Journal of Mathematical Inequalities
Volume 18, Number 2 (2024), 551–563 doi:10.7153/jmi-2024-18-30

APPROXIMATION BY PERTURBED BASKAKOV–TYPE OPERATORS

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(*Communicated by T. Burić*)

Abstract. In this paper, we introduce a new Baskakov-type operator. Firstly, we obtain the rate of convergence by using modulus of continuity and then Voronovskaja type asymptotic formula for these operators.

1. Introduction

In 1912, the classical Bernstein polynomials given by

$$B_m(f;x)=\sum_{k=0}^{m} p_{m,k}(x)\, f\left(\frac{k}{m}\right),$$

where

$$p_{m,k}(x)=\begin{cases} \binom{m}{k}x^k(1-x)^{m-k}, \\ 0,\ 0\leqslant k\leqslant m,\ k<0 \text{ or } k>m, \end{cases}$$

for any $m\in\mathbb{N}$, $f\in C[0,1]$, $x\in[0,1]$ were proposed by Bernstein [7] as one of the simplest way to prove Weierstrass approximation theorem and studied intensively by a large number of researches. It is known that the fundamental polynomials $p_{m,k}(x)$ satisfy the following recursion

$$p_{m,k}(x)=(1-x)\,p_{m-1,k}(x)+xp_{m-1,k-1}(x),\ 0\leqslant k\leqslant m. \tag{1.1}$$

Although linear positive operators have many advantages with regard to construction, simplicity and analyzing, the rate of convergence of these operators is extremely slow. So, in order to improve the degree of approximation of these operators some approaches have been given in the literature.

Very recently, Khosravian-Arab et al. [11] introduced a sequence of modified Bernstein operators to improve the order of approximation as

$$B_m^{(1)}(f;x)=\sum_{k=0}^{m} p_{m,k}^{(1)}(x)\, f\left(\frac{k}{m}\right),\ x\in[0,1], \tag{1.2}$$

Mathematics subject classification (2020): 41A36.

Keywords and phrases: Baskakov-type operator, Voronovskaja type results, weighted modulus of continuity.

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$$p_{m,k}^{(1)}(x) = \alpha(x,m)\,p_{m-1,k}(x) + \alpha(1-x,m)\,p_{m-1,k-1}(x),\ 1 \leqslant k \leqslant m-1$$
$$p_{m,0}^{(1)}(x) = \alpha(x,m)(1-x)^{m-1},\ \ p_{m,m}^{(1)}(x) = \alpha(1-x,m)\,x^{m-1}, \tag{1.3}$$

and

$$\alpha(x,m) = \alpha_1(m)\,x + \alpha_0(m),\ m = 0,1,\ldots,$$

where $\alpha_0(m)$, $\alpha_1(m)$ are unknown sequences that will be determined appropriately later. Obviously, for $\alpha_1(m) = -1$, $\alpha_0(m) = 1$, expression (1.3) becomes (1.1). Later, using their approach some generalizations of the operators given by (1.2) have been studied by many authors (see, e.g., [1]–[4], [9]). Recently, Acu and Bascanbaz-Tunca [5] considered complex form of perturbed Bernstein-type operators attached to an analytic function in a disk of radius $R > 1$ centered at the origin. The authors obtained quantitative upper estimate for complex perturbed Bernstein-type operators and their derivatives on compact disks, the qualitative Voronovskaja type result and the exact order of simultaneous approximation. Very recently, Cetin [8] introduced the Stancu variant of complex perturbed Bernstein-type operators and studied approximation properties.

In this paper, motivated by the same technique in [11], we introduce a new variant of Baskakov operators which we call as perturbed Baskakov operators.

For every $f \in C_B[0,\infty)$, the space of real valued, bounded and continuous functions defined on $[0,\infty)$, Baskakov operators are given in [6] as

$$S_m(f;x) = \sum_{k=0}^{\infty} P_{m,k}(x)\, f\left(\frac{k}{m}\right), \tag{1.4}$$

where $m \geqslant 1$, $x \in [0,\infty)$ and

$$P_{m,k}(x) = \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}}.$$

The fundamental polynomials satisfy the recursion

$$P_{m,k}(x) = (1+x)\,P_{m+1,k}(x) - x P_{m+1,k-1}(x),\ m \geqslant 1. \tag{1.5}$$

Now, we propose modified Baskakov operators as follows:

$$S_m^{M,1}(f;x) = \sum_{k=0}^{\infty} P_{m,k}^{M,1}(x)\, f\left(\frac{k}{m}\right), \tag{1.6}$$

where $m \geqslant 1$, $x \in [0,\infty)$,

$$P_{m,k}^{M,1}(x) = \alpha(-x,m)\,P_{m+1,k}(x) + \alpha(1+x,m)\,P_{m+1,k-1}(x), \tag{1.7}$$

and

$$\alpha(-x,m) = \alpha_1(m)(-x) + \alpha_0(m),\ m = 0,1,\ldots,$$

where $\alpha_0(m)$, $\alpha_1(m)$ are unknown sequences. For $\alpha_1(m) = -1$, $\alpha_0(m) = 1$, (1.7) reduces to (1.5).

