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REVERSE OSTROWSKI'S TYPE WEIGHTED INEQUALITIES FOR CONVEX FUNCTIONS ON LINEAR SPACES WITH APPLICATIONS

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Abstract. In this paper we provide several upper and lower bounds for the Ostrowski difference

$$\int_0^1 f((1-t)x+ty)\,w(t)\,dt - \left(\int_0^1 w(t)\,dt\right) f((1-\lambda)x+\lambda y),$$

where $f: C \to \mathbb{R}$ is a convex function, C is a convex subset of a vector space X and w is integrable and nonnegative a.e. on $[0,1]$. A perturbed version under some natural assumptions on the weight function w is also considered. These results are then employed to obtain several weighted integral inequalities for norms and semi-inner products. The particular case of inner product spaces is analyzed and refinements of the weighted integral midpoint inequality for norms are provided.

1. Introduction

As revealed by a simple search in MathSciNet database with the key words "Ostrowski" and "inequality" in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. Numerous extensions, generalizations in both the integral and discrete cases have been discovered. More general versions for weighted integrals, n-time differentiable functions, the corresponding versions on time scales and for vector valued functions or multiple integrals have been established as well.

In 1938, A. Ostrowski [11] proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$ in the case of differentiable functions on an open interval:

THEOREM 1.1. (Ostrowski, 1938 [11]) *Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leqslant \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b-a)$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

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In order to extend the above result to various classes of real-valued functions, the first author obtained in 2002 [3] the following version of the classical Ostrowski inequality in terms of convex functions:

Let $h:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in[a,b]$, one has the inequalities (see also [4] and [5])

$$\begin{aligned}&\frac{1}{2}\left[(b-x)^2h'_+(x)-(x-a)^2h'_-(x)\right]\\ &\leqslant \int_a^b h(t)\,dt-(b-a)h(x)\\ &\leqslant \frac{1}{2}\left[(b-x)^2h'_-(b)-(x-a)^2h'_+(a)\right],\end{aligned} \tag{1.1}$$

where h'_+ (resp. h'_-) denotes the right-hand derivative (resp. the left-hand derivative) of h. The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x=a$ or $x=b$. A particular case of interest is the following mid-point inequality

$$\begin{aligned}&\frac{1}{8}(b-a)^2\left[h'_+\left(\frac{a+b}{2}\right)-h'_-\left(\frac{a+b}{2}\right)\right]\\ &\leqslant \int_a^b h(t)\,dt-(b-a)h\left(\frac{a+b}{2}\right)\\ &\leqslant \frac{1}{8}(b-x)^2\left[h'_-(b)-h'_+(a)\right]\end{aligned}$$

in which the constant $\frac{1}{8}$ is the best possible in both sides.

In the same paper [3], the first author employed (1.1) to obtain the corresponding result for functions defined on convex subsets of vector spaces and focused his attention in providing some natural applications to norms and semi-inner products. The case of inner product spaces was also investigated. For a recent survey on Ostrowski inequality, see [7] and the references therein.

Motivated by the above results, it is then natural to explore the Ostrowski difference

$$\int_0^1 f((1-t)x+ty)w(t)\,dt-\left(\int_0^1 w(t)\,dt\right)f((1-\lambda)x+\lambda y)$$

in the case when f is a convex function on a convex subset C of a vector space X and w is integrable and nonnegative a.e. on $[0,1]$.

In this paper we provide several upper and lower bounds for the Ostrowski difference and its perturbed version under some natural assumptions on the integrable weight function w. As applications, the obtained inequalities are used to obtain several weighted integral inequalities for norms and semi-inner products. Moreover, the particular case of inner product spaces is analyzed and refinements of the weighted integral midpoint inequality for norms are derived.

The rest of the paper is organized as follows. In Section 2, we recall some basic notions and results related to Gâteaux lateral derivatives and lower and upper semi-inner products. We also recall some known inequalities that will be used throughout this paper. Our main results are stated and proved in Section 3. Some appllications to norms are provided in Section 4. Finally, the case of inner product spaces is studied in Section 5.

2. Preliminaries

In this section, for the reader's convenience, we recall some basic notions and results that will be used throughout this paper.

2.1. Gâteaux lateral derivatives

Let $f : X \to \mathbb{R}$, where X is a vector space. For all $x, y \in X$, the Gâteaux lateral derivatives of f at the point x over the direction y are defined by

$$(\nabla_+ f(x))(y) := \lim_{h \to 0^+} \frac{f(x+hy) - f(x)}{h}$$

and

$$(\nabla_- f(x))(y) := \lim_{h \to 0^-} \frac{f(x+hy) - f(x)}{h},$$

provided the above limits exist. Notice that, if f is convex on X, then the above limits exist.

Let $x, y \in X$ with $x \neq y$ and $f : [x,y] \to \mathbb{R}$, where $[x,y]$ is the segment generated by x and y, that is, $[x,y] := \{(1-t)x + ty : t \in [0,1]\}$. We consider the associated function $g(x,y) : [0,1] \to \mathbb{R}$ defined by

$$g(x,y)(t) := f((1-t)x + ty), \quad 0 \leqslant t \leqslant 1.$$

Notice that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$. Furthermore, the function $g(x,y)$ satisfies the following properties:

(i) $g'_{\pm}(x,y)(t) = (\nabla_{\pm} f[(1-t)x + ty])(y-x)$, $t \in (0,1)$;

(ii) $g'_+(x,y)(0) = (\nabla_+ f(x))(y-x)$;

(iii) $g'_-(x,y)(1) = (\nabla_- f(y))(y-x)$,

where $g'_+(x,y)(t)$ (resp. $g'_-(x,y)(t)$) denotes the right-hand derivative (resp. the left-hand derivative) of $g(x,y)$ at t.

For more properties related to Gâteaux lateral derivatives, see e.g. [6].

2.2. Lower and upper semi-inner products

Assume that $(X, \|\cdot\|)$ is a normed space. Since the function

$$f_0(x) := \frac{1}{2}\|x\|^2, \quad x \in X$$

is convex, then for all $x,y \in X$, the following limits exist:

$$\langle x,y\rangle_{s(i)} := \left(\nabla_{+(-)} f_0(y)\right)(x) = \lim_{t\to 0^{+(-)}} \frac{\|y+tx\|^2 - \|y\|^2}{2t}.$$

We call $\langle\cdot,\cdot\rangle_s$ (resp. $\langle\cdot,\cdot\rangle_i$) the lower semi-inner product (resp. the upper semi-inner product) associated to $\|\cdot\|$. For the sake of completeness, we list below some of the main properties of the above mappings that will be used in the sequel (for more details, see e.g. [2]).

Let $p,q \in \{s,i\}$ with $p \neq q$. We have the following properties:

(a) $\langle x,x\rangle_p = \|x\|^2$ for all $x \in X$;

(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta\langle x,y\rangle_p$ for all $\alpha,\beta \geqslant 0$ and $x,y \in X$;

(aaa) $|\langle x,y\rangle_p| \leqslant \|x\|\|y\|$ for all $x,y \in X$;

(b) $\langle \alpha x + y, x\rangle_p = \alpha\langle x,x\rangle_p + \langle y,x\rangle_p$ for all $x,y \in X$ and $\alpha \in \mathbb{R}$;

(bb) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all $x,y \in X$;

(bbb) $\langle x+y,z\rangle_p \leqslant \|x\|\|z\| + \langle y,z\rangle_p$ for all $x,y,z \in X$;

(c) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (resp. superadditive) in the first variable for $p = s$ (resp. $p = i$);

(cc) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0 \in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s = \langle y,x_0\rangle_i$ for all $y \in X$; in general $\langle y,x\rangle_i \leqslant \langle y,x\rangle_s$ for all $x,y \in X$;

(ccc) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i = \langle y,x\rangle = \langle y,x\rangle_s$ for all $x,\ y \in X$.

For all $1 \leqslant p < \infty$, the function

$$f_p(x) := \|x\|^p, \quad x \in X \tag{2.1}$$

is also convex. Therefore, the following limits, which are related to the upper (lower) semi-inner products

$$\left(\nabla_{+(-)} f_p(y)\right)(x) = \lim_{t\to 0^{+(-)}} \frac{\|y+tx\|^p - \|y\|^p}{t} = p\|y\|^{p-2}\langle x,y\rangle_{s(i)} \tag{2.2}$$

exist for all $x,y \in X$ whenever $p \geqslant 2$; otherwise, they exist for any $x \in X$ and $y \in X\setminus\{0\}$. In particular, if $p = 1$, then the following limits

$$\left(\nabla_{+(-)} f_1(y)\right)(x) = \lim_{t\to 0^{+(-)}} \frac{\|y+tx\| - \|y\|}{t} = \left\langle x, \frac{y}{\|y\|}\right\rangle_{s(i)}$$

exist for any $x \in X$ and $y \in X\setminus\{0\}$.

2.3. Useful inequalities

We recall below the Chebyshev inequality (see [1, 10]) that will be used later.

