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GENERALIZATIONS OF HARDY–TYPE INEQUALITIES BY THE HERMITE INTERPOLATING POLYNOMIAL

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Abstract. In this paper we obtain generalizations of Hardy-type inequalities for convex functions of the higher order by applying Hermite interpolating polynomials. The results for particular cases: Lagrange, $(m,n-m)$ and two-point Taylor interpolating polynomials are also considered. Finally, we derive the Grüss and Ostrowski type inequalities related to these generalizations.

1. Introduction

Let $(\Sigma_1,\Omega_1,\mu_1)$ and $(\Sigma_2,\Omega_2,\mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f,k)$ denote the class of functions $g:\Omega_1\to\mathbb{R}$ with the representation

$$g(x)=\int_{\Omega_2} k(x,t)f(t)d\mu_2(t),$$

and A_k be an integral operator defined by

$$A_kf(x):=\frac{g(x)}{K(x)}=\frac{1}{K(x)}\int_{\Omega_2} k(x,t)f(t)d\mu_2(t), \tag{1}$$

where $k:\Omega_1\times\Omega_2\to\mathbb{R}$ is measurable and non-negative kernel, $f:\Omega_2\to\mathbb{R}$ is measurable function and

$$0<K(x):=\int_{\Omega_2} k(x,t)d\mu_2(t), \quad x\in\Omega_1. \tag{2}$$

The following result was given in [11] (see also [13]).

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THEOREM 1. *Let u be a weight function, $k(x,y) \geqslant 0$. Assume that $\frac{k(x,y)}{K(x)}u(x)$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by*

$$v(y) := \int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{3}$$

If ϕ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \leqslant \int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) \tag{4}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Im f \subseteq I$, where A_k is defined by (1)–(2).

Inequality (4) is generalization of Hardy's inequality. G. H. Hardy [7] stated and proved that the inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\,dt \right)^p dx \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\,dx,\ p > 1, \tag{5}$$

holds for all non-negative functions f such that $f \in L^p(\mathbb{R}_+)$ and $\mathbb{R}_+ = (0,\infty)$. The constant $\left(\frac{p}{p-1}\right)^p$ is sharp. More details about Hardy's inequality can be found in [16] and [17].

We also note that (5) can be interpreted as the Hardy operator $H : Hf(x) := \frac{1}{x}\int_0^x f(t)\,dt$, maps L^p into L^p with the operator norm $p' = \frac{p}{p-1}$.

DEFINITION 1. Let f be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function f at distinct points $x_0,\ldots,x_n \in [a,b]$ is defined recursively (see [4], [18]) by

$$f[x_i] = f(x_i), \quad (i = 0,\ldots,n)$$

and

$$f[x_0,\ldots,x_n] = \frac{f[x_1,\ldots,x_n] - f[x_0,\ldots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\ldots,x_n]$ is independent of the order of the points $x_0,\ldots,x_n$.

The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\ldots,x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

The notion of n-convexity was defined in terms of divided differences by T. Popoviciu [20]. A function $\phi : [\alpha,\beta] \to \mathbb{R}$ is n-convex, $n \geqslant 0$, if its n-th order divided differences $[x_0,\ldots,x_n;\phi]$ are nonnegative for all choices of $(n+1)$ distinct points $x_i \in [\alpha,\beta]$. If ϕ is $n-$convex then we can assume that ϕ is n-times differentiable and $\phi^{(n)} \geqslant 0$ (see [18]).

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0\cdot\infty, \frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a non-negative measurable function on the actual interval or more general set.

2. Preliminaries

Let $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$, $(r \geqslant 2)$ be the given points. For $\phi \in C^n([\alpha,\beta])$ $(n \geqslant r)$ a unique polynomial $\rho_H(s)$ of degree $(n-1)$ exists, such that *Hermite conditions* hold:

$$\rho_H^{(i)}(a_j) = \phi^{(i)}(a_j), \quad 0 \leqslant i \leqslant k_j, \quad 1 \leqslant j \leqslant r,$$

where $\sum_{j=1}^{r} k_j + r = n.$

In particular, for $r = n$, $k_j = 0$ for all j, we have *Lagrange conditions:*

$$\rho_L(a_j) = \phi(a_j), \quad 1 \leqslant j \leqslant n.$$

For $r = 2, 1 \leqslant m \leqslant n-1$, $k_1 = m-1$, $k_2 = n-m-1$, we have *Type* $(m,n-m)$ *conditions:*

$$\rho_{(m,n)}^{(i)}(\alpha) = \phi^{(i)}(\alpha), \quad 0 \leqslant i \leqslant m-1,$$

$$\rho_{(m,n)}^{(i)}(\beta) = \phi^{(i)}(\beta), \quad 0 \leqslant i \leqslant n-m-1.$$