Throught the paper, we suppose that the sequences $\alpha_1(m)$ and $\alpha_0(m)$ verify the condition

$$2\alpha_0(m) + \alpha_1(m) = 1. \tag{1.8}$$

2. Approximation by perturbed Baskakov-type operators

Denote $e_i(x) = x^i,\ i = 0,1,2,3,4$. Then, we have the following.

LEMMA 2.1. *For the moments of the operator $S_m^{M,1}$ given by (1.6), one has*

i) $S_m^{M,1}(e_0;x) = 1,$

ii) $S_m^{M,1}(e_1;x) = \frac{1}{m}\left[mx + (1+2x)(1-\alpha_0(m))\right],$

iii) $S_m^{M,1}(e_2;x) = \frac{1}{m^2}\left\{m^2x^2 + mx\left[(5-4\alpha_0(m))x + 3 - 2\alpha_0(m)\right] + (1+2x)^2(1-\alpha_0(m))\right\},$

iv) $S_m^{M,1}(e_3;x) = \frac{1}{m^3}\left\{m^3x^3 + 3m^2x^2\left[(3-2\alpha_0(m))x + 2 - \alpha_0(m)\right] + mx\left[x^2(20-18\alpha_0(m)) + x(24-21\alpha_0(m)) + 7 - 6\alpha_0(m)\right] + \left(12x^3+18x^2+8x+1\right)(1-\alpha_0(m))\right\},$

v) $S_m^{M,1}(e_4;x) = \frac{1}{m^4}\left\{m^4x^4 + 2m^3x^3\left[x(7-4\alpha_0(m)) + 5 - 2\alpha_0(m)\right] + m^2x^2\left[x^2(59-48\alpha_0(m)) + x(78-60\alpha_0(m)) + 25 - 18\alpha_0(m)\right] + mx\left[x^3(94-88\alpha_0(m)) + x^2(164-152\alpha_0(m)) + x(89-82\alpha_0(m)) + 15 - 14\alpha_0(m)\right] + \left(48x^4+96x^3+64x^2+16x+1\right)(1-\alpha_0(m))\right\}.$

LEMMA 2.2. *For the central moments of the operator $S_m^{M,1}$ given by (1.6), one has*

i) $S_m^{M,1}(t-x;x) = \frac{1}{m}(1+2x)(1-\alpha_0(m)),$

ii) $S_m^{M,1}\left((t-x)^2;x\right) = \frac{1}{m^2}\left[x(x+1)m + (1+2x)^2(1-\alpha_0(m))\right],$

iii) $S_m^{M,1}\left((t-x)^4;x\right) = \frac{1}{m^4}\left\{3m^2x^2(x+1)^2 + mx\left[x^3(46-40\alpha_0(m)) + x^2(92-80\alpha_0(m)) + x(57-50\alpha_0(m)) + 11 - 10\alpha_0(m)\right] + \left(48x^4+96x^3+64x^2+16x+1\right)(1-\alpha_0(m))\right\}.$

Now, denote $B_2[0,\infty) := \left\{f : [0,\infty) \to \mathbb{R} \mid |f(x)| \leqslant K\left(1+x^2\right)\right\}$ equipped with the norm given by $\|f\|_2 = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$, where K is a positive constant. Also, let us define

$$C_2[0,\infty) = \{f \in B_2[0,\infty) : f \text{ is continuous}\}$$

and

$$C_2^*[0,\infty) = \left\{f \in C_2[0,\infty) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ is finite}\right\}.$$

In the next theorem, we give the rate of convergence by the operators $S_m^{M,1}$ given by (1.6) in terms of modulus of continuity.

THEOREM 2.1. *Suppose that $f \in C_2[0,\infty)$ and $x \in [0,a]$, $a>0$. If $\alpha_1(m)$ is a bounded sequence, then we have*

$$\left|S_m^{M,1}(f;x)-f(x)\right| \leqslant \left[\left(a+\frac{3}{2}\right)|\alpha_1(m)|+\frac{1}{2}\right]\left\{4K\left(1+x^2\right)\frac{2x(x+1)(m+2)+1}{m^2}\right.$$
$$\left.+\omega_a\left(f;\frac{1}{\sqrt{m}}\right)\left[1+\sqrt{2}\sqrt{\frac{2x(x+1)(m+2)+1}{m}}\right]\right\},$$

where ω_a is modulus of continuity of f on the finite interval $[0,a]$.

