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STEVIĆ–SHARMA TYPE OPERATORS FROM H^∞ INTO THE BLOCH SPACE

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Abstract. In this paper, we give some characterizations for the boundedness and compactness of some Stević-Sharma type operators called the polynomial differentiation composition operators from H^∞ into the Bloch space on the unit disk.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ the unit circle and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. We denote by $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a. Then $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathscr{B} = \mathscr{B}(\mathbb{D})$, if

$$\|f\|_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

It is well known that $\mathscr{B}$ is a Banach space under the norm $\|f\|_{\mathscr{B}} = |f(0)| + \|f\|_\beta$. For more results about some operators on the Bloch space, see [9, 11, 16, 17, 19, 37, 40, 41, 44]. Let $H^\infty = H^\infty(\mathbb{D})$ denote the set of all bounded analytic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. Note that $H^\infty \subset \mathscr{B}$ and that $\|f\|_{\mathscr{B}} \leqslant \|f\|_\infty$ if $f \in H^\infty$. For $\varphi \in S(\mathbb{D})$, $\|\varphi\|_{\mathscr{B}} \leqslant \|\varphi\|_\infty \leqslant 1$.

Let $\varphi \in S(\mathbb{D})$. The composition operator C_φ is defined by

$$C_\varphi f = f\circ\varphi, \quad f \in H(\mathbb{D}).$$

The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. Beside the integral type operators (see, for example, [1, 9, 22, 44] and the references therein), the composition operators have been studied the most. See [3, 41] and the references therein for the study of various properties of composition operators.

For $n \in \mathbb{N}_0$, the nth differentiation operator D^n is defined by

$$D^n f = f^{(n)}, \quad f \in H(\mathbb{D}),$$

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where $f^{(0)}=f$. If $n=1$, it is the classical differentiation operator D and typically unbounded on many holomorphic function spaces. Products of composition and differentiation operators have been studied considerably (see, for example, [12, 13, 20, 21, 25, 38, 47] and the references therein).

Let $\psi\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$. We denote the generalized weighted composition operator (also called weighted differentiation composition operator) by $D^n_{\psi,\varphi}$, i.e.,

$$D^n_{\psi,\varphi}f=\psi\cdot f^{(n)}\circ\varphi, f\in H(\mathbb{D}).$$

When $n=0$, $D^n_{\psi,\varphi}$ is the well-known weighted composition operator and always denoted by ψC_φ. The operator $D^n_{\psi,\varphi}$ was introduced by Zhu in [42]. See, for example, [24, 26, 27, 42, 43, 45, 46, 48] for more information and results on generalized weighted composition operator on analytic function spaces. A corresponding operator on the unit ball in $\mathbb{C}^n$ was introduced by Stević in [28].

In [10, 19] were obtained some characterizations for the boundedness and compactness of the weighted composition operator $\psi C_\varphi : H^\infty\to\mathscr{B}$. It was shown that $\psi C_\varphi : H^\infty\to\mathscr{B}$ is compact if and only if $\psi C_\varphi : H^\infty\to\mathscr{B}$ is bounded and

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|\psi'(z)|=0 \quad\text{and}\quad \lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}=0.$$

Colonna [2] characterized the boundedness and compactness of weighted composition operators by using two families functions and $\psi\varphi^n$. Among others, she showed that $\psi C_\varphi : H^\infty\to\mathscr{B}$ is compact if and only if $\psi C_\varphi : H^\infty\to\mathscr{B}$ is bounded and $\lim_{|\varphi(z)|\to 1}(1-|z|^2)|\psi'(z)|=0$ and $\lim_{n\to\infty}\left\|\psi\varphi^n\right\|_{\mathscr{B}}=0$.

The study of sums of generalized weighted composition operators has been proposed by Stević and Sharma (see, for example, [31, 32, 33, 34]). In [31, 32, 33], the authors studied the operator defined as follows:

$$D^m_{\psi_1,\psi_2,\varphi}f=\psi_1\cdot f^{(m)}\circ\varphi+\psi_2\cdot f^{(m+1)}\circ\varphi,$$

with $m=0$, whereas the case of arbitrary m was studied in [34]. See also [8, 14, 15, 39] for more results about this and related operators.

Having published [34], Stević proposed his collaborators to study the operator

$$T^{k,n}_{\vec{\psi},\varphi}f=\sum_{j=0}^{k}\psi_j\cdot f^{(n+j)}\circ\varphi=\sum_{j=0}^{k}D^{n+j}_{\psi_j,\varphi}f,\quad f\in H(\mathbb{D}),$$

where $n,k\in\mathbb{N}_0$, $\varphi\in S(\mathbb{D})$ and $\vec{\psi}=(\psi_0,\psi_1,\dots,\psi_k)$. Here $\psi_j\in H(\mathbb{D})$, $j=0,1,\dots,k$. When $n=0$, we denote $T^{k,n}_{\vec{\psi},\varphi}$ by $P^k_{\vec{\psi},\varphi}$ for the simplicity. The operator $P^k_{\vec{\psi},\varphi}$ has been recently studied in [5, 36, 49]. A special case was also studied in [23]. For some n-dimensional counterparts of the Stević-Sharma type operators see [29, 30, 35]. A natural question arises as to how to characterize the boundedness and compactness of $P^k_{\vec{\psi},\varphi} : H^\infty\to\mathscr{B}$.

In this paper, we obtain some characterizations for the boundedness and compactness of the Stević-Sharma type operator, that is, the polynomial differentiation composition operator $P^k_{\vec{\psi},\varphi}$ from H^∞ into the Bloch space $\mathscr{B}$, extending, among other things, some of the results, for example, in [10, 13, 19], and complementing some of the results in [25, 26].

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Boundedness of $P^k_{\vec{\psi},\varphi} : H^\infty \to \mathscr{B}$

In this section, we characterize the boundedness of the operator $P^k_{\vec{\psi},\varphi} : H^\infty \to \mathscr{B}$. For this purpose, we need the following lemma.

LEMMA 2.1. [41] *Let n be a positive integer and $f \in \mathscr{B}$. Then there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \leqslant C \frac{\|f\|_{\mathscr{B}}}{(1-|z|^2)^n}.$$

THEOREM 2.1. *Let $k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j = 0,1,\ldots,k$. Then the operator $P^k_{\vec{\psi},\varphi} : H^\infty \to \mathscr{B}$ is bounded if and only if*

$$\sum_{i=0}^{k+1} M_i < \infty.$$

Here

$$(i)\ \ M_0 = \sup_{z\in\mathbb{D}} (1-|z|^2)|\psi_0'(z)|;$$

$$(ii)\ \ M_j = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z) + \psi_j'(z)|}{(1-|\varphi(z)|^2)^j}, \quad for \quad j = 1,2,\cdots,k;$$

$$(iii)\ M_{k+1} = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}.$$

Proof. First, suppose that $\sum_{i=0}^{k+1} M_i < \infty$. Let $f \in H^\infty$. By Lemma 2.1 and the fact that $\|f\|_{\mathscr{B}} \lesssim \|f\|_\infty$ we have

$$\begin{aligned}
\|P^k_{\vec{\psi},\varphi} f\|_{\mathscr{B}} &= |P^k_{\vec{\psi},\varphi} f(0)| + \|P^k_{\vec{\psi},\varphi} f\|_\beta = \Big|\sum_{j=0}^{k} \psi_j(0) f^{(j)}(\varphi(0))\Big| \\
&+ \sup_{z\in\mathbb{D}} (1-|z|^2)\Big|\sum_{j=0}^{k} (\psi_j'(z) f^{(j)}(\varphi(z)) + \psi_j(z)\varphi'(z) f^{(j+1)}(\varphi(z)))\Big| \\
\leqslant& \sum_{j=0}^{k} |\psi_j(0)||f^{(j)}(\varphi(0))| + \sup_{z\in\mathbb{D}} (1-|z|^2)|\psi_0'(z)||f(\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}} (1-|z|^2) \sum_{j=1}^{k} |\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)||f^{(j)}(\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}} (1-|z|^2)|\psi_k(z)\varphi'(z)||f^{(k+1)}(\varphi(z))|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|f\|_\infty|\psi_0(0)| + \|f\|_{\mathscr{B}}\sum_{j=1}^{k}\frac{|\psi_j(0)|}{(1-|\varphi(0)|^2)^j} + \|f\|_\infty\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0'(z)| \\
&\quad +\|f\|_{\mathscr{B}}\left(\sum_{j=1}^{k}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j} + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}\right) \\
&\lesssim \|f\|_\infty\left(\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0'(z)| + \sum_{j=0}^{k}\frac{|\psi_j(0)|}{(1-|\varphi(0)|^2)^j}\right. \\
&\quad \left. + \sum_{j=1}^{k}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j} + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}\right) \\
&\leqslant \|f\|_\infty\left(C+\sum_{j=0}^{k+1}M_j\right) < \infty. \qquad (2.1)
\end{aligned}
$$

This proves that the operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded.