For $n = 2m$, $r = 2$ and $k_1 = k_2 = m-1$, we have *Two-point Taylor conditions:*

$$\rho_{2T}^{(i)}(\alpha) = \phi^{(i)}(\alpha), \quad \rho_{2T}^{(i)}(\beta) = \phi^{(i)}(\beta), \quad 0 \leqslant i \leqslant m-1.$$

The following theorem and remark can be found in [3].

THEOREM 2. *Let $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$, $(r \geqslant 2)$, be the given points and $\phi \in C^n([\alpha,\beta])$, $(n \geqslant r)$. Let $\rho_H(s)$ be the Hermite inrepolating polynomial. Then*

$$\begin{aligned}\phi(t) &= \rho_H(t) + R_{H,n}(\phi,t)\\ &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i)}(a_j) + \int_\alpha^\beta G_{H,n}(t,s)\phi^{(n)}(s)ds,\end{aligned}$$

where H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right)\Bigg|_{t=a_j}(t-a_j)^k, \tag{6}$$

where

$$\omega(t)=\prod_{j=1}^{r}(t-a_j)^{k_j+1},$$

and $G_{H,n}(t,s)$ *is defined by*

$$G_{H,n}(t,s)=\begin{cases}\sum_{j=1}^{l}\sum_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s\leqslant t,\\ -\sum_{j=l+1}^{r}\sum_{i=0}^{k_j}\frac{(a_j-s)^{n-i-1}}{(n-i-1)!}H_{ij}(t);\ s\geqslant t,\end{cases} \tag{7}$$

for all $a_l\leqslant s\leqslant a_{l+1}$*;* $l=0,\dots,r$ *with* $a_0=\alpha$ *and* $a_{r+1}=\beta$*.*

REMARK 1. For Lagrange conditions, from Theorem 2 we have

$$\phi(t)=\rho_L(t)+R_L(\phi,t)$$

where $\rho_L(t)$ is the Lagrange interpolating polynomial i.e.

$$\rho_L(t)=\sum_{j=1}^{n}\prod_{\substack{k=1\\k\neq j}}^{n}\left(\frac{t-a_k}{a_j-a_k}\right)\phi(a_j)$$

and the remainder $R_L(\phi,t)$ is given by

$$R_L(\phi,t)=\int_\alpha^\beta G_L(t,s)\phi^{(n)}(s)ds$$

with

$$G_L(t,s)=\frac{1}{(n-1)!}\begin{cases}\sum_{j=1}^{l}(a_j-s)^{n-1}\prod_{\substack{k=1\\k\neq j}}^{n}\left(\frac{t-a_k}{a_j-a_k}\right), & s\leqslant t\\ -\sum_{j=l+1}^{n}(a_j-s)^{n-1}\prod_{\substack{k=1\\k\neq j}}^{n}\left(\frac{t-a_k}{a_j-a_k}\right), & s\geqslant t\end{cases} \tag{8}$$

$a_l\leqslant s\leqslant a_{l+1}$, $l=1,2,\dots,n-1$ with $a_1=\alpha$ and $a_n=\beta$.

For type $(m,n-m)$ conditions, from Theorem 2 we have

$$\phi(t)=\rho_{(m,n)}(t)+R_{(m,n)}(\phi,t)$$

where $\rho_{(m,n)}(t)$ is $(m,n-m)$ interpolating polynomial, i.e.

