sup}} - \frac{\mathrm{E}(1-\varphi)}{\mathrm{E}\varphi} = \frac{1-\varphi_{\inf}}{\varphi_{\sup}} - \frac{\mathrm{E}(1-\varphi)}{\mathrm{E}\varphi} = \frac{1-b}{a+b} - \frac{1-(a\alpha^{-1}+b)}{a\alpha^{-1}+b}$$

and

$$\mathrm{F}_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(-\infty) = -\left[\frac{(1-\varphi)_{\inf}}{\varphi_{\inf}} - \frac{\mathrm{E}(1-\varphi)}{\mathrm{E}\varphi}\right] = -\frac{1-a-b}{b} + \frac{1-(a\alpha^{-1}+b)}{a\alpha^{-1}+b},$$

that is,

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\infty) = \frac{1-b}{a+b} - \frac{1-(a\alpha^{-1}+b)}{a\alpha^{-1}+b} \tag{126}$$

and

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(-\infty) = -\frac{1-a-b}{b} + \frac{1-(a\alpha^{-1}+b)}{a\alpha^{-1}+b}. \tag{127}$$

By $\mathbf{p} \backsim \mathbf{e}$, we have

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(2) = 0 \tag{128}$$

and

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(1) = 0. \tag{129}$$

Since $\mathbf{p} \backsim \mathbf{e}$, by the assertion (71), we see that (119) holds. Thus, by (119), (126) and (127), we have

$$\begin{aligned} F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\infty) < 0 &\Leftrightarrow \frac{1-b}{a+b} - \frac{1-(a\alpha^{-1}+b)}{a\alpha^{-1}+b} < 0 \Leftrightarrow \frac{1-b}{a+b} - 1 < 0 \\ &\Leftrightarrow a+2b > 1 \Leftrightarrow \frac{a}{2} > \frac{1}{2} - b = \frac{a}{\alpha} \Leftrightarrow \alpha > 2, \end{aligned}$$

and

$$\begin{aligned} F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(-\infty) < 0 &\Leftrightarrow -\frac{1-a-b}{b} + \frac{1-(a\alpha^{-1}+b)}{a\alpha^{-1}+b} < 0 \Leftrightarrow -\frac{1-a-b}{b} + 1 < 0 \\ &\Leftrightarrow a+2b < 1 \Leftrightarrow \frac{a}{2} < \frac{1}{2} - b = \frac{a}{\alpha} \Leftrightarrow 1 < \alpha < 2, \end{aligned}$$

that is,

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\infty) < 0 \Leftrightarrow \alpha > 2, \tag{130}$$

and

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(-\infty) < 0 \Leftrightarrow 1 < \alpha < 2. \tag{131}$$

(i) Let $\alpha > 2$. By (124), (130), (128) and the theory of limit , there exists a real $\gamma^* \geqslant 2$ such that

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma) < 0,\ \forall \gamma > \gamma^* \Leftrightarrow \mathbf{p} \prec_\gamma \mathbf{e},\ \forall \gamma > \gamma^*. \tag{132}$$

(ii) Let $1 < \alpha < 2$. By (124), (131), (128), (129) and the theory of limit, there exists a real $\gamma_* \leqslant 1$ such that

$$F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma) < 0,\ \forall \gamma < \gamma_* \Leftrightarrow \mathbf{p} \prec_\gamma \mathbf{e},\ \forall \gamma < \gamma_*. \tag{133}$$

This completes the proof of Assertion 6.6. □

ASSERTION 6.7. *Let* $\mathbf{p} \backsim \mathbf{e}$ *and* $b = 1/4$. *If* $\alpha = 3 > 2$, *then we can choose the parameter* $\gamma^* = 2$ *in Assertion* 6.6. *If* $1 < \alpha = 3/2 < 2$, *then we can choose the parameter* $\gamma_* = 1$ *in Assertion* 6.6.

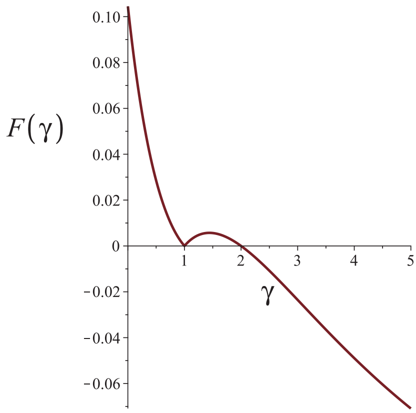

Figure 1: *The graph of the feature function* $F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma)$*, where* $\alpha = 3$.