Conversely, assume that the operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded. We shall prove that $\sum_{i=0}^{k+1} M_i < \infty$. Fix $a \in \mathbb{D}$. First, we prove the condition $M_{k+1} < \infty$ holds. For this purpose, we define $f_{k+1,\varphi(a)}(z) = \frac{1-|\varphi(a)|^2}{1-\overline{\varphi(a)}z}\sigma^{k+1}_{\varphi(a)}(z)$, $z \in \mathbb{D}$. It is clear that $f_{k+1,\varphi(a)} \in H^\infty$ with $\|f_{k+1,\varphi(a)}\|_\infty \leqslant 2$, $f^{(i)}_{k+1,\varphi(a)}(\varphi(a)) = 0$ for all $i = 0,1,\cdots,k$ and

$$
\left|f^{(k+1)}_{k+1,\varphi(a)}(\varphi(a))\right| = \frac{(k+1)!}{(1-|\varphi(a)|^2)^{k+1}}.
$$

Thus,

$$
\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} \gtrsim \|P^k_{\vec{\psi},\varphi}f_{k+1,\varphi(a)}\|_{\mathscr{B}} \geqslant (1-|a|^2)|(P^k_{\vec{\psi},\varphi}f_{k+1,\varphi(a)})'(a)| \\
&= (1-|a|^2)\Big|\sum_{j=0}^{k}\big(\psi_j'(a)f^{(j)}_{k+1,\varphi(a)}(\varphi(a)) + \psi_j(a)\varphi'(a)f^{(j+1)}_{k+1,\varphi(a)}(\varphi(a))\big)\Big| \\
&= (1-|a|^2)\Big|\psi_0'(a)f_{k+1,\varphi(a)}(\varphi(a)) + \psi_k(a)\varphi'(a)f^{(k+1)}_{k+1,\varphi(a)}(\varphi(a)) \\
&\quad + \sum_{j=1}^{k}\big(\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big)f^{(j)}_{k+1,\varphi(a)}(\varphi(a))\Big| \\
&= (1-|a|^2)|\psi_k(a)\varphi'(a)||f^{(k+1)}_{k+1,\varphi(a)}(\varphi(a))| \\
&= \frac{(1-|a|^2)|\psi_k(a)\varphi'(a)|(k+1)!}{(1-|\varphi(a)|^2)^{k+1}}. \qquad (2.2)
\end{aligned}
$$

Therefore, by the abitrariness of a, we see that

$$
M_{k+1} = \sup_{a\in\mathbb{D}}\frac{(1-|a|^2)|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{k+1}} \leqslant \frac{1}{(k+1)!}\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} < \infty. \qquad (2.3)
$$

Next, we will prove that $M_j < \infty$ for $j = 1,2,\cdots,k$. Define

$$f_{k,\varphi(a)}(z) = \frac{1-|\varphi(a)|^2}{1-\overline{\varphi(a)}z}\sigma_{\varphi(a)}^k(z), \quad z \in \mathbb{D}.$$

It is clear that $f_{k,\varphi(a)} \in H^\infty$ with $\|f_{k,\varphi(a)}\|_\infty \leqslant 2, f_{k,\varphi(a)}^{(i)}(\varphi(a)) = 0$ for all $i = 0,1,\cdots,k-1$ and

$$\left|f_{k,\varphi(a)}^{(k)}(\varphi(a))\right| = \frac{k!}{(1-|\varphi(a)|^2)^k}. \tag{2.4}$$

Using Lemma 2.1 and (2.4), we have

$$\begin{aligned}
&\|P_{\vec{\psi},\varphi}^k\|_{H^\infty\to\mathscr{B}} \gtrsim \|P_{\vec{\psi},\varphi}^k f_{k,\varphi(a)}\|_{\mathscr{B}} \geqslant (1-|a|^2)|(P_{\vec{\psi},\varphi}^k f_{k,\varphi(a)})'(a)| \\
\geqslant\, &(1-|a|^2)\left|\psi_k'(a)+\psi_{k-1}(a)\varphi'(a)\right|\left|f_{k,\varphi(a)}^{(k)}(\varphi(a))\right| \\
&-(1-|a|^2)\left|\psi_k(a)\varphi'(a)\right|\left|f_{k,\varphi(a)}^{(k+1)}(\varphi(a))\right| \\
\geqslant\, &\frac{(1-|a|^2)\left|\psi_k'(a)+\psi_{k-1}(a)\varphi'(a)\right|k!}{(1-|\varphi(a)|^2)^k} - \frac{C\|f_{k,\varphi(a)}\|_{\mathscr{B}}(1-|a|^2)|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{k+1}}.
\end{aligned} \tag{2.5}$$

Since $H^\infty \subset \mathscr{B}$ and $\|f_{k,\varphi(a)}\|_{\mathscr{B}} \leqslant \|f_{k,\varphi(a)}\|_\infty \leqslant 2$, using (2.3) and (2.5), we have

$$\begin{aligned}
M_k &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)\left|\psi_k'(a)+\psi_{k-1}(a)\varphi'(a)\right|}{(1-|\varphi(a)|^2)^k} \\
&\leqslant \frac{1}{k!}\left(\|P_{\vec{\psi},\varphi}^k\|_{H^\infty\to\mathscr{B}} + 2C\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{k+1}}\right) \\
&\lesssim \|P_{\vec{\psi},\varphi}^k\|_{H^\infty\to\mathscr{B}}.
\end{aligned} \tag{2.6}$$

Further, fix $1 \leqslant j \leqslant k-1$ and assume that

$$M_i \lesssim \|P_{\vec{\psi},\varphi}^k\|_{H^\infty\to\mathscr{B}}, \tag{2.7}$$

for all $i = j+1,\cdots,k$. We will prove

$$M_j \lesssim \|P_{\vec{\psi},\varphi}^k\|_{H^\infty\to\mathscr{B}}.$$

To prove the above estimate, we define $f_{j,\varphi(a)}(z) = \frac{1-|\varphi(a)|^2}{1-\overline{\varphi(a)}z}\sigma_{\varphi(a)}^j(z)$, $z \in \mathbb{D}$. Then, clearly $f_{j,\varphi(a)} \in H^\infty$ such that $\|f_{j,\varphi(a)}\|_\infty \leqslant 2, f_{j,\varphi(a)}^{(s)}(\varphi(a)) = 0$ for all $s < j$ and

$$\left|f_{j,\varphi(a)}^{(j)}(\varphi(a))\right| = \frac{j!}{(1-|\varphi(a)|^2)^j}. \tag{2.8}$$

Using Lemma 2.1 and (2.8), we have

$$
\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} \geqslant \|P^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}} \geqslant (1-|a|^2)|(P^k_{\vec{\psi},\varphi} f_{j,\varphi(a)})'(a)| \\
\geqslant\ & (1-|a|^2)\big|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big|\big|f^{(j)}_{j,\varphi(a)}(\varphi(a))\big| \\
&-(1-|a|^2)\big|\psi_k(a)\varphi'(a)\big|\big|f^{(k+1)}_{j,\varphi(a)}(\varphi(a))\big| \\
&-\sum_{i=j+1}^{k}(1-|a|^2)\big|\psi_i'(a)+\psi_{i-1}(a)\varphi'(a)\big|\big|f^{(i)}_{j,\varphi(a)}(\varphi(a))\big| \\
\geqslant\ & \frac{(1-|a|^2)\big|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big| j!}{(1-|\varphi(a)|^2)^j} - \frac{C\|f_{j,\varphi(a)}\|_{\mathscr{B}}(1-|a|^2)|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{k+1}} \\
&-\sum_{i=j+1}^{k}\frac{C\|f_{j,\varphi(a)}\|_{\mathscr{B}}(1-|a|^2)\big|\psi_i'(a)+\psi_{i-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^i}.
\end{aligned}
\tag{2.9}
$$

Since $H^\infty \subset \mathscr{B}$ and $\|f_{j,\varphi(a)}\|_{\mathscr{B}} \leqslant \|f_{j,\varphi(a)}\|_\infty \leqslant 2$, by (2.3), (2.7) and (2.9), we have

$$
\begin{aligned}
M_j &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)\big|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^j} \\
&\leqslant \frac{1}{j!}\Bigg(\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} + 2C\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{k+1}} \\
&\quad +2C\sum_{i=j+1}^{k}\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)\big|\psi_i'(a)+\psi_{i-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^i}\Bigg) \\
&\lesssim \|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}},
\end{aligned}
\tag{2.10}
$$

as desired.

Finally, we prove that $M_0 < \infty$. For this purpose, set $f_{0,\varphi(a)}(z) = \frac{1-|\varphi(a)|^2}{1-\overline{\varphi(a)}z}$. It is easy to see that $\|f_{0,\varphi(a)}\|_\infty \leqslant 2$ for all $a \in \mathbb{D}$, and

$$
\big|f_{0,\varphi(a)}(\varphi(a))\big| = 1. \tag{2.11}
$$

Using Lemma 2.1 and (2.11), we have

$$
\begin{aligned}
\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} &\geqslant \|P^k_{\vec{\psi},\varphi} f_{0,\varphi(a)}\|_{\mathscr{B}} \geqslant (1-|a|^2)|(P^k_{\vec{\psi},\varphi} f_{0,\varphi(a)})'(a)| \\
&= (1-|a|^2)\Big|\sum_{j=0}^{k}\big(\psi_j'(a)f^{(j)}_{0,\varphi(a)}(\varphi(a))+\psi_j(a)\varphi'(a)f^{(j+1)}_{0,\varphi(a)}(\varphi(a))\big)\Big| \\
&= (1-|a|^2)\Big|\psi_0'(a)f_{0,\varphi(a)}(\varphi(a))+\psi_k(a)\varphi'(a)f^{(k+1)}_{0,\varphi(a)}(\varphi(a)) \\
&\quad +\sum_{j=1}^{k}\big(\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big)f^{(j)}_{0,\varphi(a)}(\varphi(a))\Big|
\end{aligned}
$$

$$\geqslant (1-|a|^2)|\psi_0'(a)| - \frac{C\|f_{0,\varphi(a)}\|_{\mathscr{B}}(1-|a|^2)|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{k+1}} - \sum_{j=1}^{k} \frac{C\|f_{0,\varphi(a)}\|_{\mathscr{B}}(1-|a|^2)\left|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\right|}{(1-|\varphi(a)|^2)^j}. \tag{2.12}$$

Since $H^\infty \subset \mathscr{B}$ and $\|f_{0,\varphi(a)}\|_{\mathscr{B}} \leqslant \|f_{0,\varphi(a)}\|_\infty \leqslant 2$, using (2.3), (2.10) and (2.12), we have

$$\begin{aligned} M_0 &= \sup_{a\in\mathbb{D}}(1-|a|^2)|\psi_0'(a)| \\ &\lesssim \|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{k+1}} \\ &\quad + \sum_{j=1}^{k} \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)\left|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\right|}{(1-|\varphi(a)|^2)^j} \\ &\lesssim \|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} < \infty. \end{aligned} \tag{2.13}$$

The proof is complete. □

REMARK. Theorem 2.1 was essentially proved in [36]. We have given above a different and detailed proof for the completeness and the benefit of the reader.