$$\rho_{(m,n)}(t)=\sum_{i=0}^{m-1}\tau_i(t)\phi^{(i)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(t)\phi^{(i)}(\beta),$$

with

$$\tau_i(t)=\frac{1}{i!}(t-\alpha)^i\left(\frac{t-\beta}{\alpha-\beta}\right)^{n-m}\sum_{k=0}^{m-1-i}\binom{n-m+k-1}{k}\left(\frac{t-\alpha}{\beta-\alpha}\right)^k \tag{9}$$

and

$$\eta_i(t) = \frac{1}{i!}(t-\beta)^i \left(\frac{t-\alpha}{\beta-\alpha}\right)^m \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{t-\beta}{\alpha-\beta}\right)^k, \tag{10}$$

and the remainder $R_{(m,n)}(\phi,t)$ is given by

$$R_{(m,n)}(\phi,t) = \int_\alpha^\beta G_{(m,n)}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{(m,n)}(t,s) = \begin{cases} \sum_{j=0}^{m-1} \left[\sum_{p=0}^{m-1-j} \binom{n-m+p-1}{p} \left(\frac{t-\alpha}{\beta-\alpha}\right)^p \right] \frac{(t-\alpha)^j(\alpha-s)^{n-j-1}}{j!(n-j-1)!} \left(\frac{\beta-t}{\beta-\alpha}\right)^{n-m}, & s \leqslant t \\ -\sum_{i=0}^{n-m-1} \left[\sum_{q=0}^{n-m-i-1} \binom{m+q-1}{q} \left(\frac{\beta-t}{\beta-\alpha}\right)^q \frac{(t-\beta)^i(\beta-s)^{n-i-1}}{i!(n-i-1)!} \right] \left(\frac{t-\alpha}{\beta-\alpha}\right)^m, & t \leqslant s. \end{cases} \tag{11}$$

For type Two-point Taylor conditions, from Theorem 2 we have

$$\phi(t) = \rho_{2T}(t) + R_{2T}(\phi,t)$$

where $\rho_{2T}(t)$ is the two-point Taylor interpolating polynomial i.e,

$$\begin{aligned} \rho_{2T}(t) = \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \Bigg[& \phi^{(i)}(\alpha) \frac{(t-\alpha)^i}{i!} \left(\frac{t-\beta}{\alpha-\beta}\right)^m \left(\frac{t-\alpha}{\beta-\alpha}\right)^k \\ & + \phi^{(i)}(\beta) \frac{(t-\beta)^i}{i!} \left(\frac{t-\alpha}{\beta-\alpha}\right)^m \left(\frac{t-\beta}{\alpha-\beta}\right)^k \Bigg] \end{aligned} \tag{12}$$

and the remainder $R_{2T}(\phi,t)$ is given by

$$R_{2T}(\phi,t) = \int_\alpha^\beta G_{2T}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{2T}(t,s) = \begin{cases} \frac{(-1)^m}{(2m-1)!} p^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (t-s)^{m-1-j} q^j(t,s), & s \leqslant t; \\ \frac{(-1)^m}{(2m-1)!} q^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (s-t)^{m-1-j} p^j(t,s), & s \geqslant t; \end{cases} \tag{13}$$

where $p(t,s) = \frac{(s-\alpha)(\beta-t)}{\beta-\alpha}$, $q(t,s) = p(s,t), \forall t,s \in [\alpha,\beta]$.

New results involving the Hardy inequality involving Green functions and Lidstone interpolation polynomial can be found in [10], [12], [14], [15] and [19]. Also, new results involving the Hermite interpolation polynomial can be found in [1].

3. Main results

Applying Hermite's interpolating polynomial we obtain a generalization of Hardy type inequality which holds for non-negative weights u, v. We give our first result.

THEOREM 3. *Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $u : \Omega_1 \to \mathbb{R}$ and $v : \Omega_2 \to \mathbb{R}$ be weight functions. Let $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$ ($r \geqslant 2$) be the given points, $k_j \geqslant 0$, $j = 1, \ldots, r$, with $\sum_{j=1}^{r} k_j + r = n$. Let $\phi \in C^n([\alpha, \beta])$ be n-convex and $A_k f(x), K(x)$ be defined by (1) and (2) respectively. If*

$$\int_{\Omega_2} v(y) G_{H,n}(v(y), s) d\mu_2(y) - \int_{\Omega_1} u(x) G_{H,n}(A_k f(x), s) d\mu_1(x) \geqslant 0, \quad s \in [\alpha, \beta],$$

then

$$\int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \tag{14}$$
$$\geqslant \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) \left[\int_{\Omega_2} v(y) H_{ij}(v(y)) d\mu_2(y) - \int_{\Omega_1} u(x) v_q H_{ij}(A_k f(x)) d\mu_1(x) \right],$$

where $G_{H,n}$ and H_{ij} are defined as in (7) and (6), respectively.