Proof. Let $x \in [0,a]$ and $\frac{k}{m} > a+1$, which clearly means $\frac{k}{m}-x>1$. Then for $f \in C_2[0,\infty)$ we have

$$\begin{aligned}\left|f\left(\frac{k}{m}\right)-f(x)\right| &\leqslant \left|f\left(\frac{k}{m}\right)\right|+|f(x)| \\ &\leqslant K\left(2+x^2+\left(\frac{k}{m}\right)^2\right)=K\left[2+2x^2+\left(\frac{k}{m}-x\right)^2+2x\left(\frac{k}{m}-x\right)\right] \\ &\leqslant K\left(\frac{k}{m}-x\right)^2\left(3+2x^2+2x\right)\leqslant 4K\left(1+x^2\right)\left(\frac{k}{m}-x\right)^2. \end{aligned} \tag{2.1}$$

For $x \in [0,a]$ and $\frac{k}{m} \in [0,a+1]$ we have

$$\left|f\left(\frac{k}{m}\right)-f(x)\right| \leqslant \omega_a\left(f;\left|\frac{k}{m}-x\right|\right) \leqslant \left(1+\frac{1}{\delta}\left|\frac{k}{m}-x\right|\right)\omega_a(f;\delta). \tag{2.2}$$

From the inequalities (2.1) and (2.2), for $x \in [0,a]$ and $\frac{k}{m} \geqslant 0$ we have

$$\left|f\left(\frac{k}{m}\right)-f(x)\right| \leqslant 4K\left(1+x^2\right)\left(\frac{k}{m}-x\right)^2+\left(1+\frac{1}{\delta}\left|\frac{k}{m}-x\right|\right)\omega_a(f;\delta). \tag{2.3}$$

On the other hand, from (1.8), for $x \in [0,a]$, $a>0$, we get

$$\begin{aligned}|\alpha(-x,m)| &\leqslant |\alpha_1(m)|a+\left|\frac{1-\alpha_1(m)}{2}\right| \\ &\leqslant |\alpha_1(m)|a+\frac{1}{2}+\frac{|\alpha_1(m)|}{2}=\left(a+\frac{1}{2}\right)|\alpha_1(m)|+\frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}|\alpha(1+x,m)| &\leqslant |\alpha_1(m)|(1+a)+\left|\frac{1-\alpha_1(m)}{2}\right| \\ &\leqslant \left(a+\frac{3}{2}\right)|\alpha_1(m)|+\frac{1}{2},\end{aligned}$$

which follows

$$\begin{aligned}\left|P_{m,k}^{M,1}(x)\right| &\leqslant \left[\left(a+\frac{1}{2}\right)|\alpha_1(m)|+\frac{1}{2}\right]P_{m+1,k}(x)\\ &\quad+\left[\left(a+\frac{3}{2}\right)|\alpha_1(m)|+\frac{1}{2}\right]P_{m+1,k-1}(x).\\ &\leqslant \left[\left(a+\frac{3}{2}\right)|\alpha_1(m)|+\frac{1}{2}\right]\left[P_{m+1,k}(x)+P_{m+1,k-1}(x)\right].\end{aligned} \tag{2.4}$$

Thus, using (2.3), (2.4) and $\delta=\frac{1}{\sqrt{m}}$, we obtain

$$\begin{aligned}\left|S_m^{M,1}(f;x)-f(x)\right| &\leqslant \sum_{k=0}^{\infty}\left|P_{m,k}^{M,1}(x)\right|\left|f\left(\frac{k}{m}\right)-f(x)\right|\\ &\leqslant \left[\left(a+\frac{3}{2}\right)|\alpha_1(m)|+\frac{1}{2}\right]\sum_{k=0}^{\infty}\left[P_{m+1,k}(x)+P_{m+1,k-1}(x)\right]\\ &\quad\times\left[4K\left(1+x^2\right)\left(\frac{k}{m}-x\right)^2+\left(1+\sqrt{m}\left|\frac{k}{m}-x\right|\right)\omega_a\left(f;\frac{1}{\sqrt{m}}\right)\right].\end{aligned}$$