LEMMA 2.1. *Let $c,d \in \mathbb{R}$ with $c<d$ and $g,h \in L^1(c,d)$, both non-decreasing or both non-increasing. Then*

$$\int_c^d g(t)h(t)\,dt \geqslant \frac{1}{d-c}\int_c^d g(t)\,dt \int_c^d h(t)\,dt.$$

If one of the functions g or h is non-decreasing and the other is non-increasing, then the above inequality reverses.

We also recall the following Grüss inequality (see [9, 10]).

LEMMA 2.2. *Let $c,d \in \mathbb{R}$ with $c<d$ and $g,h:[c,d]\to\mathbb{R}$ be such that $m \leqslant g \leqslant M$ and $n \leqslant h \leqslant N$ on $[c,d]$, where m,M,n,N are constants. Then*

$$\left|\frac{1}{d-c}\int_c^d g(t)h(t)\,dt - \frac{1}{d-c}\int_c^d g(t)\,dt\,\frac{1}{d-c}\int_c^d h(t)\,dt\right| \leqslant \frac{1}{4}(M-m)(N-n).$$

3. The results

We start with the following result that is of interest in itself as well.

PROPOSITION 3.1. *Let $\varphi:[0,1]\to\mathbb{R}$ be a convex function and $\lambda\in(0,1)$. Let $w \in L^1(0,1)$ be such that*

$$\begin{aligned}
&\int_0^t w(s)\,ds \geqslant 0 \quad \textit{for almost every } t\in[0,\lambda],\\
&\int_t^1 w(s)\,ds \geqslant 0 \quad \textit{for almost every } t\in[\lambda,1].
\end{aligned} \tag{3.1}$$

Then, we have

$$\begin{aligned}
&\varphi'_+(\lambda)\int_\lambda^1 (t-\lambda)w(t)\,dt + \varphi'_-(\lambda)\int_0^\lambda (t-\lambda)w(t)\,dt\\
&\leqslant \int_0^1 w(t)\varphi(t)\,dt - \left(\int_0^1 w(t)\,dt\right)\varphi(\lambda)\\
&\leqslant \varphi'_-(1)\int_\lambda^1 (t-\lambda)w(t)\,dt - \varphi'_+(0)\int_0^\lambda (\lambda-t)w(t)\,dt,
\end{aligned} \tag{3.2}$$

where φ'_+ (resp. φ'_-) denotes the right-hand (resp. the left-hand) derivative of φ.

Proof. Using integration by parts formula, we obtain

$$\begin{aligned}\int_0^\lambda \left(\int_0^t w(s)\,ds\right)\varphi'(t)\,dt &= \left(\int_0^t w(s)\,ds\right)\varphi(t)\Big|_{t=0}^{\lambda} - \int_0^\lambda w(t)\varphi(t)\,dt \\ &= \left(\int_0^\lambda w(t)\,dt\right)\varphi(\lambda) - \int_0^\lambda w(t)\varphi(t)\,dt\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\int_\lambda^1 \left(\int_t^1 w(s)\,ds\right)\varphi'(t)\,dt &= \left(\int_t^1 w(s)\,ds\right)\varphi(t)\Big|_{t=\lambda}^{1} + \int_\lambda^1 w(t)\varphi(t)\,dt \\ &= -\left(\int_\lambda^1 w(t)\,dt\right)\varphi(\lambda) + \int_\lambda^1 w(t)\varphi(t)\,dt,\end{aligned}$$

that is,

$$-\int_\lambda^1 \left(\int_t^1 w(s)\,ds\right)\varphi'(t)\,dt = \left(\int_\lambda^1 w(t)\,dt\right)\varphi(\lambda) - \int_\lambda^1 w(t)\varphi(t)\,dt. \tag{3.4}$$

Summing (3.3)and (3.4), we get

$$\begin{aligned}&\left(\int_0^1 w(s)\,ds\right)\varphi(\lambda) - \int_0^1 w(t)\varphi(t)\,dt \\ &= \int_0^\lambda \left(\int_0^t w(s)\,ds\right)\varphi'(t)\,dt - \int_\lambda^1 \left(\int_t^1 w(s)\,ds\right)\varphi'(t)\,dt.\end{aligned} \tag{3.5}$$

On the other hand, since φ is convex, then for almost every $t \in (0,\lambda)$, we have

$$\varphi'_+(0) \leqslant \varphi'(t) \leqslant \varphi'_-(\lambda).$$

Multiplying the above double inequality by $\int_0^t w(s)\,ds \geqslant 0$ and integrate, we get

$$\begin{aligned}\varphi'_+(0)\int_0^\lambda \left(\int_0^t w(s)\,ds\right)dt &\leqslant \int_0^\lambda \left(\int_0^t w)(s)\,ds\right)\varphi'(t)\,dt \\ &\leqslant \varphi'_-(\lambda)\int_0^\lambda \left(\int_0^t w(s)\,ds\right)dt.\end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}\int_0^\lambda \left(\int_0^t w(s)\,ds\right)dt &= \left(\int_0^t w(s)\,ds\right)t\Big|_{t=0}^{\lambda} - \int_0^\lambda t w(t)\,dt \\ &= \int_0^\lambda (\lambda - t)w(t)\,dt,\end{aligned}$$

then by (3.6), we get

$$\begin{aligned} \varphi'_+(0)\int_0^{\lambda}(\lambda-t)w(t),dt &\leqslant \int_0^{\lambda}\left(\int_0^t w(s)\,ds\right)\varphi'(t)\,dt \\ &\leqslant \varphi'_-(\lambda)\int_0^{\lambda}(\lambda-t)w(t)\,dt. \end{aligned} \tag{3.7}$$

Similarly, by the convexity of φ, we have for almost $t\in(\lambda,1)$,

$$\varphi'_+(\lambda)\leqslant\varphi'(t)\leqslant\varphi'_-(1),$$

that is,

$$-\varphi'_-(1)\leqslant-\varphi'(t)\leqslant-\varphi'_+(\lambda).$$

Multiplying the above double inequality by $\int_t^1 w(s)\,ds\geqslant 0$ and integrate, we obtain

$$\begin{aligned} -\varphi'_-(1)\int_{\lambda}^1\left(\int_t^1 w(s)\,ds\right)dt &\leqslant -\int_{\lambda}^1\left(\int_t^1 w(s)\,ds\right)\varphi'(t)\,dt \\ &\leqslant -\varphi'_+(\lambda)\int_{\lambda}^1\left(\int_t^1 w(s)\,ds\right)dt. \end{aligned} \tag{3.8}$$

Since

$$\begin{aligned} \int_{\lambda}^1\left(\int_t^1 w(s)\,ds\right)dt &= \left(\int_t^1 w(s)\,ds\right)t\Big|_{t=\lambda}^1+\int_{\lambda}^1 w(t)t\,dt \\ &= \int_{\lambda}^1(t-\lambda)w(t)\,dt, \end{aligned}$$

then by (3.8), we obtain

$$\begin{aligned} -\varphi'_-(1)\int_{\lambda}^1(t-\lambda)w(t)\,dt &\leqslant -\int_{\lambda}^1\left(\int_t^1 w(s`0\,ds\right)\varphi'(t)\,dt \\ &\leqslant -\varphi'_+(\lambda)\int_{\lambda}^1(t-\lambda)w(t)\,dt. \end{aligned} \tag{3.9}$$

Summing (3.7) and (3.9), we derive

$$\begin{aligned} &\varphi'_+(0)\int_0^{\lambda}(\lambda-t)w(t)\,dt-\varphi'_-(1)\int_{\lambda}^1(t-\lambda)w(t)\,dt \\ &\leqslant \int_0^{\lambda}\left(\int_0^t w(s)\,ds\right)\varphi'(t)\,dt-\int_{\lambda}^1\left(\int_t^1 w(s)\,ds\right)\varphi'(t)\,dt \\ &\leqslant \varphi'_-(\lambda)\int_0^{\lambda}(\lambda-t)w(t)\,dt-\varphi'_+(\lambda)\int_{\lambda}^1(t-\lambda)w(t)\,dt \end{aligned}$$

and by (3.5),

$$\begin{aligned}
&\varphi_+'(0)\int_0^\lambda (\lambda - t)w(t)\,dt - \varphi_-'(1)\int_\lambda^1 (t-\lambda)w(t)\,dt \\
&\leqslant \left(\int_0^1 w(t)\,dt\right)\varphi(\lambda) - \int_0^1 w(t)\varphi(t)\,dt \\
&\leqslant \varphi_-'(\lambda)\int_0^\lambda (\lambda - t)w(t)\,dt - \varphi_+'(\lambda)\int_\lambda^1 (t-\lambda)w(t)\,dt,
\end{aligned}$$

which is equivalent to (3.2). □

COROLLARY 3.2. *Let $\varphi : [0,1] \to \mathbb{R}$ be a convex function and $w \in L^1(0,1)$. Assume that*

$$\begin{aligned}
&\int_0^t w(s)\,ds \geqslant 0 \quad \textit{for almost every } t \in \left[0, \frac{1}{2}\right], \\
&\int_t^1 w(s)\,ds \geqslant 0 \quad \textit{for almost every } t \in \left[\frac{1}{2}, 1\right].
\end{aligned} \tag{3.10}$$