Proof. (i) Since $\phi \in C^n([\alpha, \beta])$, applying Theorem 2 on

$$\int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x)$$

we get

$$\int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \tag{15}$$
$$= \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) \left[\int_{\Omega_2} v(y) H_{ij}(v(y)) d\mu_2(y) - \int_{\Omega_1} u(x) v_q H_{ij}(A_k f(x)) d\mu_1(x) \right]$$
$$+ \int_{\alpha}^{\beta} \left[\int_{\Omega_2} v(y) G_{H,n}(v(y), s) d\mu_2(y) - \int_{\Omega_1} u(x) G_{H,n}(A_k f(x), s) d\mu_1(x) \right] \phi^{(n)}(s) ds.$$

Since ϕ is n-convex on $[\alpha, \beta]$, then we have $\phi^{(n)} \geqslant 0$ on $[\alpha, \beta]$. Moreover, the inequality (14) holds. $\square$

We begin with the following result:

THEOREM 4. *Let all the assumptions of Theorem 3 be satisfied. Additionally, let v be defined by (3). If (14) holds and the function*

$$\bar{F}(\cdot) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) H_{ij}(\cdot) \tag{16}$$

is convex on $[\alpha, \beta]$ then the inequality (4) holds.

Proof. If (14) holds, the right hand side of (14) can be written in the form

$$\int_{\Omega_2} v(y) \bar{F}(f(y)) d\mu_2(y) - \int_{\Omega_1} u(x) \bar{F}(A_k f(x)) d\mu_1(x),$$

where $\bar{F}$ is defined by (16). If $\bar{F}$ is convex, then by Theorem 1 we have

$$\int_{\Omega_2} v(y) \bar{F}(f(y)) d\mu_2(y) - \int_{\Omega_1} u(x) \bar{F}(A_k f(x)) d\mu_1(x) \geqslant 0,$$

i.e. the right-hand side of (14) is non-negative, so (4) immediately follows. □

By using Lagrange conditions we get the following generalization of Theorem 1.

COROLLARY 1. *Let $\alpha \leqslant a_1 < a_2 < \ldots < a_n \leqslant \beta$ $(n \geqslant 2)$ be the given points and $\phi \in C^n([\alpha, \beta])$ be n-convex. Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $u : \Omega_1 \to \mathbb{R}$ be a weight function and v be defined by (3).*

(i) If

$$\int_{\Omega_2} v(y) G_L(v(y), s) d\mu_2(y) - \int_{\Omega_1} u(x) G_L(A_k f(x), s) d\mu_1(x) \geqslant 0, \quad s \in [\alpha, \beta],$$

then

$$\begin{aligned} &\int_{\Omega_2} \phi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x)) u(x) d\mu_1(x) \\ &\geqslant \int_{\Omega_2} v(y) \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{u=1 \\ u \neq j}}^{n} \left(\frac{f(y) - a_u}{a_j - a_u} \right) d\mu_2(y) \\ &\quad - \int_{\Omega_1} u(x) \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{u=1 \\ u \neq j}}^{n} \left(\frac{A_k f(x) - a_u}{a_j - a_u} \right) d\mu_1(x), \end{aligned} \tag{17}$$

where G_L is defined as in (8).

(ii) If (17) holds and the function

$$\tilde{F}(\cdot)=\sum_{j=1}^{n}\phi(a_j)\prod_{\substack{u=1\\u\neq j}}^{n}\left(\frac{\cdot-a_u}{a_j-a_u}\right)$$

is convex on $[\alpha,\beta]$, then

$$\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)\leqslant\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y).$$

By using type $(m,n-m)$ conditions we can give the following result.