THEOREM 2.2. *Let $k\in\mathbb{N}_0$, $\varphi\in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j=0,1,\ldots,k$. Then the following statements are equivalent:*

(i) The operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded;

(ii) $\sup_{m\in\mathbb{N}} \|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}} < \infty$, where $I^m(z) = z^m$;

(iii) $\sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{j,a}\|_{\mathscr{B}} < \infty$, for $j=0,1,\cdots,k+1$. Here

$$f_{i,a}(z) = \frac{1-|a|^2}{1-\bar{a}z}\sigma_a^i(z) = \frac{1-|a|^2}{1-\bar{a}z}\left(\frac{a-z}{1-\bar{a}z}\right)^i, \quad z\in\mathbb{D}.$$

Proof. $(i)\Rightarrow(ii)$ This implication is obvious, since for $m\in\mathbb{N}$, the function $I^m(z)$ $= z^m$ is bounded in H^∞ and $\|I^m\|_\infty = 1$.

$(ii)\Rightarrow(iii)$ Assume that (ii) holds. For each $j=0,1,\cdots,k+1$, from the definition of $f_{j,a}$, it is easy to see that $f_{j,a}$ have bounded norms in H^∞. Since

$$f_{0,a} = (1-|a|^2)\sum_{i=0}^{\infty} \bar{a}^i z^i,$$

using linearity we get

$$\|P^k_{\vec{\psi},\varphi} f_{0,a}\|_{\mathscr{B}} \leqslant (1-|a|^2)\sum_{i=0}^{\infty}|a|^i\|P^k_{\vec{\psi},\varphi} I^i\|_{\mathscr{B}} \leqslant 2\sup_{m\in\mathbb{N}}\|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}} < \infty. \tag{2.14}$$

Therefore,

$$\sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{0,a}\|_{\mathscr{B}} < \infty. \tag{2.15}$$

Noting that

$$\sigma_a(z) = a - (1-|a|^2)\sum_{i=0}^{\infty} \bar{a}^i z^{i+1}. \tag{2.16}$$

Suppose $f_{1,a}(z) = \sum_{l=0}^{\infty} c_l z^l$. Since $f_{1,a}(z) = f_{0,a}(z)\cdot\sigma_a(z)$, we write $f_{0,a}(z) = \sum_{i=0}^{\infty} a_i z^i$, $\sigma_a(z) = \sum_{t=0}^{\infty} b_t z^t$. Then we have

$$f_{1,a}(z) = \left(\sum_{i=0}^{\infty} a_i z^i\right)\left(\sum_{t=0}^{\infty} b_t z^t\right) = \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l} a_i b_{l-i}\right) z^l.$$

Thus $c_l = \sum_{i=0}^{l} a_i b_{l-i}$ and

$$\sum_{l=0}^{\infty} |c_l| = \sum_{l=0}^{\infty}\Big|\sum_{i=0}^{l} a_i b_{l-i}\Big| \leqslant \sum_{l=0}^{\infty}\sum_{i=0}^{l} |a_i||b_{l-i}| = \left(\sum_{i=0}^{\infty}|a_i|\right)\left(\sum_{i=0}^{\infty}|b_i|\right). \tag{2.17}$$

From (2.14), (2.16), (2.17) and linearity, we get

$$\sum_{l=0}^{\infty} |c_l| \leqslant \left((1-|a|^2)\sum_{i=0}^{\infty}|a|^i\right)\left(|a| + (1-|a|^2)\sum_{i=0}^{\infty}|a|^i\right) \leqslant 2\times 3 = 6 \tag{2.18}$$

and

$$\|P^k_{\vec{\psi},\varphi} f_{1,a}\|_{\mathscr{B}} \leqslant \sum_{l=0}^{\infty} |c_l| \|P^k_{\vec{\psi},\varphi} I^l\|_{\mathscr{B}} \lesssim \sup_{m\in\mathbb{N}} \|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}} < \infty. \tag{2.19}$$

Therefore,

$$\sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{1,a}\|_{\mathscr{B}} < \infty. \tag{2.20}$$

Similarly, we suppose $f_{2,a}(z) = \sum_{l=0}^{\infty} d_l z^l$. Since $f_{2,a}(z) = f_{1,a}(z)\cdot\sigma_a(z)$, we write $f_{1,a}(z) = \sum_{i=0}^{\infty} c_i z^i$, $\sigma_a(z) = \sum_{t=0}^{\infty} b_t z^t$. Then we have

$$f_{2,a}(z) = \left(\sum_{i=0}^{\infty} c_i z^i\right)\left(\sum_{t=0}^{\infty} b_t z^t\right) = \sum_{l=0}^{\infty}\left(\sum_{i=0}^{l} c_i b_{l-i}\right) z^l.$$

Thus $d_l = \sum_{i=0}^{l} c_i b_{l-i}$ and by (2.16), (2.17) and (2.18), we have

$$\begin{aligned}\sum_{l=0}^{\infty} |d_l| &= \sum_{l=0}^{\infty}\Big|\sum_{i=0}^{l} c_i b_{l-i}\Big| \leqslant \sum_{l=0}^{\infty}\sum_{i=0}^{l} |c_i||b_{l-i}| \\ &= \left(\sum_{i=0}^{\infty}|c_i|\right)\left(\sum_{i=0}^{\infty}|b_i|\right) \leqslant 6\times 3 = 18.\end{aligned} \tag{2.21}$$

From (2.21) and linearity, we get

$$\|P^k_{\vec{\psi},\varphi} f_{2,a}\|_{\mathscr{B}} \leqslant \sum_{l=0}^{\infty} |d_l| \|P^k_{\vec{\psi},\varphi} I^l\|_{\mathscr{B}} \lesssim \sup_{m\in\mathbb{N}} \|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}} < \infty.$$

Therefore,

$$\sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{2,a}\|_{\mathscr{B}} < \infty. \tag{2.22}$$

In the same manner, using (2.16), (2.17), (2.21) and linearity, we can also get

$$\sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{3,a}\|_{\mathscr{B}} < \infty. \tag{2.23}$$

By a standard inductive argument we can obtain

$$\sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{j,a}\|_{\mathscr{B}} < \infty, \quad \text{for} \quad j = 4,5,\cdots,k+1. \tag{2.24}$$

Therefore, (2.15), (2.20), (2.22), (2.23) and (2.24) imply that (iii) holds.

$(iii) \Rightarrow (i)$ Assume that (iii) holds. From the assumption we see that

$$\sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}} < \infty, \tag{2.25}$$

for all $j = 0,1,\cdots,k+1$. From the proof of Theorem 2.1, (2.2) and (2.25) imply that

$$M_{k+1} \leqslant \sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{k+1,\varphi(a)}\|_{\mathscr{B}} < \infty. \tag{2.26}$$

(2.5), (2.25) and (2.26) imply that

$$M_k \leqslant \sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{k,\varphi(a)}\|_{\mathscr{B}} + M_{k+1} < \infty. \tag{2.27}$$

Further, fix $1 \leqslant j \leqslant k-1$ and assume that

$$M_i \leqslant \sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{i,\varphi(a)}\|_{\mathscr{B}} + M_{k+1} + \sum_{t=i+1}^{k} M_t, \tag{2.28}$$

for all $i = j+1,\cdots,k$, by (2.9), (2.25), (2.26) and (2.28), we obtain the following estimate:

$$M_j \leqslant \sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}} + M_{k+1} + \sum_{i=j+1}^{k} M_i < \infty, \quad \text{for} \quad j = 1,2,\cdots,k. \tag{2.29}$$

(2.12), (2.25), (2.26) and (2.29) imply that

$$M_0 \leqslant \sup_{a\in\mathbb{D}} \|P^k_{\vec{\psi},\varphi} f_{0,\varphi(a)}\|_{\mathscr{B}} + M_{k+1} + \sum_{j=1}^{k} M_j < \infty. \tag{2.30}$$

By (2.26), (2.29), (2.30) and Theorem 2.1, we know that (i) holds. The proof is complete. $\square$

Next, motivated by [4], we give another characterization for the boundedness of the operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$. For this purpose, we state some lemmas and definitions which will be used.

Let $v : \mathbb{D} \to R_+$ be a continuous, strictly positive and bounded function. The weighted v is called radial, if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the weighted space, denoted by H^∞_v, if

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

H^∞_v is a Banach space with the norm $\|\cdot\|_v$. In particular, we denote H^∞_v by $H^\infty_{v_\alpha}$ when $v = v_\alpha(z) = (1-|z|^2)^\alpha (0 < \alpha < \infty)$. For a weight v, the associated weight $\widetilde{v}$ is defined by

$$\widetilde{v} = (\sup\{|f(z)| : f \in H^\infty_v, \|f\|_v \leqslant 1\})^{-1}, \quad z \in \mathbb{D}.$$

It is easy to check that $\widetilde{v}_\alpha(z) = v_\alpha(z)$. According to Lemma 2.2 in [6], we define $\bar{v}_\alpha(z) = \left(\sup_{n\in\mathbb{N}} \frac{|z|^{n-1}}{\|\xi^{n-1}\|_{v_\alpha}}\right)^{-1}$, where the norm of the monomial ξ^n is calculated in $H^\infty_{v_\alpha}$.

LEMMA 2.2. [18] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then $\psi C_\varphi : H^\infty_v \to H^\infty_w$ is bounded if and only if* $\sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))}|\psi(z)| < \infty$. *Moreover,*

$$\|\psi C_\varphi\|_{H^\infty_v \to H^\infty_w} = \sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))}|\psi(z)|.$$

LEMMA 2.3. [6] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then $\psi C_\varphi : H^\infty_v \to H^\infty_w$ is bounded if and only if*

$$\sup_{n\geqslant 0} \frac{\|\psi\varphi^n\|_w}{\|\xi^n\|_v} < \infty.$$

Moreover, $\|\psi C_\varphi\|_{H^\infty_v \to H^\infty_w} \approx \sup_{n\geqslant 0} \frac{\|\psi\varphi^n\|_w}{\|\xi^n\|_v}$.

LEMMA 2.4. [7] *For $\alpha > 0$, we have* $\lim_{n\to\infty} n^\alpha \|\xi^{n-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.