Then

$$\begin{aligned}
&\varphi_+'\left(\frac{1}{2}\right)\int_{\frac{1}{2}}^1 \left(t - \frac{1}{2}\right)w(t)\,dt + \varphi_-'\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}} \left(t - \frac{1}{2}\right)w(t)\,dt \\
&\leqslant \int_0^1 w(t)\varphi(t)\,dt - \left(\int_0^1 w(t)\,dt\right)\varphi\left(\frac{1}{2}\right) \\
&\leqslant \varphi_-'(1)\int_{\frac{1}{2}}^1 \left(t - \frac{1}{2}\right)w(t)\,dt - \varphi_+'(0)\int_0^{\frac{1}{2}} \left(\frac{1}{2} - t\right)w(t)\,dt.
\end{aligned}$$

Proof. Taking $\lambda = \frac{1}{2}$ in Proposition 3.1, we obtain the desired inequalities. □

Observe that for $w \equiv 1$, we have

$$\int_\lambda^1 (t-\lambda)w(t)\,dt = \int_\lambda^1 (t-\lambda)\,dt = \frac{1}{2}(1-\lambda)^2$$

and

$$\int_0^\lambda (t-\lambda)w(t)\,dt = \int_0^\lambda (t-\lambda)\,dt = -\frac{1}{2}\lambda^2.$$

Then, taking $w \equiv 1$ in (3.2), we get the following inequalities which are previously derived in [3], see also [4,5].

COROLLARY 3.3. *Let $\varphi : [0,1] \to \mathbb{R}$ be a convex function and $\lambda \in (0,1)$. Then*

$$\begin{aligned}
\frac{1}{2}\left[\varphi_+'(\lambda)(1-\lambda)^2 - \varphi_-'(\lambda)\lambda^2\right] &\leqslant \int_0^1 \varphi(t)\,dt - \varphi(\lambda) \\
&\leqslant \frac{1}{2}\left[\varphi_-'(1)(1-\lambda)^2 - \varphi_+'(0)\lambda^2\right].
\end{aligned} \tag{3.11}$$

In particular, for $\lambda = \frac{1}{2}$, (3.11) *reduces to*

$$\frac{1}{8}\left[\varphi'_+\left(\frac{1}{2}\right) - \varphi'_-\left(\frac{1}{2}\right)\right] \leqslant \int_0^1 \varphi(t)\,dt - \varphi\left(\frac{1}{2}\right) \leqslant \frac{1}{8}\left[\varphi'_-(1) - \varphi'_+(0)\right]. \qquad (3.12)$$

Under the assumptions of Proposition 3.1, if φ is differentiable at $\lambda \in (0,1)$, then

$$\varphi'_+(\lambda) = \varphi'_-(\lambda) = \varphi'(\lambda).$$

Then, form (3.2), we deduce the following result.

COROLLARY 3.4. *Let* $\varphi : [0,1] \to \mathbb{R}$ *be a convex function and* $\lambda \in (0,1)$ *be such that* φ *is differentiable at* λ. *Let* $w \in L^1(0,1)$ *satisfies* (3.1). *Then*

$$\varphi'(\lambda)\int_0^1 (t-\lambda)w(t)\,dt \leqslant \int_0^1 w(t)\varphi(t)\,dt - \left(\int_0^1 w(t)\,dt\right)\varphi(\lambda).$$

In particular, if w *satisfies* (3.10) *and* φ *is differentiable at* $\frac{1}{2}$, *then*

$$\varphi'\left(\frac{1}{2}\right)\int_0^1 \left(t-\frac{1}{2}\right)w(t)\,dt \leqslant \int_0^1 w(t)\varphi(t)\,dt - \left(\int_0^1 w(t)\,dt\right)\varphi\left(\frac{1}{2}\right). \qquad (3.13)$$

Recall that if $\varphi : [0,1] \to \mathbb{R}$ is convex, $w \in L^1(0,1)$ is nonnegative a.e. on $[0,1]$ $(w \not\equiv 0)$ and symmetric on $[0,1]$, namely $y = w(\lambda)$ is a symmetric curve with respect to the straight line containing the point $\left(\frac{1}{2},0\right)$ and is normal to the λ-axis, then the following result is known in the literature as Fejér's inequality [8]:

$$\varphi\left(\frac{1}{2}\right) \leqslant \frac{\int_0^1 \varphi(t)w(t)\,dt}{\int_0^1 w(t)\,dt} \leqslant \frac{\varphi(0)+\varphi(1)}{2}. \qquad (3.14)$$

We have the following alternative and refinement of the first Fejér's inequality (3.14).

COROLLARY 3.5. *Let* $\varphi : [0,1] \to \mathbb{R}$ *be a convex function such that* φ *is differentiable at* $\frac{1}{2}$. *Let* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$ $(w \not\equiv 0)$. *If either* w *is non-decreasing on* $[0,1]$ *and* $\varphi'\left(\frac{1}{2}\right) \geqslant 0$ *or* w *is non-increasing on* $[0,1]$ *and* $\varphi'\left(\frac{1}{2}\right) \leqslant 0$, *then*

$$0 \leqslant \frac{\varphi'\left(\frac{1}{2}\right)}{\int_0^1 w(t)\,dt}\int_0^1 \left(t-\frac{1}{2}\right)w(t)\,dt \leqslant \frac{\int_0^1 w(t)\varphi(t)\,dt}{\int_0^1 w(t)\,dt} - \varphi\left(\frac{1}{2}\right). \qquad (3.15)$$

Proof. If w is non-decreasing on $[0,1]$ and since $g(t) = t - \frac{1}{2}$ is non-decreasing, then by the Chebyshev inequality (see Lemma 2.1), we get

$$\int_0^1 \left(t-\frac{1}{2}\right)w(t)\,dt \geqslant \int_0^1 \left(t-\frac{1}{2}\right)\int_0^1 w(t)\,dt \geqslant 0.$$

Since $\varphi'\left(\frac{1}{2}\right) \geqslant 0$, then the first inequality in (3.15) is valid. The second inequality follows from (3.13).

The case w is non-increasing on $[0,1]$ and $\varphi'\left(\frac{1}{2}\right) \leqslant 0$ goes in a similar way. □

Our first main result is stated in the following theorem.

THEOREM 3.6. *Let C be a convex subset of a vector space X and $f : C \to \mathbb{R}$ be a convex function. Let $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x, y \in C$ and $x \neq y$, then for all $\lambda \in [0,1]$,*

$$\begin{aligned}
&(\nabla_+ f\,[(1-\lambda)x+\lambda y])\,(y-x)\int_\lambda^1 (t-\lambda)w(t)\,dt \\
&\quad + (\nabla_- f\,[(1-\lambda)x+\lambda y])\,(y-x)\int_0^\lambda (t-\lambda)w(t)\,dt \\
&\leqslant \int_0^1 w(t) f\,[(1-t)x+ty]\,dt - \left(\int_0^1 w(t)\,dt\right) f\,[(1-\lambda)x+\lambda y] \\
&\leqslant (\nabla_- f(y))\,(y-x)\int_\lambda^1 (t-\lambda)w(t)\,dt - (\nabla_+ f(x))\,(y-x)\int_0^\lambda (\lambda-t)w(t)\,dt.
\end{aligned} \tag{3.16}$$

Proof. If $x, y \in C$ with $x \neq y$, then by applying Proposition 3.1 to the function

$$\varphi(t) := g(x,y)(t) = f\left((1-t)x+ty\right), \quad t \in [0,1], \tag{3.17}$$

we get (3.16). □

Taking $\lambda = \frac{1}{2}$ in Theorem 3.6, we obtain the following result.

COROLLARY 3.7. *et C be a convex subset of a vector space X and $f : C \to \mathbb{R}$ be a convex function. Let $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x, y \in C$ and $x \neq y$, then*

$$\begin{aligned}
&\left(\nabla_+ f\left(\frac{x+y}{2}\right)\right)(y-x)\int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right) w(t)\,dt \\
&\quad + \left(\nabla_- f\left(\frac{x+y}{2}\right)\right)(y-x)\int_0^{\frac{1}{2}} \left(t-\frac{1}{2}\right) w(t)\,dt \\
&\leqslant \int_0^1 w(t) f\,[(1-t)x+ty]\,dt - \left(\int_0^1 w(t)\,dt\right) f\left(\frac{x+y}{2}\right) \\
&\leqslant (\nabla_- f(y))\,(y-x)\int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right) w(t)\,dt - (\nabla_+ f(x))\,(y-x)\int_0^{\frac{1}{2}} \left(\frac{1}{2}-t\right) w(t)\,dt.
\end{aligned}$$

If the function f is Gâteaux differentiable at the point $(1-\lambda)x+\lambda y$ for some $\lambda \in (0,1)$ and $x, y \in C$ with $x \neq y$, then

$$\begin{aligned}
(\nabla_+ f\,[(1-\lambda)x+\lambda y])\,(y-x) &= (\nabla_- f\,[(1-\lambda)x+\lambda y])\,(y-x) \\
&= (\nabla f\,[(1-\lambda)x+\lambda y])\,(y-x).
\end{aligned}$$

Then, from Theorem 3.6, we deduce the following result.