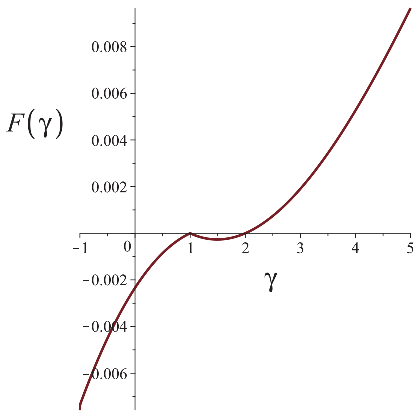

Figure 2: *The graph of the feature function* $F_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma)$*, where* $\alpha = 3/2$.

Proof. Indeed, by (119), we have $a = \alpha/4$ and, by (124), we have

$$\mathrm{F}_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma) \triangleq \operatorname{sign}(\gamma-1)\left\{\frac{\left[\int_0^1\left(1-\frac{\alpha t^{\alpha-1}+1}{4}\right)^{\gamma}\mathrm{d}t\right]^{1/\gamma}}{\left[\int_0^1\left(\frac{\alpha t^{\alpha-1}+1}{4}\right)^{\gamma}\mathrm{d}t\right]^{1/\gamma}}-1\right\}, \tag{134}$$

where $\gamma \in \mathbb{R}$. By means of the command Plot[ ] of the *Mathematica* software, we know that the graph of the feature function $\mathrm{F}_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma)$ is depicted in Figure 1 where $\alpha = 3 > 2$ and, the graph of the feature function $\mathrm{F}_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma)$ is depicted in Figure 2 where $1 < \alpha = 3/2 < 2$. By means of the commands Solve[ ] or FindMinimum[ ] of the Mathematica software, we get

$$\alpha = 3 \Rightarrow \gamma^* = 2 \ \wedge \ \alpha = 3/2 \Rightarrow \gamma_* = 1, \tag{135}$$

where γ^* and γ_* are the roots of the *feature equation*

$$F(\gamma) \triangleq \mathrm{F}_{\mathbf{p}/\mathbf{i},\mathbf{e}/\mathbf{i}}(\gamma) = 0, \ \gamma \in \mathbb{R}. \tag{136}$$

Assertion 6.7 is proved. □

7. Conclusions

This paper introduced the theory of ϕ-Jensen coefficient which is based on our previous works. By means of the functional analysis, linear algebra, discrete mathematics and inequality theories with proper hypotheses, the Jensen-type inequality $\widehat{f}|_\phi \geqslant \widehat{g}|_\phi$, Marshall-type inequality $\widetilde{f}_\gamma \geqslant \widetilde{g}_\gamma$ and the Ky Fan-type inequality $\widehat{\varphi}_\gamma \geqslant \widehat{(1-\varphi)}_\gamma$ are established, and the proofs of these inequalities are novel and interesting. We also displayed the applications of our main results in business profit management model, and some conditions such that $\mathbf{p} \prec_\gamma \mathbf{e}$ with $\mathbf{p} \backsim_\gamma \mathbf{e}$ hold are found and, these conditions can be achieved.

In particular, we weakened the conditions for the Ky Fan-type inequality $\widehat{(1-\varphi)}_\gamma \leqslant \widehat{\varphi}_\gamma$ since the traditional conditions for the inequality are very strong which are inconvenient for application.

It is worth pointing out that to find new properties of the ϕ-Jensen coefficient $\widehat{\varphi}|_\phi$ is an important research topic and, how to further weaken the conditions for the Ky Fan-type inequality $\widehat{(1-\varphi)}_\gamma \leqslant \widehat{\varphi}_\gamma$ to hold is also an important research topic and, how to further weaken the conditions for the relationship $\mathbf{p} \preccurlyeq_\gamma \mathbf{e}$ to hold is still an important research topic. These research topics are of the theoretical significance in the analysis of variance and the application value in business sector.

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