THEOREM 2.3. *Let $k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j = 0,1,\ldots,k$. Then $P^k_{\vec{\psi},\varphi} : H^\infty \to \mathscr{B}$ is bounded if and only if $\psi_0 \in \mathscr{B}$;*

$$\sup_{n\geqslant 1} n^j \|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_1} < \infty, \quad for \quad j = 1,2,\cdots,k;$$

$$\sup_{n\geqslant 1} n^{k+1} \|\psi_k \varphi' \varphi^{n-1}\|_{v_1} < \infty.$$

Proof. According to Theorem 2.1, the operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded if and only if $\Sigma_{i=0}^{k+1} M_i < \infty$. By Lemma 2.2, we see that $M_j < \infty$ is equivalent to the operator $(\psi_{j-1}\varphi' + \psi_j')C_\varphi : H^\infty_{v_j} \to H^\infty_{v_1}$ is bounded for $j = 1, 2, \cdots, k$. By Lemma 2.3,

$$\sup_{n\geqslant 1} \frac{\|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_j}} \approx M_j < \infty, \quad \text{for} \quad j = 1, 2, \cdots, k. \tag{2.31}$$

By Lemma 2.2, it is easy to see that $M_{k+1} < \infty$ is equivalent to the operator $\psi_k \varphi' C_\varphi : H^\infty_{v_{k+1}} \to H^\infty_{v_1}$ is bounded. By Lemma 2.3,

$$\sup_{n\geqslant 1} \frac{\|\psi_k \varphi' \varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_{k+1}}} \approx M_{k+1} < \infty. \tag{2.32}$$

By Lemma 2.4, we see that $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded if and only if $\psi_0 \in \mathscr{B}$,

$$\sup_{n\geqslant 1} n^j \|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_1} \approx \sup_{n\geqslant 1} \frac{n^j \|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_1}}{n^j \|\xi^{n-1}\|_{v_j}} < \infty, \quad \text{for} \quad j = 1, 2, \cdots, k \tag{2.33}$$

and

$$\sup_{n\geqslant 1} n^{k+1} \|\psi_k \varphi' \varphi^{n-1}\|_{v_1} \approx \sup_{n\geqslant 1} \frac{n^{k+1} \|\psi_k \varphi' \varphi^{n-1}\|_{v_1}}{n^{k+1} \|\xi^{n-1}\|_{v_{k+1}}} < \infty.$$

Here we used the fact that $M_0 < \infty$ if and only if $\psi_0 \in \mathscr{B}$. The proof is complete. □

3. Compactness of $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$

For proving the compactness of the operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$, we need some lemmas. The following lemma whose proof follows from Proposition 3.11 in [3].

LEMMA 3.1. *Let $k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j = 0, 1, \ldots, k$. The operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is compact if and only if $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded and for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in H^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|P^k_{\vec{\psi},\varphi} f_n\|_{\mathscr{B}} \to 0$ as $n \to \infty$.*

LEMMA 3.2. [6] *Let v be a radial, non-increasing weight which tends to zero at the boundary of $\mathbb{D}$. Then $\tilde{v}$ is equivalent to $\bar{v}$ in $\mathbb{D}$.*

THEOREM 3.1. *Let $k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j = 0, 1, \ldots, k$. Then the operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is compact if and only if the operator $P^k_{\vec{\psi},\varphi}$ is bounded and*

$$\sum_{j=0}^{k+1} Q_j = 0.$$

Here

$$(i)\ \ Q_0=\lim_{r\to 1}\sup_{|\varphi(z)|>r}(1-|z|^2)|\psi_0'(z)|;$$

$$(ii)\ \ Q_j=\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j},\ \ for\ \ j=1,2,\cdots,k;$$

$$(iii)\ \ Q_{k+1}=\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}.$$

Proof. First, we assume that the operator $P^k_{\vec{\psi},\varphi}: H^\infty\to\mathscr{B}$ is compact. Clearly $P^k_{\vec{\psi},\varphi}$ is bounded. We need to show that $\sum_{j=0}^{k+1}Q_j=0$. First, we prove $Q_{k+1}=0$. For this, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$ such that

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}=\lim_{n\to\infty}\frac{(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}}.$$

For each n, we define $f_{k+1,n}(z)=\frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}\sigma^{k+1}_{\varphi(z_n)}(z)$, $z\in\mathbb{D}$. It is easy to see that $f_{k+1,n}\in H^\infty$, $\|f_{k+1,n}\|_\infty\leqslant 2, f^{(i)}_{k+1,n}(\varphi(z_n))=0$ for all $i=0,1,\cdots,k$ and

$$\left|f^{(k+1)}_{k+1,n}(\varphi(z_n))\right|=\frac{(k+1)!}{(1-|\varphi(z_n)|^2)^{k+1}}. \tag{3.1}$$

Clearly, $\{f_{k+1,n}\}_{n\in\mathbb{N}}$ is bounded sequence in H^∞ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Then by Lemma 3.1, $\|P^k_{\vec{\psi},\varphi}f_{k+1,n}\|_{\mathscr{B}}\to 0$ as $n\to\infty$. On the other hand, by (3.1), we have

$$\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}f_{k+1,n}\|_{\mathscr{B}}\geqslant(1-|z_n|^2)|(P^k_{\vec{\psi},\varphi}f_{k+1,n})'(z_n)|\\
=&(1-|z_n|^2)\Big|\sum_{j=0}^{k}\left(\psi_j'(z_n)f^{(j)}_{k+1,n}(\varphi(z_n))+\psi_j(z_n)f^{(j+1)}_{k+1,n}(\varphi(z_n))\varphi'(z_n)\right)\Big|\\
=&(1-|z_n|^2)\Big|\psi_0'(z_n)f_{k+1,n}(\varphi(z_n))+\psi_k(z_n)\varphi'(z_n)f^{(k+1)}_{k+1,n}(\varphi(z_n))\\
&+\sum_{j=1}^{k}\left(\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right)f^{(j)}_{k+1,n}(\varphi(z_n))\Big|\\
=&(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)||f^{(k+1)}_{k+1,n}(\varphi(z_n))|\\
=&\frac{(k+1)!(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}},
\end{aligned}$$

which implies that $Q_{k+1}=0$. Now to prove that $Q_k=0$. let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$ such that

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{k-1}(z)\varphi'(z)+\psi_k'(z)|}{(1-|\varphi(z)|^2)^k}=\lim_{n\to\infty}\frac{(1-|z_n|^2)|\psi_{k-1}(z_n)\varphi'(z_n)+\psi_k'(z_n)|}{(1-|\varphi(z_n)|^2)^k}.$$

We define $f_{k,n}(z) = \frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}\sigma^k_{\varphi(z_n)}(z)$, $z \in \mathbb{D}$. It is easy to see that $f_{k,n} \in H^\infty$ and $\|f_{k,n}\|_\infty \leqslant 2, f^{(i)}_{k,n}(\varphi(z_n)) = 0$ for all $i = 0,1,\cdots,k-1$ and

$$\left|f^{(k)}_{k,n}(\varphi(z_n))\right| = \frac{k!}{(1-|\varphi(z_n)|^2)^k}. \tag{3.2}$$

Clearly, $\{f_{k,n}\}_{n\in\mathbb{N}}$ is bounded sequence in H^∞ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Then by Lemma 3.1, $\|P^k_{\vec{\psi},\varphi} f_{k,n}\|_{\mathscr{B}} \to 0$ as $n \to \infty$. Thus, using (3.2) and Lemma 2.1, we have

$$\begin{aligned}
&\|P^k_{\vec{\psi},\varphi} f_{k,n}\|_{\mathscr{B}} \geqslant (1-|z_n|^2)|(P^k_{\vec{\psi},\varphi} f_{k,n})'(z_n)| \\
\geqslant\ & (1-|z_n|^2)\left|\psi_k'(z_n)+\psi_{k-1}(z_n)\varphi'(z_n)\right|\left|f^{(k)}_{k,n}(\varphi(z_n))\right| \\
&-(1-|z_n|^2)\left|\psi_k(z_n)\varphi'(z_n)\right|\left|f^{(k+1)}_{k,n}(\varphi(z_n))\right| \\
\geqslant\ & \frac{k!(1-|z_n|^2)\left|\psi_k'(z_n)+\psi_{k-1}(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^k} - \frac{C\|f_{k,n}\|_{\mathscr{B}}(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}}.
\end{aligned}$$

Further, we get

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)|\psi_{k-1}(z_n)\varphi'(z_n)+\psi_k'(z_n)|}{(1-|\varphi(z_n)|^2)^k} = 0, \tag{3.3}$$

which implies that $Q_k = 0$. Now we fix $1 \leqslant j \leqslant k-1$ and assume that $Q_i = 0$ for $i = j+1,\cdots,k$. Then we show that $Q_j = 0$. For that, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)| \to 1$ as $n \to \infty$ such that

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j} = \lim_{n\to\infty} \frac{(1-|z_n|^2)|\psi_{j-1}(z_n)\varphi'(z_n)+\psi_j'(z_n)|}{(1-|\varphi(z_n)|^2)^j}.$$

For each n, we define $f_{j,n}(z) = \frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}\sigma^j_{\varphi(z_n)}(z)$, $z \in \mathbb{D}$. It is easy to see that $f_{j,n} \in H^\infty$ and $\|f_{j,n}\|_\infty \leqslant 2, f^{(i)}_{j,n}(\varphi(z_n)) = 0$ for all $i = 0,1,\cdots,j-1$ and

$$\left|f^{(j)}_{j,n}(\varphi(z_n))\right| = \frac{j!}{(1-|\varphi(z_n)|^2)^j}. \tag{3.4}$$

Clearly, $\{f_{j,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in H^∞ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Then by Lemma 3.1, $\|P^k_{\vec{\psi},\varphi} f_{j,n}\|_{\mathscr{B}} \to 0$ as $n \to \infty$. Thus, by (3.4) and Lemma 2.1, we have

$$\begin{aligned}
\|P^k_{\vec{\psi},\varphi} f_{j,n}\|_{\mathscr{B}} \geqslant\ & (1-|z_n|^2)\left|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right|\left|f^{(j)}_{j,n}(\varphi(z_n))\right| \\
&- \sum_{i=j+1}^{k} (1-|z_n|^2)\left|\psi_i'(z_n)+\psi_{i-1}(z_n)\varphi'(z_n)\right|\left|f^{(i)}_{j,n}(\varphi(z_n))\right| \\
&-(1-|z_n|^2)\left|\psi_k(z_n)\varphi'(z_n)\right|\left|f^{(k+1)}_{j,n}(\varphi(z_n))\right|
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{j!(1-|z_n|^2)\left|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^j}\\
&\quad -\frac{C\|f_{j,n}\|_{\mathscr{B}}(1-|z_n|^2)\left|\psi_k(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^{k+1}}\\
&\quad -\sum_{i=j+1}^{k}\frac{C\|f_{j,n}\|_{\mathscr{B}}(1-|z_n|^2)\left|\psi_i'(z_n)+\psi_{i-1}(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^i}.
\end{aligned}
$$