But

$$\begin{aligned}&\sum_{k=0}^{\infty}\left[P_{m+1,k}(x)+P_{m+1,k-1}(x)\right]\left|\frac{k}{m}-x\right|\\ &\leqslant\left\{\sum_{k=0}^{\infty}\left[P_{m+1,k}(x)+P_{m+1,k-1}(x)\right]\right\}^{1/2}\\ &\quad\times\left\{\sum_{k=0}^{\infty}\left[P_{m+1,k}(x)+P_{m+1,k-1}(x)\right]\left(\frac{k}{m}-x\right)^2\right\}^{1/2}\\ &=\sqrt{2}\frac{\sqrt{2x(x+1)(m+2)+1}}{m}.\end{aligned}$$

Therefore, from the last inequality we have

$$\begin{aligned}\left|S_m^{M,1}(f;x)-f(x)\right| &\leqslant \left[\left(a+\frac{3}{2}\right)|\alpha_1(m)|+\frac{1}{2}\right]\left\{4K\left(1+x^2\right)\frac{2x(x+1)(m+2)+1}{m^2}\right.\\ &\quad\left.+\omega_a\left(f;\frac{1}{\sqrt{m}}\right)\left[1+\sqrt{2}\sqrt{\frac{2x(x+1)(m+2)+1}{m}}\right]\right\}. \quad \square\end{aligned}$$

THEOREM 2.2. *Suppose that* $f\in C_2[0,\infty)$ *and* $\rho>0$*. If* $\alpha_1(m)$ *is a bounded sequence, then we have*

$$\lim_{m\to\infty}\sup_{x\in[0,\infty)}\frac{\left|S_m^{M,1}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\rho}}=0.$$

Proof. Since $\alpha_1(m)$ is a bounded sequence, then there is $M>0$ such that $|\alpha_1(m)| < M$. Let $x_0 \in [0,\infty)$ be an arbitrary fixed point. Then we can write

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\left|S_m^{M,1}(f;x)-f(x)\right|}{(1+x^2)^{1+\rho}} &\leqslant \sup_{x\leqslant x_0} \frac{\left|S_m^{M,1}(f;x)-f(x)\right|}{(1+x^2)^{1+\rho}} + \sup_{x>x_0} \frac{\left|S_m^{M,1}(f;x)-f(x)\right|}{(1+x^2)^{1+\rho}} \\ &\leqslant \left\|S_m^{M,1}(f;x)-f(x)\right\|_{C[0,x_0]} + \sup_{x>x_0} \frac{\left|S_m^{M,1}(f;x)\right|}{(1+x^2)^{1+\rho}} + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\rho}}. \qquad (2.5)\end{aligned}$$

Since $|f(x)| \leqslant \|f\|_2 (1+x^2)$, we have

$$\sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\rho}} \leqslant \frac{\|f\|_2}{(1+x_0^2)^{\rho}}. \qquad (2.6)$$

Let $\varepsilon>0$ be arbitrary. We choose x_0 such that

$$\frac{\|f\|_2}{(1+x_0^2)^{\rho}} < \frac{\varepsilon}{4} \frac{1}{(2a+3)M+1}. \qquad (2.7)$$

Also, we have

$$\begin{aligned}\left|S_m^{M,1}(f;x)\right| &\leqslant \sum_{k=0}^{\infty} \left|P_{m,k}^{M,1}(x)\right| \left|f\left(\frac{k}{m}\right)\right| \\ &\leqslant \left[\left(a+\frac{3}{2}\right)|\alpha_1(m)| + \frac{1}{2}\right] \sum_{k=0}^{\infty} \left[P_{m+1,k}(x)+P_{m+1,k-1}(x)\right] \left|f\left(\frac{k}{m}\right)\right| \\ &\leqslant \left[\left(a+\frac{3}{2}\right)M + \frac{1}{2}\right] \sum_{k=0}^{\infty} \left[P_{m+1,k}(x)+P_{m+1,k-1}(x)\right] \|f\|_2 \left(1+\frac{k^2}{m^2}\right) \\ &= \left[\left(a+\frac{3}{2}\right)M + \frac{1}{2}\right] \|f\|_2 \left\{2(1+x^2) + \frac{2x(2+3x)m+(2x+1)^2}{m^2}\right\}.\end{aligned}$$

Therefore,

$$\lim_{m\to\infty} \sup_{x>x_0} \frac{\left|S_m^{M,1}(f;x)\right|}{1+x^2} \leqslant 2\left[\left(a+\frac{3}{2}\right)M+\frac{1}{2}\right]\|f\|_2 = [(2a+3)M+1]\|f\|_2,$$

which follows that there exists a positive integer $m_1 \in \mathbb{N}$ such that

$$\sup_{x>x_0} \frac{\left|S_m^{M,1}(f;x)\right|}{1+x^2} \leqslant [(2a+3)M+1]\|f\|_2 + (1+x_0^2)^{\rho} \frac{\varepsilon}{4}.$$