COROLLARY 3.8. *Let C be a convex subset of a vector space X and $f : C \to \mathbb{R}$ be a convex function. Assume that f is Gâteaux differentiable at the point $(1-\lambda)x+\lambda y$ for some $\lambda \in (0,1)$ and $x, y \in C$ with $x \neq y$. Let $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. Then*

$$\begin{aligned}&(\nabla f[(1-\lambda)x+\lambda y])(y-x)\int_0^1 (t-\lambda)w(t)\,dt\\&\leqslant \int_0^1 w(t)f[(1-t)x+ty]\,dt - \left(\int_0^1 w(t)\,dt\right) f[(1-\lambda)x+\lambda y].\end{aligned}$$

In particular, for $\lambda = \frac{1}{2}$, we deduce from Corollary 3.8 the following result.

COROLLARY 3.9. *Let C be a convex subset of a vector space X and $f : C \to \mathbb{R}$ be a convex function. Assume that f is Gâteaux differentiable at the point $\frac{x+y}{2}$ for some $x, y \in C$ with $x \neq y$. Let $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. Then*

$$\begin{aligned}&\left(\nabla f\left(\frac{x+y}{2}\right)\right)(y-x)\int_0^1 \left(t-\frac{1}{2}\right) w(t)\,dt\\&\leqslant \int_0^1 w(t)f[(1-t)x+ty]\,dt - \left(\int_0^1 w(t)\,dt\right) f\left(\frac{x+y}{2}\right).\end{aligned}$$

REMARK 3.10. Notice that by the Chebyshev inequality (see Lemma 2.1), we have the following Fejér's type inequality

$$\begin{aligned}0 &\leqslant \left(\nabla f\left(\frac{x+y}{2}\right)\right)(y-x)\frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left(t-\frac{1}{2}\right) w(t)\,dt\\&\leqslant \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)f[(1-t)x+ty]\,dt - f\left(\frac{x+y}{2}\right)\end{aligned} \tag{3.18}$$

provided that either w $(w \not\equiv 0)$ is non-decreasing on $[0,1]$ and $\left(\triangledown f\left(\frac{x+y}{2}\right)\right)(y-x) \geqslant 0$ or w $(w \not\equiv 0)$ is non-increasing on $[0,1]$ and $\left(\triangledown f\left(\frac{x+y}{2}\right)\right) \leqslant 0$.

PROPOSITION 3.11. *Let $\varphi : [0,1] \to \mathbb{R}$ be a convex function and $\lambda \in (0,1)$. Let $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. Then*

$$\begin{aligned}0 &\leqslant \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (t-\lambda)w(t)\,dt - \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda (\lambda-t)w(t)\,dt\\&\quad - \left(\int_0^1 w(t)\varphi(t)\,dt - \varphi(\lambda)\int_0^1 w(t)\,dt\right)\\&\leqslant \frac{1}{4}\left[\lambda(\varphi'_-(\lambda)-\varphi'_+(0))\int_0^\lambda w(t)\,dt + (1-\lambda)(\varphi'_-(1)-\varphi'_+(\lambda))\int_\lambda^1 w(t)\,dt\right].\end{aligned} \tag{3.19}$$

Proof. By the Chebyshev inequality for two non-decreasing functions (see Lemma 2.1), we have

$$\int_0^\lambda \left(\int_0^t w(s)\,ds\right) \varphi'(t)\,dt \geqslant \frac{1}{\lambda}\int_0^\lambda \left(\int_0^t w(s)\,ds\right) dt \int_0^\lambda \varphi'(t)\,dt \tag{3.20}$$

and

$$\int_\lambda^1 \left(-\int_t^1 w(s)\,ds\right) \varphi'(t)\,dt \geqslant \frac{1}{1-\lambda}\int_\lambda^1 \left(-\int_t^1 w(s)\,ds\right) dt \int_\lambda^1 \varphi'(t)\,dt. \tag{3.21}$$

On the other hand, from the proof of Proposition 3.1, we have

$$\int_0^\lambda \left(\int_0^t w(s)\,ds\right) dt = \int_0^\lambda (\lambda - t)w(t)\,dt$$

and

$$\int_\lambda^1 \left(-\int_t^1 w(s)\,ds\right) dt = \int_\lambda^1 (\lambda - t)w(t)\,dt.$$

Then, by (3.20) and (3.21), we get

$$\int_0^\lambda \left(\int_0^t w(s)\,ds\right) \varphi'(t)\,dt \geqslant \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda (\lambda - t)w(t)\,dt$$

and

$$\int_\lambda^1 \left(-\int_t^1 w(s)\,ds\right) \varphi'(t)\,dt \geqslant \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (\lambda - t)w(t)\,dt.$$

Now, if we sum the above two inequalities, we get

$$\begin{aligned}
&\int_0^\lambda \left(\int_0^t w(s)\,ds\right) \varphi'(t)\,dt - \int_\lambda^1 \left(\int_t^1 w(s)\,ds\right) \varphi'(t)\,dt \\
&\geqslant \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda (\lambda - t)w(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (t-\lambda)w(t)\,dt,
\end{aligned}$$

which implies by (3.5) that

$$\begin{aligned}
&\left(\int_0^1 w(t)\,dt\right) \varphi(\lambda) - \int_0^1 w(t)\varphi(t)\,dt \\
&\geqslant \frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda (\lambda - t)w(t)\,dt - \frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1 (t-\lambda)w(t)\,dt,
\end{aligned}$$

which is equivalent to the first inequality in (3.19).

On the other hand, observe that for almost everywhere $t \in (0,\lambda)$, we have

$$0 \leqslant \int_0^t w(s)\,ds \leqslant \int_0^\lambda w(s)\,ds$$

and

$$\varphi'_+(0) \leqslant \varphi'(t) \leqslant \varphi'_-(\lambda).$$

Then, by the Grüss inequality (see Lemma 2.2), we have

$$\begin{aligned} &\frac{1}{4}\lambda\left(\varphi'_-(\lambda)-\varphi'_+(0)\right)\int_0^\lambda w(t)\,dt+\frac{1}{\lambda}\int_0^\lambda\left(\int_0^t w(s)\,ds\right)dt\int_0^\lambda \varphi'(t)\,dt \\ &\geqslant \int_0^\lambda\left(\int_0^t w(s)\,ds\right)\varphi'(t)\,dt. \end{aligned} \tag{3.22}$$

Similarly, for almost everywhere $t\in(\lambda,1)$, we have

$$-\int_\lambda^1 w(s)\,ds \leqslant -\int_t^1 w(s)\,ds \leqslant 0$$

and

$$\varphi'_+(\lambda) \leqslant \varphi'(t) \leqslant \varphi'_-(1).$$

Using again the Grüss inequality, we get

$$\begin{aligned} &\frac{1}{4}(1-\lambda)\left(\varphi'_-(1)-\varphi'_+(\lambda)\right)\int_\lambda^1 w(t)\,dt+\frac{1}{1-\lambda}\int_\lambda^1\left(-\int_t^1 w(s)\,ds\right)dt\int_\lambda^1 \varphi'(t)\,dt \\ &\geqslant \int_\lambda^1\left(-\int_t^1 w(s)\,ds\right)\varphi'(t)\,dt. \end{aligned} \tag{3.23}$$

Summing (3.22) and (3.23), we obtain

$$\begin{aligned} &\frac{1}{4}\lambda\left(\varphi'_-(\lambda)-\varphi'_+(0)\right)\int_0^\lambda w(t)\,dt+\frac{1}{4}(1-\lambda)\left(\varphi'_-(1)-\varphi'_+(\lambda)\right)\int_\lambda^1 w(t)\,dt \\ &\quad+\frac{1}{\lambda}\int_0^\lambda\left(\int_0^t w(s)\,ds\right)dt\int_0^\lambda \varphi'(t)\,dt-\frac{1}{1-\lambda}\int_\lambda^1\left(\int_t^1 w(s)\,ds\right)dt\int_\lambda^1 \varphi'(t)\,dt \\ &\geqslant \int_0^\lambda\left(\int_0^t w(s)\,ds\right)\varphi'(t)\,dt-\int_\lambda^1\left(\int_t^1 w(s)\,ds\right)\varphi'(t)\,dt, \end{aligned}$$

that is,

$$\begin{aligned} &\frac{1}{4}\lambda\left(\varphi'_-(\lambda)-\varphi'_+(0)\right)\int_0^\lambda w(t)\,dt+\frac{1}{4}(1-\lambda)\left(\varphi'_-(1)-\varphi'_+(\lambda)\right)\int_\lambda^1 w(t)\,dt \\ &\quad+\frac{\varphi(\lambda)-\varphi(0)}{\lambda}\int_0^\lambda(\lambda-t)w(t)\,dt-\frac{\varphi(1)-\varphi(\lambda)}{1-\lambda}\int_\lambda^1(t-\lambda)w(t)\,dt \\ &\geqslant \left(\int_0^1 w(t)\,dt\right)\varphi(\lambda)-\int_0^1 w(t)\varphi(t)\,dt, \end{aligned}$$

which is equivalent to the second inequality in (3.19). □

Taking $\lambda=\frac{1}{2}$ in Proposition 3.11, we deduce the following result.