COROLLARY 2. *Let $n\geqslant 2$, $1\leqslant m\leqslant n-1$ and $\phi\in C^n([\alpha,\beta])$ be n-convex. Let $(\Sigma_1,\Omega_1,\mu_1)$ and $(\Sigma_2,\Omega_2,\mu_2)$ be measure spaces with positive σ-finite measures. Let $u:\Omega_1\to\mathbb{R}$ be a weight function and v be defined by (3).*

(i) If

$$\int_{\Omega_2}G_{(m,n)}(f(y),s)d\mu_2(y)-\int_{\Omega_1}u(x)G_{(m,n)}(A_kf(x),s)\geqslant 0,\quad s\in[\alpha,\beta],$$

then

$$\begin{aligned}&\int_{\Omega_2}v(y)\phi(f(y))d\mu_2(y)-\int_{\Omega_1}u(x)\phi(A_kf(x))d\mu_1(x)\\&\geqslant\int_{\Omega_2}v(y)\left(\sum_{i=0}^{m-1}\tau_i(f(y))\phi^{(i)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(f(y))\phi^{(i)}(\beta)\right)d\mu_2(y)\\&\quad-\int_{\Omega_1}u(x)\left(\sum_{i=0}^{m-1}\tau_i(A_k(f(x))\phi^{i}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(A_kf(x))\phi^{(i)}(\beta)\right)d\mu_1(x),\end{aligned}\tag{18}$$

where τ_i, η_i and $G_{(m,n)}$ are defined as in (9), (10) and (11), respectively.

(ii) If (18) holds and the function

$$\hat{F}(\cdot)=\sum_{i=0}^{m-1}\tau_i(\cdot)\phi^{(i)}(\alpha)+\sum_{i=0}^{n-m-1}\eta_i(\cdot)\phi^{(i)}(\beta)$$

is convex on $[\alpha,\beta]$, then

$$\int_{\Omega_1}u(x)\phi(A_kf(x))d\mu_1(x)\leqslant\int_{\Omega_1}u(x)\phi(A_kf(x))d\mu_1(x).$$

By using Two-point Taylor conditions we can give the following result.

COROLLARY 3. *Let* $m \geqslant 1$ *and* $\phi \in C^{2m}([\alpha,\beta])$ *be* $2m$*-convex. Let* $(\Sigma_1,\Omega_1,\mu_1)$ *and* $(\Sigma_2,\Omega_2,\mu_2)$ *be measure spaces with positive* σ*-finite measures. Let* $u:\Omega_1 \to \mathbb{R}$ *be a weight function and* v *be defined by* (3)*.*

(i) If

$$\int_{\Omega_2} v(y)G_{2T}(f(y),s)d\mu_2(y) - \int_{\Omega_1} u(x)G_{2T}(A_k f(x),s)d\mu_1(x) \geqslant 0, \quad s \in [\alpha,\beta],$$

then

$$\begin{aligned}&\int_{\Omega_2} v(y)\phi(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)\phi(A_k f(x))d\mu_1(x)\\ &\geqslant \int_{\Omega_2} v(y)\rho_{2T}(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)\rho_{2T}(A_k f(x))d\mu_1(x),\end{aligned}$$

where ρ_{2T} *and* G_{2T} *are defined as in* (12) *and* (13)*, respectively.*

(ii) Moreover, if the function ρ_{2T} *is convex on* $[\alpha,\beta]$*, then*

$$\int_{\Omega_1} u(x)\phi(A_k f(x))d\mu_1(x) \leqslant \int_{\Omega_1} u(x)\phi(A_k f(x))d\mu_1(x).$$

REMARK 2. Motivated by the inequality (14), under the assumptions of Theorem 3, we define the linear functional $A: C^n([\alpha,\beta]) \to \mathbb{R}$ by

$$\begin{aligned}A(\phi) = &\int_{\Omega_2} \phi(f(y))v(y)d\mu_2(y) - \int_{\Omega_1} \phi(A_k f(x))u(x)d\mu_1(x)\\ &- \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j)\left[\int_{\Omega_2} v(y)H_{ij}(v(y))d\mu_2(y) - \int_{\Omega_1} u(x)v_q H_{ij}(A_k f(x))d\mu_1(x)\right],\end{aligned}$$

Then for every n-convex functions $\phi \in C^n([\alpha,\beta])$ we have $A(\phi) \geqslant 0$. Using the linearity and positivity of this functional we may derive corresponding mean-value theorems applying the same method as given in [2] and [19]. Moreover, we could produce new classes of exponentially convex functions and as outcome we get new means of the Cauchy type. Here we also refer to [9] with related results.

