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ON A POSITIVITY PROPERTY OF THE REAL PART OF THE LOGARITHMIC DERIVATIVE OF THE RIEMANN ξ–FUNCTION

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Abstract. In this paper, we investigate the positivity of the real part of the logarithmic derivative of the Riemann ξ-function when $1/2 < \sigma < 1$ and t is sufficiently large. We consider explicit upper and lower bounds of $\Re\sum_\rho 1/(s-\rho)$, where the summation runs over the zeros of $\zeta(s)$ on the line $1/2+it$. We also examine the positivity of $\Re\xi'/\xi(s)$ in the strip $1/2<\sigma<1$ assuming that there occur non-trivial zeros of $\zeta(s)$ off the critical line.

1. Introduction

For the complex variable $s=\sigma+it$ the Riemann ξ-function is defined by

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $\zeta(s)$ is the Riemann ζ-function. The functions $\xi(s)$ and $\zeta(s)$ have the same zeros in the strip $0<\sigma<1$, called the critical strip, and the famous Riemann's hypothesis states that they all are located on the line $1/2+it$ which is called the critical line. Zeros in the strip $0<\sigma<1$ are known as non-trivial zeros of $\zeta(s)$. The Riemann ζ-function also has zeros at each even negative integer $s=-2n$, these zeros are known as the trivial zeros of $\zeta(s)$. The function $\xi(s)$ also satisfies $\xi(s)=\xi(1-s)$ and $\overline{\xi(s)}=\xi(\overline{s})$. From this, it is clear that $\xi(\sigma+it)=0$ iff $\xi(1-\sigma+it)=0$. Also, if s is a non-trivial zero of $\xi(s)$ off the critical line then the four numbers $\{s,\overline{s},1-s,1-\overline{s}\}$ would all be non-trivial zeros off the line.

By $\rho=\beta+i\gamma$ we denote a non-trivial zero of $\zeta(s)$, i.e. $\zeta(\rho)=0$. The function $\xi(s)$ can be expanded as an infinite product over ρ, see Edwards [5, p. 39],

$$\xi(s)=\xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right)=\frac{1}{2}\prod_\rho\left(1-\frac{s}{\rho}\right), \tag{1}$$

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where the product is taken in an order that pairs each root ρ with the corresponding root $1-\rho$. The logarithmic derivative of $\xi(s)$ is

$$\frac{\xi'}{\xi}(s)=\sum_{\rho}\frac{1}{s-\rho}, \tag{2}$$

where the summation is understood the same way as defining the product (1). There is a direct relation between the location of zeros of the complex function f and the behavior of its modulus or real part of the logarithmic derivative. Matiyasevich, Saidak, and Zvengrowski [15] note that "... strict decrease of the modulus of any continuous complex function f along any curve in the complex plane implies that f can have no zero along that curve." The relation between the monotonicity of modulus of the complex function $|f|$ and the sign of its real part of logarithmic derivative $\Re f'/f$ is provided in Lemma 6.

It is known that (see for example Hinkkanen [10])

$$\Re\frac{\xi'}{\xi}(s)>0 \text{ when } \Re s>1$$

and the Riemann hypothesis is equivalent to

$$\Re\frac{\xi'}{\xi}(s)>0 \text{ when } \Re s>\frac{1}{2}.$$

Lagarias [11] proved that

$$\inf\left\{\Re\frac{\xi'}{\xi}(s):-\infty<t<\infty\right\}=\frac{\xi'}{\xi}(\sigma) \tag{3}$$

for $\sigma>10$ and Garunkštis [7] later improved (3) for $\sigma>a$, where $\sigma>a$ is a zero-free region of $\zeta(s)$. See also Broughan [2] on the subject. The following reformulation of the Riemann hypothesis was given in the paper by Sondow and Dumitrescu [21].

THEOREM 1. (Sondow, Dumitrescu) *The following statements are equivalent.*
I. *If t is any fixed real number, then $|\xi(\sigma+it)|$ is increasing for $1/2<\sigma<\infty$.*
II. *If t is any fixed real number, then $|\xi(\sigma+it)|$ is decreasing for $-\infty<\sigma<1/2$.*
III. *The Riemann hypothesis is true.*

In the same paper [21] it was proved the following theorem.