COROLLARY 3.12. *Let $\varphi:[0,1]\to\mathbb{R}$ be a convex function and $w\in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. Then*

$$\begin{aligned}
0 &\leqslant 2\left[\left(\varphi(1)-\varphi\left(\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right)w(t)\,dt-\left(\varphi\left(\frac{1}{2}\right)-\varphi(0)\right)\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right]\\
&\quad-\left(\int_0^1 w(t)\varphi(t)\,dt-\varphi\left(\frac{1}{2}\right)\int_0^1 w(t)\,dt\right)\\
&\leqslant \frac{1}{8}\left[\left(\varphi'_-\left(\frac{1}{2}\right)-\varphi'_+(0)\right)\int_0^{\frac{1}{2}} w(t)\,dt+\left(\varphi'_-(1)-\varphi'_+\left(\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{1} w(t)\,dt\right].
\end{aligned}$$

Observe that for $w\equiv 1$, we have

$$\int_\lambda^1 (t-\lambda)w(t)\,dt=\frac{1}{2}(1-\lambda)^2,\ \int_0^\lambda (\lambda-t)w(t)\,dt=\frac{1}{2}\lambda^2.$$

Then, by (3.19), we obtain the following result.

COROLLARY 3.13. *Let $\varphi:[0,1]\to\mathbb{R}$ be a convex function and $\lambda\in(0,1)$. Then*

$$\begin{aligned}
0 &\leqslant \frac{1}{2}\left[(1-\lambda)\varphi(1)+\lambda\varphi(0)+\varphi(\lambda)\right]-\int_0^1\varphi(t)\,dt\\
&\leqslant \frac{1}{4}\left[\lambda^2\left(\varphi'_-(\lambda)-\varphi'_+(0)\right)+(1-\lambda)^2\left(\varphi'_-(1)-\varphi'_+(\lambda)\right)\right].
\end{aligned}$$

In particular, for $\lambda=\frac{1}{2}$, we deduce from Corollary 3.13 the following result.

COROLLARY 3.14. *Let $\varphi:[0,1]\to\mathbb{R}$ be a convex function. Then*

$$\begin{aligned}
0 &\leqslant \frac{1}{2}\left(\frac{\varphi(1)+\varphi(0)}{2}+\varphi\left(\frac{1}{2}\right)\right)-\int_0^1\varphi(t)\,dt\\
&\leqslant \frac{1}{16}\left[\varphi'_-(1)-\varphi'_+(0)-\left(\varphi'_+\left(\frac{1}{2}\right)-\varphi'_-\left(\frac{1}{2}\right)\right)\right]\\
&\leqslant \frac{1}{16}\left[\varphi'_-(1)-\varphi'_+(0)\right].
\end{aligned}$$

By applying Proposition 3.11 to the function φ defined by (3.17), we get the following result.

THEOREM 3.15. *Let C be a convex subset of a vector space X and $f:C\to\mathbb{R}$ be a convex function. Let $w\in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y\in C$ and*

$x \neq y$, *then for all* $\lambda \in (0,1)$,

$$\begin{aligned}
0 \leqslant\ & \frac{f(y) - f((1-\lambda)x+\lambda y)}{1-\lambda} \int_{\lambda}^{1} (t-\lambda) w(t)\, dt \\
& - \frac{f((1-\lambda)x+\lambda y) - f(x)}{\lambda} \int_{0}^{\lambda} (\lambda - t) w(t)\, dt \\
& - \left(\int_{0}^{1} w(t) f((1-t)x+ty)\, dt - f((1-\lambda)x+\lambda y) \int_{0}^{1} w(t)\, dt \right) \\
\leqslant\ & \frac{1}{4}\lambda \big((\nabla_{-} f[(1-\lambda)x+\lambda y])(y-x) - (\nabla_{+} f(x))(y-x) \big) \int_{0}^{\lambda} w(t)\, dt \\
& + \frac{1}{4}(1-\lambda)\big((\nabla_{-} f(y))(y-x) - (\nabla_{+} f[(1-\lambda)x+\lambda y])(y-x) \big) \int_{\lambda}^{1} w(t)\, dt.
\end{aligned} \tag{3.24}$$

In particular, for $\lambda = \frac{1}{2}$, we have the following result.

COROLLARY 3.16. *Let* C *be a convex subset of a vector space* X *and* $f : C \to \mathbb{R}$ *be a convex function. Let* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$. *If* $x, y \in C$ *and* $x \neq y$, *then*

$$\begin{aligned}
0 \leqslant\ & 2\left(f(y) - f\left(\frac{x+y}{2}\right) \right) \int_{\frac{1}{2}}^{1} \left(t - \frac{1}{2} \right) w(t)\, dt \\
& - 2\left(f\left(\frac{x+y}{2}\right) - f(x) \right) \int_{0}^{\frac{1}{2}} \left(\frac{1}{2} - t \right) w(t)\, dt \\
& - \left(\int_{0}^{1} w(t) f((1-t)x+ty)\, dt - f\left(\frac{x+y}{2}\right) \int_{0}^{1} w(t)\, dt \right) \\
\leqslant\ & \frac{1}{8}\left(\left(\nabla_{-} f\left(\frac{x+y}{2}\right) \right)(y-x) - (\nabla_{+} f(x))(y-x) \right) \int_{0}^{\frac{1}{2}} w(t)\, dt \\
& + \frac{1}{8}\left((\nabla_{-} f(y))(y-x) - \left(\nabla_{+} f\left(\frac{x+y}{2}\right) \right)(y-x) \right) \int_{\frac{1}{2}}^{1} w(t)\, dt.
\end{aligned} \tag{3.25}$$

Taking $w \equiv 1$ in (3.24), we get the following result.

COROLLARY 3.17. *Let* C *be a convex subset of a vector space* X *and* $f : C \to \mathbb{R}$ *be a convex function. If* $x, y \in C$ *and* $x \neq y$, *then for all* $\lambda \in (0,1)$,

$$\begin{aligned}
0 \leqslant\ & \frac{1}{2}[(1-\lambda) f(y) + \lambda f(x) + f((1-\lambda)x+\lambda y)] - \int_{0}^{1} f((1-t)x+ty)\, dt \\
\leqslant\ & \frac{1}{4}\big[\lambda^2 ((\nabla_{-} f[(1-\lambda)x+\lambda y])(y-x) - (\nabla_{+} f(x))(y-x)) \\
& + (1-\lambda)^2 ((\nabla_{-} f(y))(y-x) - (\nabla_{+} f[(1-\lambda)x+\lambda y])(y-x))\big].
\end{aligned}$$

Taking $w \equiv 1$ in (3.25), we obtain the following result.

COROLLARY 3.18. *Let C be a convex subset of a vector space X and $f : C \to \mathbb{R}$ be a convex function. If $x, y \in C$ and $x \neq y$, then*

$$\begin{aligned}
0 &\leqslant \frac{1}{2}\left[\frac{f(y)+f(x)}{2}+f\left(\frac{x+y}{2}\right)\right]-\int_0^1 f((1-t)x+ty)\,dt \\
&\leqslant \frac{1}{16}\Bigg[(\nabla_- f(y))(y-x)-(\nabla_+ f(x))(y-x) \\
&\quad -\left(\left(\nabla_+ f\left(\frac{x+y}{2}\right)\right)(y-x)-\left(\nabla_- f\left(\frac{x+y}{2}\right)\right)(y-x)\right)\Bigg] \\
&\leqslant \frac{1}{16}\left[(\nabla_- f(y))(y-x)-(\nabla_+ f(x))(y-x)\right].
\end{aligned}$$

4. Applications to norms

Some applications to norms are provided in this section.

Let $(X, \|\cdot\|)$ be a normed vector space. For $1 \leqslant p < \infty$, we consider the subset Y_p of the product space $X \times X$ defined by

$$Y_p := \begin{cases} X \times X, \text{ if } p \geqslant 2, \\ \{(x,y) \in X \times X : x \text{ and } y \text{ are linearly independent}\}, \text{ otherwise.} \end{cases} \tag{4.1}$$

We have the following result.