4. Grüss and Ostrowski type inequalities

P. L. Chebyshev [6] obtained the following inequality

$$|T(f,g)| \leqslant \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty$$

where $f,g:[\alpha,\beta]\to\mathbb{R}$ are absolutely continuous functions whose derivatives f' and g' are bounded and $T(f,g)$ is so-called Chebyshev functional defined as

$$T(f,g):=\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)g(t)dt-\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)dt. \tag{19}$$

Here $\|\cdot\|_\infty$ denotes the norm in $L_\infty[\alpha,\beta]$, the space of essentially bounded functions on $[\alpha,\beta]$, defined by $\|f\|_\infty=\underset{t\in[\alpha,\beta]}{ess\sup}|f(t)|$. We also use notation $\|\cdot\|_p$, $p\geqslant 1$, for L_p norm.

P. Cerone and S. S. Dragomir [5], considering the Chebyshev functional (19), obtained the following two related results.

THEOREM 5. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be Lebesgue integrable and $g:[\alpha,\beta]\to\mathbb{R}$ be absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(g')^2\in L_1[\alpha,\beta]$. Then*

$$|T(f,g)|\leqslant\frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta(x-\alpha)(\beta-x)[g'(x)]^2dx\right)^{\frac{1}{2}}. \tag{20}$$

The constant $\frac{1}{\sqrt{2}}$ in (20) is the best possible.

THEOREM 6. *Let $g:[\alpha,\beta]\to\mathbb{R}$ be monotonic nondecreasing and $f:[\alpha,\beta]\to\mathbb{R}$ be absolutely continuous with $f'\in L_\infty[\alpha,\beta]$. Then*

$$|T(f,g)|\leqslant\frac{1}{2(\beta-\alpha)}\|f'\|_\infty\int_\alpha^\beta(x-\alpha)(\beta-x)dg(x). \tag{21}$$

The constant $\frac{1}{2}$ in (21) is the best possible.

We consider the function $\mathscr{B}:[\alpha,\beta]\to\mathbb{R}$, defined under assumptions of Theorem 3, by

$$\mathscr{B}(s)=\int_{\Omega_2}v(y)G_{H,n}(f(y),s)d\mu_2(y)-\int_{\Omega_1}u(x)G_{H,n}(A_kf(x),s)d\mu_1(x), \tag{22}$$

where $G_{H,n}$ is defined as in (7).

THEOREM 7. *Let $(\Sigma_1,\Omega_1,\mu_1)$ and $(\Sigma_2,\Omega_2,\mu_2)$ be measure spaces with positive σ-finite measures. Let $u:\Omega_1\to\mathbb{R}$ and $v:\Omega_2\to\mathbb{R}$ be weight functions. Let $\alpha\leqslant a_1<a_2<\ldots<a_r\leqslant\beta$ $(r\geqslant 2)$ be the given points, $k_j\geqslant 0$, $j=1,\ldots,r$, with $\sum_{j=1}^r k_j+r=n$. Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n)}$ is an absolutely continuous on $[\alpha,\beta]$ with $(\cdot-\alpha)(\beta-\cdot)(\phi^{(n+1)})^2\in L_1[\alpha,\beta]$ and $A_kf(x),K(x)$ be defined by (1) and (2)*

respectively. Let H_{ij} and $\mathscr{B}$ be defined as in (6) and (22), respectively. Then the remainder $R(\phi;\alpha,\beta)$ defined by

$$R(\phi;\alpha,\beta) = \int_{\Omega_2} v(y)\phi(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)\phi(A_k f(x))d\mu_1(x) - \sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\int_{\Omega_2} v(y)H_{ij}(f(y))d\mu_2(y) - \int_{\Omega_1} u(x)H_{ij}(A_k f(x))d\mu_1(x)\right] - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta \mathscr{B}(s)ds \tag{23}$$

satisfies the estimation

$$|R(\phi;\alpha,\beta)| \leqslant \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}[T(\mathscr{B},\mathscr{B})]^{\frac{1}{2}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}. \tag{24}$$

Proof. Comparing (15) and (23) we have

$$R(\phi;\alpha,\beta) = \int_\alpha^\beta \mathscr{B}(s)\phi^{(n)}(s)ds - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta \mathscr{B}(s)ds = \int_\alpha^\beta \mathscr{B}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}ds\int_\alpha^\beta \mathscr{B}(s)ds = (\beta-\alpha)T(\mathscr{B},\phi^{(n)}).$$

Applying Theorem 5 on the functions $\mathscr{B}$ and $\phi^{(n)}$ we obtain (24). □

Using Theorem 6 we obtain the Grüss type inequality.