Then we have

$$\sup_{x>x_0} \frac{\left|S_m^{M,1}(f;x)\right|}{(1+x^2)^{\rho+1}} \leqslant [(2a+3)M+1] \frac{\|f\|_2}{(1+x_0^2)^{\rho}} + \frac{\varepsilon}{4} \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \qquad (2.8)$$

From Theorem 2.1, it follows that there is $m_2 \in \mathbb{N}$ such that for $m > m_2$

$$\left\| S_m^{M,1}(f;x) - f(x) \right\|_{C[0,x_0]} < \frac{\varepsilon}{4}.$$

Let $\widetilde{m} = \max\{m_1, m_2\}$. From (2.5)–(2.8) we get

$$\sup_{x\in[0,\infty)} \frac{\left| S_m^{M,1}(f;x) - f(x) \right|}{(1+x^2)^{1+\rho}} < \varepsilon,$$

which completes the proof. □

Let $\beta \in \mathbb{N}$. Denote the space of functions

$$B_\beta[0,\infty) = \left\{ f \in C[0,\infty) : \lim_{x\to\infty} \frac{f(x)}{1+x^\beta} \in \mathbb{R} \right\}$$

equipped with the norm

$$\|f\|_\beta = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^\beta}.$$

López-Moreno [12] introduced the following weighted modulus as

$$\Omega_\beta(f;\delta) = \sup_{x\in[0,\infty),\, 0<h\leqslant\delta} \frac{|f(x+h) - f(x)|}{1+(h+x)^\beta},\ f \in B_\beta. \tag{2.9}$$

The modulus of continuity given by (2.9) has the following,

$$\lim_{\delta\to 0} \Omega_\beta(f;\delta) = 0,$$

for given $f \in B_\beta$ (see, [12]).

THEOREM 2.3. *For $f \in B_1[0,\infty)$, $x \in [0,\infty)$, $m \geqslant 1$, we have*

$$\left\| S_m^{M,1} f - f \right\|_2 \leqslant K\Omega_1\left(f; \frac{1}{\sqrt{m}}\right) + \frac{25}{4} |1-\alpha_0(m)| \, \Omega_1\left(f; \frac{1}{m}\right),$$

where $K > 0$.

Proof. We have

$$\left| S_m^{M,1}(f;x) - f(x) \right| \leqslant |S_m(f;x) - f(x)| + \left| S_m^{M,1}(f;x) - S_m(f;x) \right|. \tag{2.10}$$

Now, we will estimate $S_m^{M,1}(f;x)-S_m(f;x)$. Therefore

$$\begin{aligned}
&S_m^{M,1}(f;x)-S_m(f;x)\\
&=\sum_{k=0}^{\infty}P_{m,k}^{M,1}(x)f\left(\frac{k}{m}\right)-\sum_{k=0}^{\infty}P_{m,k}(x)f\left(\frac{k}{m}\right)\\
&=\sum_{k=0}^{\infty}\left[\alpha(-x,m)P_{m+1,k}(x)+\alpha(1+x,m)P_{m+1,k-1}(x)\right]f\left(\frac{k}{m}\right)\\
&\quad-\sum_{k=0}^{\infty}\left[(1+x)P_{m+1,k}(x)-xP_{m+1,k-1}(x)\right]f\left(\frac{k}{m}\right)\\
&=\sum_{k=0}^{\infty}\left[-\alpha_1(m)x+\alpha_0(m)-(1+x)\right]P_{m+1,k}(x)f\left(\frac{k}{m}\right)\\
&\quad+\sum_{k=1}^{\infty}\left[(1+x)\alpha_1(m)+\alpha_0(m)+x\right]P_{m+1,k-1}(x)f\left(\frac{k}{m}\right)\\
&=-\sum_{k=0}^{\infty}\left[(\alpha_1(m)+1)x+1-\alpha_0(m)\right]P_{m+1,k}(x)f\left(\frac{k}{m}\right)\\
&\quad+\sum_{k=0}^{\infty}\left[(1+\alpha_1(m))x+1-\alpha_0(m)\right]P_{m+1,k}(x)f\left(\frac{k+1}{m}\right)\\
&=\sum_{k=0}^{\infty}(1+2x)(1-\alpha_0(m))P_{m+1,k}(x)\left[f\left(\frac{k+1}{m}\right)-f\left(\frac{k}{m}\right)\right], \qquad (2.11)
\end{aligned}$$

which follows

$$\left|S_m^{M,1}(f;x)-S_m(f;x)\right|\leqslant(1+2x)\left|1-\alpha_0(m)\right|\sum_{k=0}^{\infty}P_{m+1,k}(x)\left|f\left(\frac{k+1}{m}\right)-f\left(\frac{k}{m}\right)\right|.$$