COROLLARY 4.1. *Let $(X, \|\cdot\|)$ be a normed vector space, $1 \leqslant p < \infty$ and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $(x,y) \in Y_p$, then for all $\lambda \in [0,1]$,*

$$\begin{aligned}
& p\langle y-x,(1-\lambda)x+\lambda y\rangle_s \|(1-\lambda)x+\lambda y\|^{p-2}\int_\lambda^1 (t-\lambda)w(t)\,dt \\
&\quad + p\langle y-x,(1-\lambda)x+\lambda y\rangle_i \|(1-\lambda)x+\lambda y\|^{p-2}\int_0^\lambda (t-\lambda)w(t)\,dt \\
&\leqslant \int_0^1 w(t)\|(1-t)x+ty\|^p\,dt-\left(\int_0^1 w(t)\,dt\right)\|(1-\lambda)x+\lambda y\|^p \\
&\leqslant p\langle y-x,y\rangle_i \|y\|^{p-2}\int_\lambda^1 (t-\lambda)w(t)\,dt-p\langle y-x,x\rangle_s \|x\|^{p-2}\int_0^\lambda (\lambda-t)w(t)\,dt.
\end{aligned} \tag{4.2}$$

Proof. Observe first that, if $p \geqslant 2$ and $x = y \in X$, then (4.2) is obvious. Now, for all $1 \leqslant p < \infty$, applying Theorem 3.6 (with $C = X$) to the function f_p defined by (2.1) and using (2.2), we get (4.2). □

Taking $\lambda = \frac{1}{2}$ in Corollary 4.1, we deduce the following result.

COROLLARY 4.2. *Let* $(X, \|\cdot\|)$ *be a normed vector space,* $1 \leqslant p < \infty$ *and* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$*. If* $(x,y) \in Y_p$*, then*

$$\begin{aligned}
&\frac{p}{2}\langle y-x, x+y\rangle_s \left\|\frac{x+y}{2}\right\|^{p-2} \int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right) w(t)\,dt \\
&\quad + \frac{p}{2}\langle y-x, x+y\rangle_i \left\|\frac{x+y}{2}\right\|^{p-2} \int_{0}^{\frac{1}{2}}\left(t-\frac{1}{2}\right) w(t)\,dt \\
&\leqslant \int_0^1 w(t)\,\|(1-t)x+ty\|^p\,dt - \left(\int_0^1 w(t)\,dt\right)\left\|\frac{x+y}{2}\right\|^p \\
&\leqslant p\,\langle y-x, y\rangle_i \|y\|^{p-2} \int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right) w(t)\,dt - p\,\langle y-x, x\rangle_s \|x\|^{p-2} \int_{0}^{\frac{1}{2}}\left(\frac{1}{2}-t\right) w(t)\,dt.
\end{aligned}$$

Taking $p = 2$ in Corollary 4.1, we deduce the following result.

COROLLARY 4.3. *Let* $(X, \|\cdot\|)$ *be a normed vector space and* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$*. If* $x, y \in X$*, then for all* $\lambda \in [0,1]$*,*

$$\begin{aligned}
&2\left[\langle y-x, (1-\lambda)x+\lambda y\rangle_s \int_\lambda^1 (t-\lambda)\,w(t)\,dt + \langle y-x, (1-\lambda)x+\lambda y\rangle_i \int_0^\lambda (t-\lambda)\,w(t)\,dt\right] \\
&\leqslant \int_0^1 w(t)\,\|(1-t)x+ty\|^2\,dt - \left(\int_0^1 w(t)\,dt\right)\|(1-\lambda)x+\lambda y\|^2 \\
&\leqslant 2\left[\langle y-x, y\rangle_i \int_\lambda^1 (t-\lambda)\,w(t)\,dt - \langle y-x, x\rangle_s \int_0^\lambda (\lambda-t)\,w(t)\,dt\right].
\end{aligned}$$

In particular, for $\lambda = \frac{1}{2}$, we deduce from Corollary 4.3 the following result.

COROLLARY 4.4. *Let* $(X, \|\cdot\|)$ *be a normed vector space and* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$*. If* $x, y \in X$*, then,*

$$\begin{aligned}
&\langle y-x, x+y\rangle_s \int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right) w(t)\,dt + \langle y-x, x+y\rangle_i \int_{0}^{\frac{1}{2}}\left(t-\frac{1}{2}\right) w(t)\,dt \\
&\leqslant \int_0^1 w(t)\,\|(1-t)x+ty\|^2\,dt - \left(\int_0^1 w(t)\,dt\right)\left\|\frac{x+y}{2}\right\|^2 \\
&\leqslant 2\left[\langle y-x, y\rangle_i \int_{\frac{1}{2}}^{1}\left(t-\frac{1}{2}\right) w(t)\,dt - \langle y-x, x\rangle_s \int_{0}^{\frac{1}{2}}\left(\frac{1}{2}-t\right) w(t)\,dt\right].
\end{aligned}$$

Taking $p = 1$ in Corollary 4.1, we get the following result.

COROLLARY 4.5. *Let $(X,\|\cdot\|)$ be a normed vector space and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y \in X$ are linearly independent, then for all $\lambda \in [0,1]$,*

$$\begin{aligned}
&\left\langle y-x, \frac{(1-\lambda)x+\lambda y}{\|(1-\lambda)x+\lambda y\|}\right\rangle_s \int_\lambda^1 (t-\lambda)\,w(t)\,dt \\
&\quad + \left\langle y-x, \frac{(1-\lambda)x+\lambda y}{\|(1-\lambda)x+\lambda y\|}\right\rangle_i \int_0^\lambda (t-\lambda)\,w(t)\,dt \\
&\leqslant \int_0^1 w(t)\,\|(1-t)x+ty\|\,dt - \left(\int_0^1 w(t)\,dt\right)\|(1-\lambda)x+\lambda y\| \\
&\leqslant \left\langle y-x, \frac{y}{\|y\|}\right\rangle_i \int_\lambda^1 (t-\lambda)\,w(t)\,dt - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s \int_0^\lambda (\lambda-t)\,w(t)\,dt.
\end{aligned}$$

In particular, for $\lambda = \frac{1}{2}$, we get the following result.

COROLLARY 4.6. *Let $(X,\|\cdot\|)$ be a normed vector space and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y \in X$ are linearly independent, then*

$$\begin{aligned}
&\left\langle y-x, \frac{x+y}{\|x+y\|}\right\rangle_s \int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right) w(t)\,dt + \left\langle y-x, \frac{x+y}{\|x+y\|}\right\rangle_i \int_0^{\frac{1}{2}} \left(t-\frac{1}{2}\right) w(t)\,dt \\
&\leqslant \int_0^1 w(t)\,\|(1-t)x+ty\|\,dt - \left(\int_0^1 w(t)\,dt\right)\left\|\frac{x+y}{2}\right\| \\
&\leqslant \left\langle y-x, \frac{y}{\|y\|}\right\rangle_i \int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right) w(t)\,dt - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s \int_0^{\frac{1}{2}} \left(\frac{1}{2}-t\right) w(t)\,dt.
\end{aligned}$$

From (3.18) and Corollary 4.2, we deduce the following result.

COROLLARY 4.7. *Let $(X,\|\cdot\|)$ be a normed vector space, $1 \leqslant p < \infty$ and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$ $(w \not\equiv 0)$. Assume that there exists $(x,y) \in Y_p$ such that $\|\cdot\|$ is smooth at $x+y$, w is non-decreasing (non-increasing) on $[0,1]$ and*

$$\langle y-x, x+y\rangle := \langle y-x, x+y\rangle_{i(s)} \geqslant (\leqslant)\, 0. \tag{4.3}$$

Then

$$\begin{aligned}
0 &\leqslant \frac{p}{2}\langle y-x, x+y\rangle \left\|\frac{x+y}{2}\right\|^{p-2} \frac{1}{\int_0^1 w(t)\,dt} \int_0^1 \left(t-\frac{1}{2}\right) w(t)\,dt \\
&\leqslant \frac{1}{\int_0^1 w(t)\,dt} \int_0^1 w(t)\,\|(1-t)x+ty\|^p\,dt - \left\|\frac{x+y}{2}\right\|^p.
\end{aligned}$$

In particular, for $p = 2$, we deduce from Corollary 4.7 the following result.

COROLLARY 4.8. *Let* $(X,\|\cdot\|)$ *be a normed vector space and* $w\in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$ $(w\not\equiv 0)$. *Assume that there exist* $x,y\in X$ *such that* $\|\cdot\|$ *is smooth at* $x+y$, w *is non-decreasing (non-increasing) on* $[0,1]$ *and* (4.3) *holds. Then*

$$\begin{aligned}0 &\leqslant \langle y-x,x+y\rangle \frac{1}{\int_0^1 w(t)\,dt}\int_0^1\left(t-\frac{1}{2}\right)w(t)\,dt\\ &\leqslant \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,\|(1-t)x+ty\|^2\,dt-\left\|\frac{x+y}{2}\right\|^2.\end{aligned}$$

Similarly, for $p=1$, we deduce from Corollary 4.7 the following result.