THEOREM 8. *Let $(\Sigma_1,\Omega_1,\mu_1)$ and $(\Sigma_2,\Omega_2,\mu_2)$ be measure spaces with positive σ-finite measures. Let $u:\Omega_1\to\mathbb{R}$ and $v:\Omega_2\to\mathbb{R}$ be weight functions. Let $\alpha\leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$ $(r\geqslant 2)$ be the given points, $k_j\geqslant 0$, $j=1,\ldots,r$, with $\sum_{j=1}^{r} k_j + r = n$. Let $\phi\in C^n([\alpha,\beta])$ be such that $\phi^{(n+1)}\geqslant 0$ on $[\alpha,\beta]$, H_{ij} and $\mathscr{B}$ be defined as in (6) and (22), respectively. Then the remainder $R(\phi;\alpha,\beta)$ defined by (23) satisfies the estimation*

$$|R(\phi;\alpha,\beta)| \leqslant \|\mathscr{B}'\|_\infty\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{25}$$

Proof. Since $R(\phi;\alpha,\beta) = (\beta-\alpha)T(\mathscr{B},\phi^{(n)})$, applying Theorem 6 on the functions $\mathscr{B}$ and $\phi^{(n)}$ we obtain (25). □

We present the Ostrowski type inequality related to generalizations of Sherman's inequality.

THEOREM 9. *Let* $(\Sigma_1,\Omega_1,\mu_1)$ *and* $(\Sigma_2,\Omega_2,\mu_2)$ *be measure spaces with positive* σ*-finite measures. Let* $u:\Omega_1\to\mathbb{R}$ *and* $v:\Omega_2\to\mathbb{R}$ *be weight functions. Let* $\alpha\leqslant a_1<a_2<\ldots<a_r\leqslant\beta$ $(r\geqslant 2)$ *be the given points,* $k_j\geqslant 0$, $j=1,\ldots,r$, *with* $\sum_{j=1}^{r}k_j+r=n$. *Let* $\phi\in C^n([\alpha,\beta])$, $1\leqslant p,q\leqslant\infty$, $1/p+1/q=1$ *and* $\left|\phi^{(n)}\right|^p\in L_p[\alpha,\beta]$. *Then*

$$\left|\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)\right.$$
$$\left.-\sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\int_{\Omega_2}v(y)H_{ij}(v(y))d\mu_2(y)-\int_{\Omega_1}u(x)v_qH_{ij}(A_kf(x))d\mu_1(x)\right]\right|$$
$$\leqslant\left\|\phi^{(n)}\right\|_p\|\mathscr{B}\|_q,$$

where H_{ij} *and* $\mathscr{B}$ *are defined as in* (6) *and* (22)*, respectively.*

The constant $\|\mathscr{B}\|_q$ *is sharp for* $1<p\leqslant\infty$ *and the best possible for* $p=1$.

Proof. Under assumption of theorem the identity (15) holds. Applying the well-known Hölder inequality to (15), we have

$$\left|\int_{\Omega_2}\phi(f(y))v(y)d\mu_2(y)-\int_{\Omega_1}\phi(A_kf(x))u(x)d\mu_1(x)\right.$$
$$\left.-\sum_{j=1}^{r}\sum_{i=0}^{k_j}\phi^{(i)}(a_j)\left[\int_{\Omega_2}v(y)H_{ij}(v(y))d\mu_2(y)-\int_{\Omega_1}u(x)v_qH_{ij}(A_kf(x))d\mu_1(x)\right]\right|$$
$$=\left|\int_\alpha^\beta\left[\int_{\Omega_2}v(y)G_{H,n}(f(y),s)d\mu_2(y)-\int_{\Omega_1}u(x)G_{H,n}(A_kf(x),s)d\mu_1(x)\right]\phi^{(n)}(s)ds\right|$$
$$=\left|\int_\alpha^\beta\mathscr{B}(s)\phi^{(n)}(s)ds\right|\leqslant\left\|\phi^{(n)}\right\|_p\left(\int_\alpha^\beta|\mathscr{B}(s)|^qds\right)^{\frac{1}{q}}.$$

The proof of the sharpness is analog to one in proof of Theorem 11 in [8]. □

REFERENCES

[1] M. ADIL KHAN, S. IVELIĆ BRADANOVIĆ AND J. PEČARIĆ, *Generalizations of Sherman's inequality by Hermite's interpolating polynomial*, Math. Inequal. Appl., **19**, no. 4 (2016), 1181–1192.