Therefore, by the fact that $Q_i = 0$ for $i = j+1,\cdots,k+1$, the last inequality implies that

$$
\lim_{n\to\infty}\frac{(1-|z_n|^2)|\psi_{j-1}(z_n)\varphi'(z_n)+\psi_j'(z_n)|}{(1-|\varphi(z_n)|^2)^j}=0.
$$

This proves that $Q_j = 0$ for $1 \leqslant j \leqslant k-1$. In the same manner, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)| \to 1$ as $n \to \infty$ such that

$$
\lim_{r\to 1}\sup_{|\varphi(z)|>r}(1-|z|^2)|\psi_0'(z)| = \lim_{n\to\infty}(1-|z_n|^2)|\psi_0'(z_n)|.
$$

For each n, set $f_{0,n}(z) = \frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}$, $z \in \mathbb{D}$. It is easy to see that $f_{0,n} \in H^\infty$ and $\|f_{0,n}\|_\infty \leqslant 2$, and

$$
\left|f_{0,n}(\varphi(z_n))\right| = 1.
$$

Clearly, $\{f_{0,n}\}_{n\in\mathbb{N}}$ is bounded sequence in H^∞ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Then by Lemma 3.1, $\|P^k_{\vec{\psi},\varphi}f_{0,n}\|_{\mathscr{B}} \to 0$ as $n \to \infty$. Using Lemma 2.1, we have

$$
\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}f_{0,n}\|_{\mathscr{B}} \geqslant (1-|z_n|^2)|(P^k_{\vec{\psi},\varphi}f_{0,n})'(z_n)|\\
&= (1-|z_n|^2)\Big|\sum_{j=0}^{k}\left(\psi_j'(z_n)f_{0,n}^{(j)}(\varphi(z_n))+\psi_j(z_n)f_{0,n}^{(j+1)}(\varphi(z_n))\varphi'(z_n)\right)\Big|\\
&= (1-|z_n|^2)\big|\psi_0'(z_n)f_{0,n}(\varphi(z_n))+\psi_k(z_n)\varphi'(z_n)f_{0,n}^{(k+1)}(\varphi(z_n))\\
&\quad +\sum_{j=1}^{k}\left(\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right)f_{0,n}^{(j)}(\varphi(z_n))\big|\\
&\geqslant (1-|z_n|^2)|\psi_0'(z_n)||f_{0,n}(\varphi(z_n))|\\
&\quad -\sum_{j=1}^{k}(1-|z_n|^2)\left|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right|\left|f_{0,n}^{(j)}(\varphi(z_n))\right|\\
&\quad -(1-|z_n|^2)\left|\psi_k(z_n)\varphi'(z_n)\right|\left|f_{0,n}^{(k+1)}(\varphi(z_n))\right|\\
&\geqslant (1-|z_n|^2)|\psi_0'(z_n)| - \frac{C\|f_{0,n}\|_{\mathscr{B}}(1-|z_n|^2)\left|\psi_k(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^{k+1}}\\
&\quad -\sum_{j=1}^{k}\frac{C\|f_{0,n}\|_{\mathscr{B}}(1-|z_n|^2)\left|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^j}.
\end{aligned}
$$

Therefore $\lim_{n\to\infty}(1-|z_n|^2)|\psi_0'(z_n)|=0$, which implies that $Q_0=0$.

Conversely, assume that $\sum_{j=0}^{k+1} Q_j=0$ and $P^k_{\vec{\psi},\varphi}: H^\infty\to\mathscr{B}$ is bounded. Let $f_0(z)=1$, for every $z\in\mathbb{D}$. Then $f_0\in H^\infty$ and

$$\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} \geqslant \|P^k_{\vec{\psi},\varphi}f_0\|_{\mathscr{B}} \geqslant \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0'(z)|=I_0. \tag{3.5}$$

Now let $f_1(z)=z$, for every $z\in\mathbb{D}$. Then, we have

$$\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} \geqslant \|P^k_{\vec{\psi},\varphi}f_1\|_{\mathscr{B}}\\
\geqslant& \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0'(z)\varphi(z)+\psi_0(z)\varphi'(z)+\psi_1'(z)|\\
\geqslant& \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0(z)\varphi'(z)+\psi_1'(z)|-\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0'(z)\varphi(z)|.
\end{aligned}$$

Using the boundedness of φ, (3.5) and the last inequality, we obtain

$$I_1=\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0(z)\varphi'(z)+\psi_1'(z)| \leqslant 2\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}}. \tag{3.6}$$

Next, we let $1<j\leqslant k$ and assume that for each $1\leqslant i\leqslant j-1$, there exists a constant $C_i>0$ such that

$$I_i=\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_{i-1}(z)\varphi'(z)+\psi_i'(z)| \leqslant C_i\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}}. \tag{3.7}$$

We prove the above inequality for $i=j$. Define $f_j(z)=z^j$ for every $z\in\mathbb{D}$. Then $f_j\in H^\infty$ and we have

$$\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}} \geqslant \|P^k_{\vec{\psi},\varphi}f_j\|_{\mathscr{B}}\\
\geqslant& \sup_{z\in\mathbb{D}}(1-|z|^2)\big|\psi_0'(z)(\varphi(z))^j+\big(\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)\big)j!\\
&+\sum_{i=1}^{j-1}\big(\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)\big)\frac{j!}{(j-i)!}(\varphi(z))^{j-i}\big|\\
\geqslant& \sup_{z\in\mathbb{D}}(1-|z|^2)\big|\big(\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)\big)j!\big|-\sup_{z\in\mathbb{D}}(1-|z|^2)\big|\psi_0'(z)(\varphi(z))^j\big|\\
&-\sup_{z\in\mathbb{D}}(1-|z|^2)\big|\sum_{i=1}^{j-1}\big(\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)\big)\frac{j!}{(j-i)!}(\varphi(z))^{j-i}\big|.
\end{aligned}$$

Using the boundedness of φ, (3.5), (3.7) and the above inequality, we obtain

$$\begin{aligned}
I_j&=\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|\\
&\leqslant \frac{\big(2+\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}C_i\big)\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}}}{j!}=C_j\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}}.
\end{aligned} \tag{3.8}$$

Similarly, using $f_{k+1}(z)=z^{k+1}$ for every $z\in\mathbb{D}$. Then $f_{k+1}\in H^\infty$, using the boundedness of φ, we have

$$\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}}\geqslant\|P^k_{\vec{\psi},\varphi}f_{k+1}\|_{\mathscr{B}}\\
\geqslant&\sup_{z\in\mathbb{D}}(1-|z|^2)\big|\psi_0'(z)(\varphi(z))^{k+1}+\big(\psi_k(z)\varphi'(z)\big)(k+1)!\\
&+\sum_{i=1}^{k}\big(\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)\big)\frac{(k+1)!}{(k+1-i)!}(\varphi(z))^{k+1-i}\big|\\
\geqslant&\sup_{z\in\mathbb{D}}(1-|z|^2)\big|\psi_k(z)\varphi'(z)(k+1)!\big|-\sup_{z\in\mathbb{D}}(1-|z|^2)\big|\psi_0'(z)(\varphi(z))^{k+1}\big|\\
&-\sup_{z\in\mathbb{D}}(1-|z|^2)\big|\sum_{i=1}^{k}\big(\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)\big)\frac{(k+1)!}{(k+1-i)!}(\varphi(z))^{k+1-i}\big|.
\end{aligned}$$

Then, by (3.5), (3.6) and (3.8), we get

$$I_{k+1}=\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_k(z)\varphi'(z)|\leqslant C_{k+1}\|P^k_{\vec{\psi},\varphi}\|_{H^\infty\to\mathscr{B}}. \tag{3.9}$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H^∞ such that it converges to zero uniformly on compact subsets of $\mathbb{D}$. To prove that $P^k_{\vec{\psi},\varphi}$ is compact, according to Lemma 3.1, we need to show that $\|P^k_{\vec{\psi},\varphi}f_n\|_{\mathscr{B}}\to 0$ as $n\to\infty$. Fix $\varepsilon>0$. Since $\sum_{j=0}^{k+1}Q_j=0$, there exists $r\in(0,1)$ such that whenever $r<|\varphi(z)|<1$, we have

$$(1-|z|^2)|\psi_0'(z)|<\varepsilon; \tag{3.10}$$

$$\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j}<\varepsilon,\ \text{ for } j=1,2,\cdots,k; \tag{3.11}$$

$$\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}<\varepsilon. \tag{3.12}$$

Since $\{f_n\}_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimates imply that $\{f_n^{(i)}\}_{n\in\mathbb{N}}$, $i=0,1,\cdots,k+1$ also converges to zero uniformly on compact subsets of $\mathbb{D}$. Hence there is an $n_0\in\mathbb{N}$ such that, if $|\varphi(z)|\leqslant r$ and $n>n_0$, then

$$\left|f_n^{(i)}(\varphi(z))\right|<\varepsilon,\ \ i=0,1,\cdots,k+1. \tag{3.13}$$