COROLLARY 4.9. *Let* $(X,\|\cdot\|)$ *be a normed vector space and* $w\in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$ $(w\not\equiv 0)$. *Assume that there exist* $x,y\in X$ *such that* x *and* y *are linearly independent,* $\|\cdot\|$ *is smooth at* $x+y$, w *is non-decreasing (non-increasing) on* $[0,1]$ *and* (4.3) *holds. Then*

$$\begin{aligned}0 &\leqslant \left\langle y-x,\frac{x+y}{\|x+y\|}\right\rangle \frac{1}{\int_0^1 w(t)\,dt}\int_0^1\left(t-\frac{1}{2}\right)w(t)\,dt\\ &\leqslant \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\,\|(1-t)x+ty\|\,dt-\left\|\frac{x+y}{2}\right\|.\end{aligned}$$

By applying Theorem 3.15 (with $C=X$) to the function f_p defined by (2.1), we obtain

COROLLARY 4.10. *Let* $(X,\|\cdot\|)$ *be a normed vector space,* $1\leqslant p<\infty$ *and* $w\in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$. *If* $(x,y)\in Y_p$, *then for all* $\lambda\in(0,1)$,

$$\begin{aligned}0 &\leqslant \frac{\|y\|^p-\|(1-\lambda)x+\lambda y\|^p}{1-\lambda}\int_\lambda^1 (t-\lambda)\,w(t)\,dt\\ &\quad -\frac{\|(1-\lambda)x+\lambda y\|^p-\|x\|^p}{\lambda}\int_0^\lambda (\lambda-t)\,w(t)\,dt\\ &\quad -\left(\int_0^1 w(t)\|(1-t)x+ty\|^p\,dt-\|(1-\lambda)x+\lambda y\|^p\int_0^1 w(t)\,dt\right)\\ &\leqslant \frac{p}{4}\left[\langle y-x,(1-\lambda)x+\lambda y\rangle_i\,\|(1-\lambda)x+\lambda y\|^{p-2}-\langle y-x,x\rangle_s\,\|x\|^{p-2}\right]\lambda\int_0^\lambda w(t)\,dt\\ &\quad +\frac{p}{4}\left[\langle y-x,y\rangle_i\,\|y\|^{p-2}-\langle y-x,(1-\lambda)x+\lambda y\rangle_s\,\|(1-\lambda)x+\lambda y\|^{p-2}\right]\\ &\quad \times(1-\lambda)\int_\lambda^1 w(t)\,dt.\end{aligned}$$

In particular, for $\lambda=\frac{1}{2}$, we deduce from Corollary 4.10 the following result.

COROLLARY 4.11. *Let $(X,\|\cdot\|)$ be a normed vectors space, $1 \leqslant p < \infty$ and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $(x,y) \in Y_p$, then*

$$\begin{aligned}
0 \leqslant\ & 2\left[\left(\|y\|^p - \left\|\frac{x+y}{2}\right\|^p\right)\int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right)w(t)\,dt\right.\\
&\left.-\left(\left\|\frac{x+y}{2}\right\|^p - \|x\|^p\right)\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right]\\
&-\left(\int_0^1 w(t)\|(1-t)x+ty\|^p\,dt - \left\|\frac{x+y}{2}\right\|^p\int_0^1 w(t)\,dt\right)\\
\leqslant\ & \frac{p}{8}\left(\left\|\frac{x+y}{2}\right\|^{p-2}\left\langle y-x,\frac{x+y}{2}\right\rangle_i - \langle y-x,x\rangle_s\|x\|^{p-2}\right)\int_0^{\frac{1}{2}} w(t)\,dt\\
&+\frac{p}{8}\left(\langle y-x,y\rangle_i\|y\|^{p-2} - \left\langle y-x,\frac{x+y}{2}\right\rangle_s\left\|\frac{x+y}{2}\right\|^{p-2}\right)\int_{\frac{1}{2}}^1 w(t)\,dt.
\end{aligned}$$

For $p=2$, we deduce from Corollary 4.10 the following result.

COROLLARY 4.12. *Let $(X,\|\cdot\|)$ be a normed vectors space and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y \in X$, then for all $\lambda \in (0,1)$,*

$$\begin{aligned}
0 \leqslant\ & \frac{\|y\|^2 - \|(1-\lambda)x+\lambda y\|^2}{1-\lambda}\int_\lambda^1 (t-\lambda)w(t)\,dt\\
&-\frac{\|(1-\lambda)x+\lambda y\|^2 - \|x\|^2}{\lambda}\int_0^\lambda (\lambda-t)w(t)\,dt\\
&-\left(\int_0^1 w(t)\|(1-t)x+ty\|^2\,dt - \|(1-\lambda)x+\lambda y\|^2\int_0^1 w(t)\,dt\right)\\
\leqslant\ & \frac{1}{2}\left[\langle y-x,(1-\lambda)x+\lambda y\rangle_i - \langle y-x,x\rangle_s\right]\lambda\int_0^\lambda w(t)\,dt\\
&+\frac{1}{2}\left[\langle y-x,y\rangle_i - \langle y-x,(1-\lambda)x+\lambda y\rangle_s\right](1-\lambda)\int_\lambda^1 w(t)\,dt.
\end{aligned}$$

Taking $\lambda = \frac{1}{2}$ in Corollary 4.12, we get

COROLLARY 4.13. *Let $(X,\|\cdot\|)$ be a normed vectors space and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y \in X$, then*

$$\begin{aligned}
0 \leqslant\ & 2\left[\left(\|y\|^2 - \left\|\frac{x+y}{2}\right\|^2\right)\int_{\frac{1}{2}}^1 \left(t-\frac{1}{2}\right)w(t)\,dt\right.\\
&\left.-\left(\left\|\frac{x+y}{2}\right\|^2 - \|x\|^2\right)\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right]\\
&-\left(\int_0^1 w(t)\|(1-t)x+ty\|^2\,dt - \left\|\frac{x+y}{2}\right\|^2\int_0^1 w(t)\,dt\right)
\end{aligned}$$

$$\leqslant \frac{1}{4}\left(\left\langle y-x, \frac{x+y}{2}\right\rangle_i - \langle y-x, x\rangle_s\right)\int_0^{\frac{1}{2}} w(t)\,dt$$
$$+\frac{1}{4}\left(\langle y-x, y\rangle_i - \left\langle y-x, \frac{x+y}{2}\right\rangle_s\right)\int_{\frac{1}{2}}^1 w(t)\,dt.$$

For $p=1$, we deduce from Corollary 4.10 the following result.

COROLLARY 4.14. *Let* $(X,\|\cdot\|)$ *be a normed vector space and* $w\in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$. *If* $x,y\in X$ *are linearly independent, then for all* $\lambda\in(0,1)$,

$$\begin{aligned}
0 &\leqslant \frac{\|y\|-\|(1-\lambda)x+\lambda y\|}{1-\lambda}\int_\lambda^1 (t-\lambda)w(t)\,dt \\
&\quad - \frac{\|(1-\lambda)x+\lambda y\|-\|x\|}{\lambda}\int_0^\lambda (\lambda-t)w(t)\,dt \\
&\quad - \left(\int_0^1 w(t)\|(1-t)x+ty\|\,dt - \|(1-\lambda)x+\lambda y\|\int_0^1 w(t)\,dt\right) \\
&\leqslant \frac{\lambda}{4}\left(\left\langle y-x, \frac{(1-\lambda)x+\lambda y}{\|(1-\lambda)x+\lambda y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right)\int_0^\lambda w(t)\,dt \\
&\quad + \frac{(1-\lambda)}{4}\left(\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{(1-\lambda)x+\lambda y}{\|(1-\lambda)x+\lambda y\|}\right\rangle_s\right)\int_\lambda^1 w(t)\,dt.
\end{aligned}$$

In particular, for $\lambda=\frac{1}{2}$, we deduce from Corollary 4.14 the following result.

COROLLARY 4.15. *Let* $(X,\|\cdot\|)$ *be a normed vector space and* $w\in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$. *If* $x,y\in X$ *are linearly independent, then*

$$\begin{aligned}
0 &\leqslant 2\left[\left(\|y\|-\left\|\frac{x+y}{2}\right\|\right)\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)w(t)\,dt - \left(\left\|\frac{x+y}{2}\right\|-\|x\|\right)\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right] \\
&\quad - \left(\int_0^1 w(t)\|(1-t)x+ty\|\,dt - \left\|\frac{x+y}{2}\right\|\int_0^1 w(t)\,dt\right) \\
&\leqslant \frac{1}{8}\left(\left\langle y-x, \frac{x+y}{\|x+y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right)\int_0^{\frac{1}{2}} w(t)\,dt \\
&\quad + \frac{1}{8}\left(\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x+y}{\|x+y\|}\right\rangle_s\right)\int_{\frac{1}{2}}^1 w(t)\,dt.
\end{aligned}$$

5. The case of inner product spaces

We now consider the case when $(H,\langle\cdot,\cdot\rangle)$ is an inner product space. In this case, one has

$$\langle x,y\rangle_s = \langle x,y\rangle_i = \operatorname{Re}\langle x,y\rangle, \quad x,y\in H.$$

For all $1\leqslant p<\infty$, let Y_p be the subset of $H\times H$ defined by (4.1) with $X=H$.

From Corollary 4.1, we deduce the following result.