[2] R. P. AGARWAL, S. IVELIĆ BRADANOVIĆ AND J. PEČARIĆ, *Generalizations of Sherman's inequality by Lidstone's interpolationg polynomial*, J. Inequal. Appl. **6**, 2016 (2016).

[3] R. P. AGARWAL, P. J. Y. WONG, *Error Inequalities in Polynomial Interpolation and Their Applications*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[4] K. E. ATKINSON, *An Introduction to Numerical Analysis*, 2nd ed., Wiley, New York, 1989.

[5] P. CERONE, S. S. DRAGOMIR, *Some new Ostrowski-type bounds for the Cebysev functional and applications*, J. Math. Inequal. **8** (1) (2014), 159–170.

[6] P. L. CHEBYSHEV, *Sur les expressions approximatives des integrales definies par les autres prises entre les mémes limites*, Proc. Math. Soc. Charkov, **2**, (1882) 93–98.

[7] G. H. HARDY, *Notes on some points in the integral calculus LX: An inequality between integrals* (60), Messenger of Math. **54** (1925), 150–156.

[8] S. IVELIĆ BRADANOVIĆ, NAVEED LATIF, J. PEČARIĆ, *Generalizations of Sherman's inequality by Taylor's formula*, Journal of Inequalities and Special functions, vol. 8, issue 2 (2017), 18–30.

[9] S. IVELIĆ BRADANOVIĆ, J. PEČARIĆ, *Generalizations of Sherman's inequality*, Per. Math. Hung, **74**, 197–219 (2017).

[10] K. KRULIĆ HIMMELREICH, *Generalizations of Hardy Type Inequalities by Taylor's Formula*, Mathematica Slovaca **72** (1) (2022), 67–84.

[11] K. KRULIĆ, J. PEČARIĆ, AND L.-E. PERSSON, *Some new Hardy–type inequalities with general kernels*, Math. Inequal. Appl. **12** (3) (2009), 473–485.

[12] K. KRULIĆ, J. PEČARIĆ, *Some new Hardy–type inequalities with general kernels II*, Math. Inequal. Appl. **19** (1) (2016), 73–84.

[13] K. KRULIĆ HIMMELREICH, J. PEČARIĆ, D. POKAZ, *Inequalities of Hardy and Jensen*, Element, Zagreb, 2013.

[14] K. KRULIĆ HIMMELREICH, J. PEČARIĆ, D. POKAZ, M. PRALJAK, *Generalizations of Hardy Type Inequalities by Abel-Gontscharoff's Interpolating Polynomial*, Mathematics 2021, **9**, 1724.

[15] K. KRULIĆ HIMMELREICH, J. PEČARIĆ, D. POKAZ, M. PRALJAK, *Hardy-type inequalities generalized via Montgomery identity*, accepted for publication, Montes Taurus J. Pure Appl. Math. **6** (3), (2024), 62–71.

[16] A. KUFNER, L. MALIGRANDA, AND L.-E. PERSSON, *The prehistory of the Hardy inequality*, Amer. Math. Monthly **113** (2006), no. 8, 715–732.

[17] A. KUFNER, L. MALIGRANDA, AND L.-E. PERSSON, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[18] J. E. PEČARIĆ, F. PROSCHAN, AND Y. L. TONG, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, San Diego, 1992.

[19] D. POKAZ, *Inequality of Hardy-type for n-convex function via interpolation polynomial and Green functions*, Math. Inequal. Appl. **26** (4) (2023), 965–976.

[20] T. POPOVICIU, *Introduction á la théorie des différences divisées*, Bull. Mathem., Societea Romana de Stiinte, Bukharest 42, 1940, 65–78.

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