We know that $H^\infty\subset\mathscr{B}$ and $\|f\|_{\mathscr{B}}\leqslant\|f\|_\infty$, using (3.5)–(3.13) and Lemma 2.1, we have

$$\begin{aligned}
&\|P^k_{\vec{\psi},\varphi}f_n\|_{\mathscr{B}}=|P^k_{\vec{\psi},\varphi}f_n(0)|+\|P^k_{\vec{\psi},\varphi}f_n\|_\beta\\
=&|P^k_{\vec{\psi},\varphi}f_n(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)\big|\big(\sum_{j=0}^{k}\psi_j(z)f_n^{(j)}(\varphi(z))\big)'\big|
\end{aligned}$$

$$
\begin{aligned}
&= |P^k_{\vec{\psi},\varphi} f_n(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2) \Big| \sum_{j=0}^{k} \big(\psi_j'(z) f_n^{(j)}(\varphi(z)) + \psi_j(z)\varphi'(z) f_n^{(j+1)}(\varphi(z))\big)\Big| \\
&= |P^k_{\vec{\psi},\varphi} f_n(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2) \big| \psi_0'(z) f_n(\varphi(z)) \\
&\quad + \sum_{j=1}^{k} \big(\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)\big) f_n^{(j)}(\varphi(z)) + \psi_k(z)\varphi'(z) f_n^{(k+1)}(\varphi(z))\big| \\
&\leqslant |P^k_{\vec{\psi},\varphi} f_n(0)| + \sup_{|\varphi(z)|\leqslant r} (1-|z|^2) \Big| \psi_0'(z) f_n(\varphi(z)) \\
&\quad + \sum_{j=1}^{k} \big(\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)\big) f_n^{(j)}(\varphi(z)) + \psi_k(z)\varphi'(z) f_n^{(k+1)}(\varphi(z))\Big| \\
&\quad + \sup_{|\varphi(z)|> r} (1-|z|^2) \Big| \psi_0'(z) f_n(\varphi(z)) + \psi_k(z)\varphi'(z) f_n^{(k+1)}(\varphi(z)) \\
&\quad + \sum_{j=1}^{k} \big(\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)\big) f_n^{(j)}(\varphi(z))\Big| \\
&\leqslant |P^k_{\vec{\psi},\varphi} f_n(0)| + \sup_{|\varphi(z)|\leqslant r} (1-|z|^2)|\psi_0'(z)||f_n(\varphi(z))| \\
&\quad + \sum_{j=1}^{k} \sup_{|\varphi(z)|\leqslant r} (1-|z|^2)|\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)||f_n^{(j)}(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leqslant r} (1-|z|^2)|\psi_k(z)\varphi'(z)||f_n^{(k+1)}(\varphi(z))| + \|f_n\|_\infty \sup_{|\varphi(z)|> r} (1-|z|^2)|\psi_0'(z)| \\
&\quad + C\|f\|_{\mathscr{B}} \Bigg(\sum_{j=1}^{k} \sup_{|\varphi(z)|> r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z) + \psi_j'(z)|}{(1-|\varphi(z)|^2)^j} \\
&\quad + \sup_{|\varphi(z)|> r} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}} \Bigg) \\
&\lesssim |P^k_{\vec{\psi},\varphi} f_n(0)| + \sup_{|\varphi(z)|\leqslant r} (1-|z|^2)|\psi_0'(z)||f_n(\varphi(z))| \\
&\quad + \sum_{j=1}^{k} \sup_{|\varphi(z)|\leqslant r} (1-|z|^2)|\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)||f_n^{(j)}(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leqslant r} (1-|z|^2)|\psi_k(z)\varphi'(z)||f_n^{(k+1)}(\varphi(z))| \\
&\quad + \|f_n\|_\infty \Bigg(\sup_{|\varphi(z)|> r} (1-|z|^2)|\psi_0'(z)| + \sup_{|\varphi(z)|> r} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}} \\
&\quad + \sum_{j=1}^{k} \sup_{|\varphi(z)|> r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z) + \psi_j'(z)|}{(1-|\varphi(z)|^2)^j} \Bigg)
\end{aligned}
$$

$$\leqslant |P^k_{\vec{\psi},\varphi} f_n(0)| + \varepsilon\Bigg(\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0'(z)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_k(z)\varphi'(z)| + \sum_{j=1}^{k}\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|\Bigg) + \varepsilon(k+2)\|f_n\|_\infty$$

$$\leqslant |P^k_{\vec{\psi},\varphi} f_n(0)| + \varepsilon\left(\sum_{i=0}^{k+1} I_i + (k+2)\|f_n\|_\infty\right).$$

Since $\{f_n^{(i)}\}_{n\in\mathbb{N}}, i=0,1,\cdots,k+1$ converges to zero uniformly on compact subsets of $\mathbb{D}$, it can be seen that $|P^k_{\vec{\psi},\varphi} f_n(0)| \to 0$ as $n\to\infty$. Thus, $\|P^k_{\vec{\psi},\varphi} f_n\|_{\mathscr{B}} \to 0$ as $n\to\infty$ and by Lemma 3.1, the operator $P^k_{\vec{\psi},\varphi}$ is compact. The proof is complete. □

THEOREM 3.2. *Let $k\in\mathbb{N}_0$, $\varphi\in S(\mathbb{D})$ and $\psi_j\in H(\mathbb{D})$, $j=0,1,\ldots,k$. If $P^k_{\vec{\psi},\varphi}: H^\infty\to\mathscr{B}$ is bounded, then the following statements are equivalent.*

(a) The operator $P^k_{\vec{\psi},\varphi}: H^\infty\to\mathscr{B}$ is compact.

(b) $\lim_{m\to\infty}\|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}}=0$, where $I^m(z)=z^m$.

(c) $\lim_{|\varphi(a)|\to 1}\|P^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}}=0$, for $j=0,1,\cdots,k+1$.

Proof. $(a)\Rightarrow(b)$ Assume $P^k_{\vec{\psi},\varphi}: H^\infty\to\mathscr{B}$ is compact. Since the sequence $\{I^m\}$ is bounded in H^∞ and converges to 0 uniformly on compact subsets, by Lemma 3.1, it follows that $\|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}}\to 0$ as $m\to\infty$.

$(b)\Rightarrow(c)$ Suppose (b) holds. Fix $\varepsilon>0$ and choose $N\in\mathbb{N}$ such that $\|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}}<\varepsilon$ for all $m\geqslant N$. Let $z_n\in\mathbb{D}$ such that $|\varphi(z_n)|\to 1$ as $n\to\infty$. Arguing as in Theorem 2.2, we have

$$\begin{aligned}\|P^k_{\vec{\psi},\varphi} f_{0,\varphi(z_n)}\|_{\mathscr{B}} &\leqslant (1-|\varphi(z_n)|^2)\sum_{i=0}^{\infty}|\varphi(z_n)|^i\|P^k_{\vec{\psi},\varphi} I^i\|_{\mathscr{B}}\\ &= (1-|\varphi(z_n)|^2)\sum_{i=0}^{N-1}|\varphi(z_n)|^i\|P^k_{\vec{\psi},\varphi} I^i\|_{\mathscr{B}} + (1-|\varphi(z_n)|^2)\sum_{i=N}^{\infty}|\varphi(z_n)|^i\|P^k_{\vec{\psi},\varphi} I^i\|_{\mathscr{B}}\\ &\leqslant 2N(1-|\varphi(z_n)|^2)\sup_{m\in\mathbb{N}}\|P^k_{\vec{\psi},\varphi} I^m\|_{\mathscr{B}}+2\varepsilon. \end{aligned} \tag{3.14}$$

Since $|\varphi(z_n)|\to 1$ as $n\to\infty$, by the arbitrary of ε and using (3.14), we get

$$\lim_{n\to\infty}\|P^k_{\vec{\psi},\varphi} f_{0,\varphi(z_n)}\|_{\mathscr{B}}=0,$$

i.e.,we obtain

$$\lim_{|\varphi(a)|\to 1}\|P^k_{\vec{\psi},\varphi} f_{0,\varphi(a)}\|_{\mathscr{B}}=0, \tag{3.15}$$

Arguing as Theorem 2.2, suppose

$$f_{1,\varphi(z_n)}(z)=\sum_{l=0}^{\infty}c_{n,l}z^l, \qquad f_{1,\varphi(z_n)}(z)=f_{0,\varphi(z_n)}(z)\sigma_{\varphi(z_n)}(z).$$

Then

$$\left((1-|\varphi(z_n)|^2)\sum_{i=0}^{N}\overline{\varphi(z_n)}^i z^i\right)\left(\varphi(z_n)-(1-|\varphi(z_n)|^2)\sum_{i=0}^{N}\overline{\varphi(z_n)}^i z^{i+1}\right)$$
$$= c_{n,0}+c_{n,1}z^1+c_{n,2}z^2+\cdots+c_{n,N}z^N+z^{N+1}q_1(z),$$

where $q_1(z)$ is a polynomial. Therefore, by (2.17), we obtain

$$\sum_{l=0}^{N}|c_{n,l}| \leqslant \left((1-|\varphi(z_n)|^2)\sum_{i=0}^{N}|\varphi(z_n)|^i\right)\left(|\varphi(z_n)|+(1-|\varphi(z_n)|^2)\sum_{i=0}^{N}|\varphi(z_n)|^i\right)$$
$$\to 0 \text{ as } n\to\infty. \tag{3.16}$$

$$\sum_{l=0}^{\infty}|c_{n,l}| \leqslant \left((1-|\varphi(z_n)|^2)\sum_{i=0}^{\infty}|\varphi(z_n)|^i\right)\left(|\varphi(z_n)|+(1-|\varphi(z_n)|^2)\sum_{i=0}^{\infty}|\varphi(z_n)|^i\right)$$
$$\leqslant 2\cdot 3=6. \tag{3.17}$$

Thus we get

$$\begin{aligned}\|P^k_{\vec{\psi},\varphi}f_{1,\varphi(z_n)}\|_{\mathscr{B}} &\leqslant \sum_{l=0}^{\infty}|c_{n,l}|\|P^k_{\vec{\psi},\varphi}I^l\|_{\mathscr{B}}\\ &= \sum_{l=0}^{N}|c_{n,l}|\|P^k_{\vec{\psi},\varphi}I^l\|_{\mathscr{B}}+\sum_{l=N+1}^{\infty}|c_{n,l}|\|P^k_{\vec{\psi},\varphi}I^l\|_{\mathscr{B}}\\ &\leqslant \sum_{l=0}^{N}|c_{n,l}|\sup_{m\in\mathbb{N}}\|P^k_{\vec{\psi},\varphi}I^m\|_{\mathscr{B}}+\sum_{l=N+1}^{\infty}|c_{n,l}|\varepsilon.\end{aligned}$$

From (3.16) and (3.17), letting $n\to\infty$, by the arbitrary of ε, we get

$$\lim_{n\to\infty}\|P^k_{\vec{\psi},\varphi}f_{1,\varphi(z_n)}\|_{\mathscr{B}}=0,$$

i.e.,

$$\lim_{|\varphi(a)|\to 1}\|P^k_{\vec{\psi},\varphi}f_{1,\varphi(a)}\|_{\mathscr{B}}=0. \tag{3.18}$$