COROLLARY 5.1. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a an inner product space,* $1 \leqslant p < \infty$ *and* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$*. If* $(x,y) \in Y_p$*, then for all* $\lambda \in [0,1]$*,*

$$\begin{aligned}
&p\mathrm{Re}\langle y-x,(1-\lambda)x+\lambda y\rangle \|(1-\lambda)x+\lambda y\|^{p-2}\int_0^1 (t-\lambda)w(t)\,dt\\
&\leqslant \int_0^1 w(t)\|(1-t)x+ty\|^p\,dt-\left(\int_0^1 w(t)\,dt\right)\|(1-\lambda)x+\lambda y\|^p\\
&\leqslant p\mathrm{Re}\left\langle y-x,\left(\|y\|^{p-2}\int_\lambda^1 (t-\lambda)w(t)\,dt\right)y-\left(\|x\|^{p-2}\int_0^\lambda (\lambda-t)w(t)\,dt\right)x\right\rangle\\
&\leqslant p\|y-x\|\left\|\left(\|y|\|^{p-2}\int_\lambda^1 (t-\lambda)w(t)\,dt\right)y-\left(\|x\|^{p-2}\int_0^\lambda (\lambda-t)w(t)\,dt\right)x\right\|.
\end{aligned}$$

For $\lambda=\frac{1}{2}$, we deduce from Corollary 5.1 the following result.

COROLLARY 5.2. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a an inner product space,* $1 \leqslant p < \infty$ *and* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$*. If* $(x,y) \in Y_p$*, then*

$$\begin{aligned}
&\frac{p}{2}\left(\|y\|^2-\|x\|^2\right)\left\|\frac{x+y}{2}\right\|^{p-2}\int_0^1\left(t-\frac{1}{2}\right)w(t)\,dt\\
&\leqslant \int_0^1 w(t)\|(1-t)x+ty\|^p\,dt-\left(\int_0^1 w(t)\,dt\right)\left\|\frac{x+y}{2}\right\|^p\\
&\leqslant p\mathrm{Re}\left\langle y-x,\left(\|y\|^{p-2}\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)w(t)\,dt\right)y-\left(\|x\|^{p-2}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right)x\right\rangle\\
&\leqslant p\|y-x\|\left\|\left(\|y\|^{p-2}\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)w(t)\,dt\right)y-\left(\|x\|^{p-2}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right)x\right\|.
\end{aligned}$$

Taking $p=2$ in Corollary 5.1, we get the following result.

COROLLARY 5.3. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a an inner product space and* $w \in L^1(0,1)$ *be nonnegative a.e. on* $[0,1]$*. If* $x,y \in H$*, then for all* $\lambda \in [0,1]$*,*

$$\begin{aligned}
&2\mathrm{Re}\langle y-x,(1-\lambda)x+\lambda y\rangle\int_0^1 (t-\lambda)w(t)\,dt\\
&\leqslant \int_0^1 w(t)\|(1-t)x+ty\|^2\,dt-\left(\int_0^1 w(t)\,dt\right)\|(1-\lambda)x+\lambda y\|^2\\
&\leqslant 2\mathrm{Re}\left\langle y-x,\left(\int_\lambda^1 (t-\lambda)w(t)\,dt\right)y-\left(\int_0^\lambda (\lambda-t)w(t)\,dt\right)x\right\rangle\\
&\leqslant 2\|y-x\|\left\|\left(\int_\lambda^1 (t-\lambda)w(t)\,dt\right)y-\left(\int_0^\lambda (\lambda-t)w(t)\,dt\right)x\right\|.
\end{aligned}$$

Taking $\lambda=\frac{1}{2}$ in Corollary 5.3, we get the following result.

COROLLARY 5.4. *Let $(H,\langle\cdot,\cdot\rangle)$ be a an inner product space and $w\in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y\in H$, then*

$$\begin{aligned}
&\left(\|y\|^2-\|x\|^2\right)\int_0^1\left(t-\frac{1}{2}\right)w(t)\,dt\\
&\leqslant \int_0^1 w(t)\|(1-t)x+ty\|^2\,dt-\left(\int_0^1 w(t)\,dt\right)\left\|\frac{x+y}{2}\right\|^2\\
&\leqslant 2\mathrm{Re}\left\langle y-x,\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)w(t)\,dt\right)y-\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right)x\right\rangle\\
&\leqslant 2\|y-x\|\left\|\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)w(t)\,dt\right)y-\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right)x\right\|.
\end{aligned}$$

For $p=1$, we deduce from Corollary 5.1 the following result.

COROLLARY 5.5. *Let $(H,\langle\cdot,\cdot\rangle)$ be a an inner product space and $w\in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y\in H$ are linearly independent, then for all $\lambda\in[0,1]$,*

$$\begin{aligned}
&\mathrm{Re}\left\langle y-x,\frac{(1-\lambda)x+\lambda y}{\|(1-\lambda)x+\lambda y\|}\right\rangle\int_0^1(t-\lambda)w(t)\,dt\\
&\leqslant \int_0^1 w(t)\|(1-t)x+ty\|\,dt-\left(\int_0^1 w(t)\,dt\right)\|(1-\lambda)x+\lambda y\|\\
&\leqslant \mathrm{Re}\left\langle y-x,\left(\int_\lambda^1(t-\lambda)w(t)\,dt\right)\frac{y}{\|y\|}-\left(\int_0^\lambda(\lambda-t)w(t)\,dt\right)\frac{x}{\|x\|}\right\rangle\\
&\leqslant \|y-x\|\left\|\left(\int_\lambda^1(t-\lambda)w(t)\,dt\right)\frac{y}{\|y\|}-\left(\int_0^\lambda(\lambda-t)w(t)\,dt\right)\frac{x}{\|x\|}\right\|.
\end{aligned}$$

Taking $\lambda=\frac{1}{2}$ in Corollary 5.5, we derive the following result.

COROLLARY 5.6. *Let $(H,\langle\cdot,\cdot\rangle)$ be a an inner product space and $w\in L^1(0,1)$ be nonnegative a.e. on $[0,1]$. If $x,y\in H$ are linearly independent, then*

$$\begin{aligned}
&\frac{\|y\|^2-\|x\|^2}{\|x+y\|}\int_0^1\left(t-\frac{1}{2}\right)w(t)\,dt\\
&\leqslant \int_0^1 w(t)\|(1-t)x+ty\|\,dt-\left(\int_0^1 w(t)\,dt\right)\left\|\frac{x+y}{2}\right\|\\
&\leqslant \mathrm{Re}\left\langle y-x,\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)w(t)\,dt\right)\frac{y}{\|y\|}-\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right)\frac{x}{\|x\|}\right\rangle\\
&\leqslant \|y-x\|\left\|\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{2}\right)w(t)\,dt\right)\frac{y}{\|y\|}-\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-t\right)w(t)\,dt\right)\frac{x}{\|x\|}\right\|.
\end{aligned}$$

From (3.18) and Corollary 5.2, we deduce the following result.

COROLLARY 5.7. *Let $(H,\langle\cdot,\cdot\rangle)$ be a an inner product space, $1 \leqslant p < \infty$ and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$ $(w \not\equiv 0)$. Assume that there exists $(x,y) \in Y_p$ such that $\|y\| \geqslant (\leqslant) \|x\|$ and w is non-decreasing (non-increasing) on $[0,1]$. Then*

$$0 \leqslant \frac{p}{2}\left(\|y\|^2 - \|x\|^2\right)\left\|\frac{x+y}{2}\right\|^{p-2} \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left(t-\frac{1}{2}\right) w(t)\,dt$$
$$\leqslant \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\|(1-t)x+ty\|^p\,dt - \left\|\frac{x+y}{2}\right\|^p.$$

In particular, for $p=2$, we obtain

COROLLARY 5.8. *Let $(H,\langle\cdot,\cdot\rangle)$ be a an inner product space and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$ $(w \not\equiv 0)$. If there exist $x,y \in H$ such that $\|y\| \geqslant (\leqslant) \|x\|$ and w is non-decreasing (non-increasing) on $[0,1]$, then*

$$0 \leqslant \left(\|y\|^2 - \|x\|^2\right) \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left(t-\frac{1}{2}\right) w(t)\,dt$$
$$\leqslant \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\|(1-t)x+ty\|^2\,dt - \left\|\frac{x+y}{2}\right\|^2.$$

For $p=1$, Corollary 5.7 yields the following result.

COROLLARY 5.9. *Let $(H,\langle\cdot,\cdot\rangle)$ be a an inner product space and $w \in L^1(0,1)$ be nonnegative a.e. on $[0,1]$ $(w \not\equiv 0)$. Assume that there exist $x,y \in H$ linearly independent such that $\|y\| \geqslant (\leqslant) \|x\|$ and w is non-decreasing (non-increasing) on $[0,1]$. Then*

$$0 \leqslant \frac{\|y\|^2 - \|x\|^2}{2\|x+y\|} \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 \left(t-\frac{1}{2}\right) w(t)\,dt$$
$$\leqslant \frac{1}{\int_0^1 w(t)\,dt}\int_0^1 w(t)\|(1-t)x+ty\|\,dt - \left\|\frac{x+y}{2}\right\|.$$

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