From definition of $\Omega_1(f;\delta)$ the last inequality gives

$$\begin{aligned}
&\left|S_m^{M,1}(f;x)-S_m(f;x)\right|\\
&\leqslant(1+2x)\left|1-\alpha_0(m)\right|\sum_{k=0}^{\infty}P_{m+1,k}(x)\Omega_1\left(f;\frac{1}{m}\right)\left(1+\frac{k+1}{m}\right)\\
&=(1+2x)\left|1-\alpha_0(m)\right|\Omega_1\left(f;\frac{1}{m}\right)\left(1+\frac{1}{m}+\frac{1}{m}\sum_{k=1}^{\infty}kP_{m+1,k}(x)\right)\\
&=(1+2x)\left|1-\alpha_0(m)\right|\left(1+\frac{1}{m}\right)(1+x)\Omega_1\left(f;\frac{1}{m}\right). \qquad (2.12)
\end{aligned}$$

Using Theorem 3 in [12], we get

$$\|S_mf-f\|_2\leqslant K\Omega_1\left(f;\frac{1}{\sqrt{m}}\right), \qquad (2.13)$$

where K is independent of f and m. From (2.10), (2.12) and (2.13), we obtain

$$\left\|S_m^{M,1}f-f\right\|_2 \leqslant K\Omega_1\left(f;\frac{1}{\sqrt{m}}\right) + \sup_{x\in[0,\infty)}\frac{1+2x}{1+x^2}\left(1+\frac{1}{m}\right)(1+x)\left|1-\alpha_0(m)\right|\Omega_1\left(f;\frac{1}{m}\right).$$

So,

$$\left\|S_m^{M,1}f-f\right\|_2 \leqslant K\Omega_1\left(f;\frac{1}{\sqrt{m}}\right)+\frac{25}{4}\left|1-\alpha_0(m)\right|\Omega_1\left(f;\frac{1}{m}\right). \qquad \square$$

REMARK 2.1. 1) For $\alpha_0(m)=1$ we recover the estimate of the classical Baskakov operator given by (1.4).

2) If $\alpha_0(m)$ is bounded, e.g., $|\alpha_0(m)|\leqslant M$, then

$$\left\|S_m^{M,1}f-f\right\|_2 \leqslant K\Omega_1\left(f;\frac{1}{\sqrt{m}}\right)+\frac{25}{4}(1+M)\,\Omega_1\left(f;\frac{1}{m}\right).$$

THEOREM 2.4. *Suppose that* $f\in C_2[0,\infty)$ *and* $L=\lim_{m\to\infty}\alpha_0(m)$ *exists. Then we have*

$$\lim_{m\to\infty} m\left[S_m^{M,1}(f;x)-f(x)\right]=\frac{x(1+x)}{2}f''(x)+(1+2x)(1-L)f'(x).$$

Proof. We have

$$m\left[S_m^{M,1}(f;x)-f(x)\right]=m\left[S_m(f;x)-f(x)\right]+m\left[S_m^{M,1}(f;x)-S_m(f;x)\right] \tag{2.14}$$

It is known

$$\lim_{m\to\infty} m\left[S_m(f;x)-f(x)\right]=\frac{x(1+x)}{2}f''(x). \tag{2.15}$$

Using relation (2.11) we get

$$\begin{aligned} & m\left[S_m^{M,1}(f;x)-S_m(f;x)\right] \\ &= \sum_{k=0}^{\infty}(1+2x)(1-\alpha_0(m))\,mP_{m+1,k}(x)\left[f\left(\frac{k+1}{m}\right)-f\left(\frac{k}{m}\right)\right] \\ &= (1+2x)(1-\alpha_0(m))\left(S_m(f;x)\right)'. \end{aligned} \tag{2.16}$$

Therefore

$$\lim_{m\to\infty} m\left[S_m^{M,1}(f;x)-S_m(f;x)\right]=(1+2x)(1-L)f'(x). \tag{2.17}$$

From (2.14), (2.15) and (2.17), the proof is completed.