THEOREM 2. (Sondow, Dumitrescu) *The ξ-function is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no ξ zeros. Similarly, the modulus decreases on each horizontal half-line in any zero-free, open left half-plane.*

Matiyasevich, Saidak, and Zvengrowski [15] slightly reformulated Theorem 2.

THEOREM 3. (Matiyasevich, Saidak, Zvengrowski) *Let* σ_0 *be greater than or equal to the real part of any zero of* ξ. *Then* $|\xi(s)|$ *is strictly increasing*[1] *in the half-plane* $\sigma > \sigma_0$.

In this paper, we further investigate the function $\xi'/\xi(s)$. Let $\rho = 1/2 + i\gamma$ denote the zero of $\zeta(s)$ lying on the critical line, $\tilde{\rho} = \tilde{\beta} + i\tilde{\gamma}$, $\tilde{\beta} \neq 1/2$ be the hypothetical nontrivial zero of $\zeta(s)$ lying off the critical line, and define

$$\frac{\xi'}{\xi}(s) = \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} + \sum_{\tilde{\rho}=\tilde{\beta}+i\tilde{\gamma}} \frac{1}{s-\tilde{\rho}} =: \Sigma_1 + \Sigma_2, \tag{4}$$

where the summation again is understood as defining (1). This ensures an absolute convergence of the series in (4) for $s : \zeta(s) \neq 0$, see Edwards [5, p. 42]. It is clear that the sum Σ_1 exists, while Σ_2 might be vacuous as the Riemann hypothesis is unsolved.

For $1/2 < \sigma < 1$ and sufficiently large t, in Theorem 4 below, we give explicit lower and upper bounds for $\Re\Sigma_1$. The lower bound of $\Re\Sigma_1$ in Theorem 4 suggests that $\Re\xi'/\xi(s)$ may remain positive asymptotically close to the critical line despite that $\Re\Sigma_2$ might occur if the Riemann hypothesis fails. In Section 4 we test the positivity of $\Re(\Sigma_1 + \Sigma_2)$ assuming that certain versions of Σ_2 exist. We show that the obtained results widen Theorems 2 and 3, see Figures 1 and 2 in Section 4.

We start the investigation of Σ_1 by mentioning the fact that there are infinitely many zeros of $\zeta(s)$ lying on the line $1/2 + it$ (see Hardy [8]), however, we do not know the number of zeros of $\zeta(s)$ lying in the strip $1/2 < \sigma < 1$. The initial result on the part of non-trivial zeros on the critical line of the Riemann zeta function was obtained by Selberg [20]. Selberg proved that at least a positive proportion of all non-trivial zeros lie on the critical line. Later this result was improved by several authors, see for example Levinson [12], Conrey [4], Feng [6], Pratt et al. [19].

Let $N(T)$ denote the number of zeros of $\zeta(s)$ in the rectangle $0 < \sigma < 1$, $0 < t < T$, and $N_{1/2}(T)$ denote the number of zeros of $\zeta(s)$ on the critical line $1/2 + it$, $0 < t < T$. Then, it is clear that there exists such $0 < c \leqslant 1$ that

$$cN(T) \leqslant N_{1/2}(T) \leqslant N(T) \tag{5}$$

for all $T \geqslant 0$. The mentioned results on the part of non-trivial zeros on the critical line of the Riemann zeta-function consist of finding the lower estimate of

$$\liminf_{T\to\infty} \frac{N_{1/2}(T)}{N(T)}.$$

In addition, let us mention several facts about the number of known non-trivial zeros of $\zeta(s)$ on the critical line. G.F.B. Riemann was the first to compute a few of such zeros, see Edwards [5, Chap. 7]. Later, this number was improved by several authors such as E. C. Titchmarsh and others, see Matiyasevich [14, Table 1]. However the most significant methodological breakthrough in such type of computation was achieved by Alan Turing, called the father of theoretical computer science; see, for example, Cooper

[1] With respect to σ.

and Leeuwen (Eds.) [3] and references therein. In these days, the Riemann hypothesis is verified numerically within the rectangle $0<\sigma<1$, $0<t\leqslant 3\times 10^{12}=:\gamma_N$, where $N=1.236\times 10^{13}$ denotes the number of zeros of $\zeta(s)$ that all lie on the critical line within the provided rectangle and $\zeta(1/2+i\gamma_N)=0$, see Platt, Trudgian [18].