Similarly, we suppose $f_{2,\varphi(z_n)}(z)=\sum_{l=0}^{\infty}d_{n,l}z^l$. Since $f_{2,\varphi(z_n)}(z)=f_{1,\varphi(z_n)}(z)\sigma_{\varphi(z_n)}(z)$, then we have

$$\left((1-|\varphi(z_n)|^2)\sum_{i=0}^{N}\overline{\varphi(z_n)}^i z^i\right)\left(\varphi(z_n)-(1-|\varphi(z_n)|^2)\sum_{i=0}^{N}\overline{\varphi(z_n)}^i z^{i+1}\right)^2$$
$$= d_{n,0}+d_{n,1}z^1+d_{n,2}z^2+\cdots+d_{n,N}z^N+z^{N+1}q_2(z),$$

where $q_2(z)$ is a polynomial. Therefore, by (2.17), we obtain

$$\sum_{l=0}^{N}|d_{n,l}| \leqslant \left((1-|\varphi(z_n)|^2)\sum_{i=0}^{N}|\varphi(z_n)|^i\right)\left(|\varphi(z_n)|+(1-|\varphi(z_n)|^2)\sum_{i=0}^{N}|\varphi(z_n)|^i\right)^2$$
$$\to 0 \text{ as } n\to\infty. \tag{3.19}$$

$$\sum_{l=0}^{\infty}|d_{n,l}| \leqslant \left((1-|\varphi(z_n)|^2)\sum_{i=0}^{\infty}|\varphi(z_n)|^i\right)\left(|\varphi(z_n)|+(1-|\varphi(z_n)|^2)\sum_{i=0}^{\infty}|\varphi(z_n)|^i\right)^2$$
$$\leqslant 2\cdot 3^2 = 18. \tag{3.20}$$

Thus we get

$$\begin{aligned}\|P^k_{\vec{\psi},\varphi}f_{2,\varphi(z_n)}\|_{\mathscr{B}} &\leqslant \sum_{l=0}^{\infty}|d_{n,l}|\|P^k_{\vec{\psi},\varphi}I^l\|_{\mathscr{B}}\\ &= \sum_{l=0}^{N}|d_{n,l}|\|P^k_{\vec{\psi},\varphi}I^l\|_{\mathscr{B}} + \sum_{l=N+1}^{\infty}|d_{n,l}|\|P^k_{\vec{\psi},\varphi}I^l\|_{\mathscr{B}}\\ &\leqslant \sum_{l=0}^{N}|d_{n,l}|\sup_{m\in\mathbb{N}}\|P^k_{\vec{\psi},\varphi}I^m\|_{\mathscr{B}} + \sum_{l=N+1}^{\infty}|d_{n,l}|\varepsilon.\end{aligned}$$

From (3.19) and (3.20), letting $n\to\infty$, by the arbitrary of ε, we get

$$\lim_{n\to\infty}\|P^k_{\vec{\psi},\varphi}f_{2,\varphi(z_n)}\|_{\mathscr{B}} = 0,$$

i.e.,

$$\lim_{|\varphi(a)|\to 1}\|P^k_{\vec{\psi},\varphi}f_{2,\varphi(a)}\|_{\mathscr{B}} = 0. \tag{3.21}$$

By a standard inductive argument, arguing as (3.18) and (3.21), it is easy to get

$$\lim_{|\varphi(a)|\to 1}\|P^k_{\vec{\psi},\varphi}f_{i,\varphi(a)}\|_{\mathscr{B}} = 0, \text{ for } i=3,4,\cdots,k+1, \tag{3.22}$$

as desired.

$(c)\Rightarrow(a)$ Suppose (c) holds. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$. From the proof of Theorem 2.1, we notice that (2.2) and Lemma 3.1 imply

$$\frac{(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}} \leqslant \|P^k_{\vec{\psi},\varphi}f_{k+1,\varphi(z_n)}\|_{\mathscr{B}} \to 0$$

as $n\to\infty$. Therefore

$$\lim_{n\to\infty}\frac{(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}} = 0. \tag{3.23}$$

(2.5) and Lemma 3.1 imply that

$$\begin{aligned}&\frac{(1-|z_n|^2)|\psi_{k-1}(z_n)\varphi'(z_n)+\psi_k'(z_n)|}{(1-|\varphi(z_n)|^2)^k}\\ &\leqslant \|P^k_{\vec{\psi},\varphi}f_{k,\varphi(z_n)}\|_{\mathscr{B}} + \frac{(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}}\\ &\leqslant \|P^k_{\vec{\psi},\varphi}f_{k,\varphi(z_n)}\|_{\mathscr{B}} + \|P^k_{\vec{\psi},\varphi}f_{k+1,\varphi(z_n)}\|_{\mathscr{B}} \to 0\end{aligned}$$

as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)|\psi_{k-1}(z_n)\varphi'(z_n)+\psi_k'(z_n)|}{(1-|\varphi(z_n)|^2)^k} = 0. \tag{3.24}$$

(2.9) and Lemma 3.1 imply that

$$\begin{aligned}
&\frac{(1-|z_n|^2)|\psi_{j-1}(z_n)\varphi'(z_n)+\psi_j'(z_n)|}{(1-|\varphi(z_n)|^2)^j} \\
&\leqslant \|P^k_{\vec{\psi},\varphi} f_{j,\varphi(z_n)}\|_{\mathscr{B}} + \frac{(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}} \\
&\quad + \sum_{i=j+1}^{k} \frac{(1-|z_n|^2)|\psi_{i-1}(z_n)\varphi'(z_n)+\psi_i'(z_n)|}{(1-|\varphi(z_n)|^2)^i} \\
&\leqslant \|P^k_{\vec{\psi},\varphi} f_{j,\varphi(z_n)}\|_{\mathscr{B}} + \|P^k_{\vec{\psi},\varphi} f_{k+1,\varphi(z_n)}\|_{\mathscr{B}} + \sum_{i=j+1}^{k} \|P^k_{\vec{\psi},\varphi} f_{i,\varphi(z_n)}\|_{\mathscr{B}} \\
&\to 0 \quad \text{for} \quad j = 1,2,\cdots,k,
\end{aligned}$$

as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)|\psi_{j-1}(z_n)\varphi'(z_n)+\psi_j'(z_n)|}{(1-|\varphi(z_n)|^2)^j} = 0 \quad \text{for} \quad j = 1,2,\cdots,k. \tag{3.25}$$

(2.12) and Lemma 3.1 imply that

$$\begin{aligned}
&(1-|z_n|^2)|\psi_0'(z_n)| \\
&\leqslant \|P^k_{\vec{\psi},\varphi} f_{0,\varphi(z_n)}\|_{\mathscr{B}} + \frac{(1-|z_n|^2)|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{k+1}} \\
&\quad + \sum_{j=1}^{k} \frac{(1-|z_n|^2)|\psi_{j-1}(z_n)\varphi'(z_n)+\psi_j'(z_n)|}{(1-|\varphi(z_n)|^2)^j} \\
&\leqslant \|P^k_{\vec{\psi},\varphi} f_{j,\varphi(z_n)}\|_{\mathscr{B}} + \|P^k_{\vec{\psi},\varphi} f_{k+1,\varphi(z_n)}\|_{\mathscr{B}} + \sum_{j=1}^{\infty} \|P^k_{\vec{\psi},\varphi} f_{j,\varphi(z_n)}\|_{\mathscr{B}} \to 0
\end{aligned}$$

as $n \to \infty$. Therefore

$$\lim_{n\to\infty} (1-|z_n|^2)|\psi_0'(z_n)| = 0. \tag{3.26}$$

By (3.23)–(3.26) and Theorem 3.1, we know that (a) holds.

The proof is complete. □

THEOREM 3.3. *Let* $k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ *and* $\psi_j \in H(\mathbb{D})$, $j = 0,1,\ldots,k$. *Then* $P^k_{\vec{\psi},\varphi} : H^\infty \to \mathscr{B}$ *is compact if and only if* $P^k_{\vec{\psi},\varphi} : H^\infty \to \mathscr{B}$ *is bounded and*

$$\limsup_{n\to\infty} \|\psi_0'\varphi^n\|_{v_1} = 0; \quad \limsup_{n\to\infty} n^{k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_1} = 0;$$

$$\limsup_{n\to\infty} n^j \|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_1} = 0, \quad \textit{for} \quad j = 1,2,\cdots,k.$$

Proof. According to Theorem 3.1, the operator $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is compact if and only if the operator $P^k_{\vec{\psi},\varphi}$ is bounded and $\sum_{j=0}^{k+1} Q_j = 0$. In order to prove the theorem, it is enough to show that

$$\limsup_{n\to\infty} \|\psi_0'\varphi^n\|_{v_1} \approx Q_0; \tag{3.27}$$

$$\limsup_{n\to\infty} n^j \|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_1} \approx Q_j, \quad \text{for} \quad j = 1,2,\cdots,k; \tag{3.28}$$

$$\limsup_{n\to\infty} n^{k+1} \|\psi_k\varphi'\varphi^{n-1}\|_{v_1} \approx Q_{k+1}. \tag{3.29}$$

We first prove that (3.27) holds. It is obvious that for every positive integer $n \geqslant 1$,

$$\begin{aligned} \|\psi_0'\varphi^n\|_{v_1} &\gtrsim \sup_{|\varphi(z)|\geqslant(1-\frac{1}{n})} (1-|z|^2)|\varphi(z)|^n|\psi_0'(z)| \\ &\gtrsim \left(1-\frac{1}{n}\right)^n \sup_{|\varphi(z)|\geqslant(1-\frac{1}{n})} (1-|z|^2)|\psi_0'(z)|. \end{aligned} \tag{3.30}$$

Taking $n\to\infty$, we get

$$\limsup_{n\to\infty} \|\psi_0'\varphi^n\|_{v_1} \geqslant \frac{1}{e} Q_0. \tag{3.31}$$

On the other hand, for $0<r<1$,

$$\begin{aligned} \|\psi_0'\varphi^n\|_{v_1} &\lesssim \sup_{|\varphi(z)|>r} (1-|z|^2)|\varphi(z)|^n|\psi_0'(z)| + \sup_{|\varphi(z)|\leqslant r} (1-|z|^2)|\varphi(z)|^n|\psi_0'(z)| \\ &\lesssim \sup_{|\varphi(z)|>r} (1-|z|^2)|\psi_0'(z)| + r^n\|\psi_0\|_{\mathscr{B}}. \end{aligned}$$

Since $\limsup_{n\to\infty} r^n\|\psi_0\|_{\mathscr{B}} = 0$, we get

$$\limsup_{n\to\infty} \|\psi_0'\varphi^n\|_{v_1} \lesssim \sup_{|\varphi(z)|>r} (1-|z|^2)|\psi_0'(z)|$$

for any $r\in(0,1)$. Letting $r\to 1$ we have $\limsup_{n\to\infty} \|\psi_0'\varphi^n\|_{v_1} \lesssim Q_0$, which together with (3.31) gives $\limsup_{n\to\infty} \|\psi_0'\varphi^n\|_{v_1} \approx Q_0$.