To obtain a quantitative Voronovskaja-type result for Baskakov operators S_m given by (1.4), we need the subsequent results

$$S_m(1;x) = 1, \tag{2.18}$$

$$S_m(t-x;x) = 0, \tag{2.19}$$

$$S_m\left((t-x)^2;x\right) = \frac{x(1+x)}{m}, \tag{2.20}$$

$$S_m\left((t-x)^4;x\right)=\frac{x(1+x)}{m^3}+\frac{6x^2(1+x)^2}{m^3}+\frac{3x^2(1+x)^2}{m^2} \tag{2.21}$$

$$\begin{aligned}S_m\left((t-x)^6;x\right)=&\frac{x+31x^2+180x^3+390x^4+360x^5+120x^6}{m^5}\\&+\frac{25x^2+288x^3+667x^4+534x^5+130x^6}{m^4}\\&+\frac{15x^3+105x^4+105x^5+15x^6}{m^3},\end{aligned} \tag{2.22}$$

(see [10, p. 96]). □

THEOREM 2.5. *If $f\in B_1[0,\infty)\cap C_2[0,\infty)$, then we have for $x\in[0,\infty)$ that*

$$\begin{aligned}&\left|m\left[S_m(f;x)-f(x)\right]-\frac{x(1+x)}{2}f''(x)\right|\\&\leqslant K\left(1+x^2\right)^2\Omega_1\left(f;\frac{1}{\sqrt{m}}\right),\end{aligned}$$

where K is a positive constant.

Proof. Using the Taylor's expansion of f, we can write

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2}(t-x)^2+\frac{(t-x)^2}{2}\left(f''(\xi)-f''(x)\right), \tag{2.23}$$

where ξ is a number lying between t and x. Applying the operator S_m on both sides of (2.23) and taking (2.18)–(2.20) into account, we get

$$S_m(f;x)-f(x)-\frac{x(1+x)}{2m}f''(x)=S_m\left(\frac{(t-x)^2}{2}\left(f''(\xi)-f''(x)\right);x\right).$$

Using the weighted modulus of continuity given by (2.9), we can write

$$\begin{aligned}\left|f''(\xi)-f''(x)\right|&\leqslant\left[(1+2x+\xi)+(1+2x+\xi)\frac{|\xi-x|}{\delta}\right]\Omega_1(f;\delta)\\&\leqslant\left[(1+3x+t)+(1+3x+t)\frac{|t-x|}{\delta}\right]\Omega_1(f;\delta).\end{aligned}$$

Thus, from the last inequality we have

$$\begin{aligned}&\left|S_m\left(\frac{(t-x)^2}{2}\left(f''(\xi)-f''(x)\right);x\right)\right|\\&\leqslant S_m\left(\frac{(t-x)^2}{2}\left|f''(\xi)-f''(x)\right|;x\right)\\&=\frac{1}{2}\Omega_1(f;\delta)\left\{S_m\left((1+3x+t)(t-x)^2;x\right)\right.\\&\quad\left.+\frac{1}{\delta}S_m\left((1+3x+t)|t-x|^3;x\right)\right\}.\end{aligned} \tag{2.24}$$

Using the Cauchy-Schwarz inequality, (2.24) gives

$$\begin{aligned}&\left|S_m\left(\frac{(t-x)^2}{2}\left(f''(\xi)-f''(x)\right);x\right)\right|\\ &\leqslant \frac{1}{2}\Omega_1(f;\delta)\left\{\sqrt{S_m\left((1+3x+t)^2;x\right)}\sqrt{S_m\left((t-x)^4;x\right)}\right.\\ &\left.+\frac{1}{\delta}\sqrt{S_m\left((1+3x+t)^2;x\right)}\sqrt{S_m\left((t-x)^6;x\right)}\right\}.\end{aligned}$$

We have $S_m\left((1+3x+t)^2;x\right)\leqslant K_1\left(1+x^2\right),\ K_1>0.$ From (2.21) and (2.22) we get

$$\begin{aligned}S_m\left((t-x)^4;x\right) &\leqslant \frac{K_2}{m^2}\left(1+x^2\right)^2,\\ S_m\left((t-x)^6;x\right) &\leqslant \frac{K_3}{m^3}\left(1+x^2\right)^3,\end{aligned}$$

for $K_2,\ K_3>0.$ Therefore

$$\left|S_m\left(\frac{(t-x)^2}{2}\left(f''(\xi)-f''(x)\right);x\right)\right|\leqslant\frac{K}{2}\Omega_1(f;\delta)\frac{\left(1+x^2\right)^2}{m}\left(1+\frac{1}{\delta}m^{-1/2}\right),$$

for $K>0.$ If we take $\delta=m^{-1/2}$, we obtain

$$\left|S_m\left(\frac{(t-x)^2}{2}\left(f''(\xi)-f''(x)\right);x\right)\right|\leqslant K\Omega_1\left(f;\frac{1}{\sqrt{m}}\right)\frac{\left(1+x^2\right)^2}{m},$$

which completes the proof. □

THEOREM 2.6. *If $f\in B_1[0,\infty)\cap C_2[0,\infty)$ and $L_1=\lim_{m\to\infty}\alpha_1(m)$ exists. Then for $x\in[0,\infty)$ there holds*

$$\begin{aligned}&\left|m\left[S_m^{M,1}(f;x)-f(x)\right]-\frac{x(1+x)}{2}f''(x)-(1+2x)(1-L_1)f'(x)\right|\\ &\leqslant(1+2x)\left\{(1+x)\left|L_1-\alpha_0(m)\right|\left\|f'\right\|_1+K_2\left|1-L_1\right|\left(1+x^2\right)\left[\Omega_1\left(f';\frac{1}{\sqrt{m}}\right)+\frac{\|f'\|_1}{m}\right]\right\}\\ &+K_1\left(1+x^2\right)^2\Omega_1\left(f;\frac{1}{\sqrt{m}}\right),\end{aligned}$$

where $K_1,\ K_2>0.$

Proof. Denote

$$V_m^s:=m\left[S_m^{M,1}(f;x)-f(x)\right]-\frac{x(1+x)}{2}f''(x)-(1+2x)(1-L_1)f'(x).$$

The following inequality holds

$$\begin{aligned}|V_m^s| &\leqslant \left|m\left[S_m(f;x)-f(x)\right]-\frac{x(x+1)}{2}f''(x)\right|\\ &\quad+\left|m\left[S_m^{M,1}(f;x)-S_m(f;x)\right]-(1+2x)(1-L_1)f'(x)\right|\\ &:=A_1(x)+A_2(x).\end{aligned}$$

Using relation (2.16), it follows

$$\begin{aligned}A_2(x) &= \left|(1+2x)(1-\alpha_0(m))(S_m(f;x))'-(1+2x)(1-L_1)f'(x)\right|\\ &= \left|(1-\alpha_0(m))(1+2x)(S_m(f;x))'-(1+2x)(1-L_1)(S_m(f;x))'\right.\\ &\quad\left.+(1+2x)(1-L_1)(S_m(f;x))'-(1+2x)(1-L_1)f'(x)\right|\\ &\leqslant (1+2x)|L_1-\alpha_0(m)|\left|(S_m(f;x))'\right|+(1+2x)|1-L_1|\left|(S_m(f;x))'-f'(x)\right|\\ &= (1+2x)\left\{|L_1-\alpha_0(m)|\left|(S_m(f;x))'\right|+|1-L_1|\left|(S_m(f;x))'-f'(x)\right|\right\}.\end{aligned}\tag{2.25}$$

From Theorem 4 in [12], we have

$$\left\|(S_m(f;x))'-f'(x)\right\|_2\leqslant K\Omega_1\left(f';\frac{1}{\sqrt{m}}\right)+K_1\frac{\|f'\|_1}{m}.\tag{2.26}$$

On the other hand, we can write

$$(S_m(f;x))'=\sum_{k=0}^{\infty}mP_{m+1,k}(x)\left[f\left(\frac{k+1}{m}\right)-f\left(\frac{k}{m}\right)\right].$$

Therefore

$$\begin{aligned}\left|(S_m(f;x))'\right| &\leqslant \sum_{k=0}^{\infty}mP_{m+1,k}(x)\left|f\left(\frac{k+1}{m}\right)-f\left(\frac{k}{m}\right)\right|\\ &= \sum_{k=0}^{\infty}P_{m+1,k}(x)\left|f'\left(\zeta_{m,k}\right)\right|,\end{aligned}$$

where $\frac{k}{m}\leqslant\zeta_{m,k}\leqslant\frac{k+1}{m}$. So,

$$\left|(S_m(f;x))'\right|\leqslant(1+x)\left\|f'\right\|_1,\tag{2.27}$$

for $f'\in C^1[0,\infty)$. From (2.26) and (2.27), (2.25) yields

$$\begin{aligned}A_2(x)\leqslant(1+2x)\Big\{&(1+x)|L_1-\alpha_0(m)|\left\|f'\right\|_1\\ &+K_2|1-L_1|\left(1+x^2\right)\left[\Omega_1\left(f';\frac{1}{\sqrt{m}}\right)+\frac{\|f'\|_1}{m}\right]\Big\},\end{aligned}$$

for $K_2>0$. Using Theorem 2.5, we have the desired result. □

Acknowledgement. The work was supported by Lucian Blaga University of Sibiu (Knowledge Transfer Center) & Hasso Plattner Foundation research grants LBUS-HPI-ERG-2023-01.

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(Received September 2, 2023)

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