Next we prove (3.28) and (3.29). We fix $1\leqslant j\leqslant k$. Then for any $0<r<1$, by Lemma 3.2, we have

$$\begin{aligned} &\sup_{n\geqslant 1}\sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}} \\ &= \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{\bar{v}_j(\varphi(z))} \\ &\leqslant C \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{\tilde{v}_j(\varphi(z))} \end{aligned}$$

$$= C \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{v_j(\varphi(z))}$$
$$= C \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j}, \quad \text{for} \quad j=1,2,\cdots,k. \tag{3.32}$$

Similarly, we get

$$\sup_{n\geqslant 1}\sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}} \leqslant C \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}. \tag{3.33}$$

Letting $n\to\infty$ in (3.32) and (3.33), we get

$$\limsup_{n\to\infty}\sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}$$
$$\leqslant C \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j}, \quad \text{for} \quad j=1,2,\cdots,k, \tag{3.34}$$

and

$$\limsup_{n\to\infty}\sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}} \lesssim \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}. \tag{3.35}$$

Since $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded, from Theorem 2.1, we obtain

$$T_0 = \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_0'(z)| < \infty, \quad T_{k+1} = \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_k(z)\varphi'(z)| < \infty,$$

$$T_j = \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)| < \infty, \quad \text{for} \quad j=1,2,\cdots,k.$$

Now for $|\varphi(z)|\leqslant r$, we may choose δ such that $0<r<\delta<1$. Then we have

$$\sup_{|\varphi(z)|\leqslant r} \frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}$$
$$\leqslant \frac{T_j r^{n-1}}{\|\xi^{n-1}\|_{v_j}} = T_j\left(\frac{r}{\delta}\right)^{n-1}\frac{\delta^{n-1}}{\|\xi^{n-1}\|_{v_j}}$$
$$= \frac{T_j}{\bar{v}_j(\delta)}\left(\frac{r}{\delta}\right)^{n-1}, \quad \text{for} \quad j=1,2,\cdots,k, \tag{3.36}$$

and

$$\sup_{|\varphi(z)|\leqslant r} \frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}}$$
$$\leqslant \frac{T_{k+1} r^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}} = T_{k+1}\left(\frac{r}{\delta}\right)^{n-1}\frac{\delta^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}} = \frac{T_{k+1}}{\bar{v}_{k+1}(\delta)}\left(\frac{r}{\delta}\right)^{n-1}. \tag{3.37}$$

Letting $n\to\infty$ in (3.36) and (3.37), we get

$$\limsup_{n\to\infty}\sup_{|\varphi(z)|\leqslant r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}=0,\quad \text{for}\quad j=1,2,\cdots,k, \tag{3.38}$$

and

$$\limsup_{n\to\infty}\sup_{|\varphi(z)|\leqslant r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}}=0. \tag{3.39}$$

Using (3.34) and (3.38), by Lemma 2.4 we have

$$\begin{aligned}
&\limsup_{n\to\infty} n^j\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1}\approx\limsup_{n\to\infty}\frac{\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_j}}\\
&=\limsup_{n\to\infty}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}\\
&\leqslant\limsup_{n\to\infty}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}\\
&\quad+\limsup_{n\to\infty}\sup_{|\varphi(z)|\leqslant r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}\\
&\leqslant C\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j},\quad\text{for}\quad j=1,2,\cdots,k.
\end{aligned}\tag{3.40}$$

Using (3.35) and (3.39), by Lemma 2.4, similarly we have

$$\begin{aligned}
\limsup_{n\to\infty} n^{k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}&\approx\limsup_{n\to\infty}\frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_{k+1}}}\\
&\leqslant C\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}.
\end{aligned}\tag{3.41}$$

Since (3.40) and (3.41) hold for every $r\in(0,1)$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} n^j\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1}\\
&\lesssim\lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j}=Q_j,\quad\text{for}\quad j=1,2,\cdots,k,
\end{aligned}\tag{3.42}$$

and

$$\limsup_{n\to\infty} n^{k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}\lesssim\lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}=Q_{k+1}. \tag{3.43}$$

In order to obtain the reverse inequality, we use some ideas of [6]. It follows from the proof of Lemma 2.2 of [6] that

$$\frac{1}{v_i(t)}=\frac{1}{\tilde{v}_i(t)}\leqslant\frac{1}{t\,\bar{v}_i(t)},\quad i=1,2,\cdots,k+1\quad\text{for each}\quad t\in(0,1). \tag{3.44}$$

Fix $m\in\mathbb{N}$ and $r\in(0,1)$. Then using (3.44), we have

$$\begin{aligned}
&\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j}\\
&=\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{\tilde{v}_j(\varphi(z))}\\
&\leqslant C\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{|\varphi(z)|\bar{v}_j(\varphi(z))}\\
&=C\sup_{|\varphi(z)|>r}\frac{1}{|\varphi(z)|}\sup_{n\geqslant1}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}\\
&\leqslant\frac{C}{r}\Bigg(\sup_{|\varphi(z)|>r}\sup_{1\leqslant n\leqslant m}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}\\
&\quad+\sup_{|\varphi(z)|>r}\sup_{n>m}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}\Bigg)\\
&\leqslant\frac{C}{r}\Bigg(\sup_{|\varphi(z)|>r}\sup_{1\leqslant n\leqslant m}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_j}}\\
&\quad+\sup_{n>m}\frac{\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_j}}\Bigg),\quad\text{for}\quad j=1,2,\cdots,k,
\end{aligned}\tag{3.45}$$

and

$$\begin{aligned}
&\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}=\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{\tilde{v}_{k+1}(\varphi(z))}\\
&\leqslant C\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{|\varphi(z)|\bar{v}_{k+1}(\varphi(z))}\\
&=C\sup_{|\varphi(z)|>r}\frac{1}{|\varphi(z)|}\sup_{n\geqslant1}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}}\\
&\leqslant\frac{C}{r}\Bigg(\sup_{|\varphi(z)|>r}\sup_{1\leqslant n\leqslant m}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}}\\
&\quad+\sup_{|\varphi(z)|>r}\sup_{n>m}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}}\Bigg)\\
&\leqslant\frac{C}{r}\Bigg(\sup_{|\varphi(z)|>r}\sup_{1\leqslant n\leqslant m}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)||\varphi(z)|^{n-1}}{\|\xi^{n-1}\|_{v_{k+1}}}+\sup_{n>m}\frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_{k+1}}}\Bigg).
\end{aligned}\tag{3.46}$$

Since $P^k_{\vec{\psi},\varphi}: H^\infty \to \mathscr{B}$ is bounded, from Theorem 2.1 we see that $M_j < \infty$ for $j = 1,2,\cdots,k+1$. Thus, we have

$$(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)| \leqslant M_j v_j(\varphi(z)),\ \ z\in\mathbb{D},\ \ j=1,2,\cdots,k, \tag{3.47}$$

and

$$(1-|z|^2)|\psi_k(z)\varphi'(z)| \leqslant M_{k+1}v_{k+1}(\varphi(z)),\ \ z\in\mathbb{D}. \tag{3.48}$$

For some $1 \leqslant n_0 \leqslant m$, using (3.45) and (3.47), we have

$$\begin{aligned}&\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^j}\\ &\leqslant \frac{C}{r}\left(\sup_{|\varphi(z)|>r}\frac{M_j v_j(\varphi(z))}{\|\xi^{n_0-1}\|_{v_j}}+\sup_{n>m}\frac{\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_j}}\right),\ \ j=1,2,\cdots,k.\end{aligned} \tag{3.49}$$

For some $1 \leqslant n_0 \leqslant m$, using (3.46) and (3.48), we have

$$\begin{aligned}&\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+1}}\\ &\leqslant \frac{C}{r}\left(\sup_{|\varphi(z)|>r}\frac{M_{k+1}v_{k+1}(\varphi(z))}{\|\xi^{n_0-1}\|_{v_{k+1}}}+\sup_{n>m}\frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_{k+1}}}\right).\end{aligned} \tag{3.50}$$

Since $\lim_{|\varphi(z)|\to 1} v_i(\varphi(z)) = 0$, for each $i = 1,2,\cdots,k+1$, letting $r \to 1$ in (3.49) and (3.50), we get

$$Q_j \leqslant C\sup_{n>m}\frac{\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_j}},\ \ \text{for}\ \ j=1,2,\cdots,k,\ \ \text{for every}\ \ m\in\mathbb{N}, \tag{3.51}$$

and

$$Q_{k+1} \leqslant C\sup_{n>m}\frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_{k+1}}},\ \ \text{for every}\ \ m\in\mathbb{N}. \tag{3.52}$$

Letting $m \to \infty$ in (3.51) and (3.52), using Lemma 2.4, we have

$$\begin{aligned}Q_j &\lesssim \lim_{n\to\infty}\frac{\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_j}}\\ &\approx \lim_{n\to\infty} n^j\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_1},\ \ \text{for}\quad j=1,2,\cdots,k,\end{aligned} \tag{3.53}$$

and

$$Q_{k+1} \lesssim \lim_{n\to\infty}\frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_{k+1}}} \approx \lim_{n\to\infty} n^{k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}. \tag{3.54}$$

From (3.42) and (3.53), it follows that

$$\lim_{n\to\infty} \frac{\|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_j}} \approx Q_j$$

for all $j = 1,2,\cdots,k$. From (3.43) and (3.54), we get

$$\lim_{n\to\infty} \frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_1}}{\|\xi^{n-1}\|_{v_{k+1}}} \approx Q_{k+1}$$

The proof is complete. □

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