Based on the mentioned facts and definitions, we formulate the following theorem on the estimates of $\Re\Sigma_1$.

THEOREM 4. *Let* $1/2<\sigma<1$ *and* $0<c\leqslant 1$ *be such that* $c\leqslant N_{1/2}(T)/N(T)$ *for all* $T\geqslant\gamma_1=14.134725\ldots$, *where* $\zeta(1/2+i\gamma_1)=0$. *Let*

$$\begin{aligned} A(t) &= 0.12\log\frac{t}{2\pi}-2.32\log\log t-18.432-\varepsilon_1(t),\\ B(t) &= 0.49\log\frac{t}{2\pi}+0.58\log\log t+4.603+\varepsilon_2(t), \end{aligned}$$

where $\varepsilon_1(t)$ *and* $\varepsilon_2(t)$ *are known explicit* t *functions* (*see* (15) *and* (16) *below*) *both vanishing as* $t^{-1}\log t$, $t\to\infty$.

Then

$$0<c\left(\sigma-\frac{1}{2}\right)A(t)<\Re\sum_{\rho=1/2+i\gamma}\frac{1}{s-\rho},\quad t>1.984\times 10^{114},$$
$$\Re\sum_{\rho=1/2+i\gamma}\frac{1}{s-\rho}<\frac{B(t)}{\sigma-1/2},\quad t>14.635.$$

We prove Theorem 4 in Section 3. This theorem implies the following corollary.

COROLLARY 5. *The function*

$$\Re\frac{\xi'}{\xi}(s)=-\Re\frac{\xi'}{\xi}(1-s)>0$$

if

$$\Re\sum_{\tilde{\rho}=\tilde{\beta}+i\tilde{\gamma}}\frac{1}{s-\tilde{\rho}}+c\left(\sigma-\frac{1}{2}\right)A(t)>0. \tag{6}$$

The remaining structure of this article is the following: in Section 2 we formulate and prove auxiliary statements, while in the last Section 4 we depict the condition (6) assuming that the Riemann hypothesis fails. All the necessary computations and visualizations are implemented using the software [24].

2. Lemmas

In this section, we formulate several auxiliary lemmas that are needed for the proof of Theorem 4.

LEMMA 6. *(a) Let f be holomorphic in an open domain D and not identically zero. Let us also suppose $\Re f'/f(s) < 0$ for all $s \in D$ such that $f(s) \neq 0$. Then $|f(s)|$ is strictly decreasing with respect to σ in D, i.e. for each $s_0 \in D$ there exists a $\delta > 0$ such that $|f(s)|$ is strictly monotonically decreasing with respect to σ on the horizontal interval from $s_0 - \delta$ to $s_0 + \delta$.*

(b) Conversely, if $|f(s)|$ is decreasing with respect to σ in D, then $\Re f'/f(s) \leqslant 0$ for all $s \in D$ such that $f(s) \neq 0$.

Proof. See Matiyasevich, Saidak, Zvengrowski [15] for the proof. □

NOTE 1. Of course, the analogous results hold for monotone increasing $|f(s)|$ and $\Re f'/f(s) > 0$.

LEMMA 7. *Let $N(T)$ be the number of zeros of $\zeta(s)$ in the rectangle $0 < \sigma < 1$, $0 < t < T$. If $T \geqslant e$, then*

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leqslant 0.110 \log T + 0.290 \log\log T + 2.290 + \frac{25}{48\pi T}. \tag{7}$$

Proof. In the paper by Trudgian [23, p. 283] it is derived that, for $T \geqslant 1$

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leqslant |S(T)| + \frac{1}{4\pi} \arctan\left(\frac{1}{2T}\right) + \frac{T}{4\pi} \log\left(1 + \frac{1}{4T^2}\right) + \frac{1}{3\pi T},$$

where $\pi S(T)$ is the argument of the Riemann zeta-function along the critical line. From the paper by Platt and Trudgian [17, Cor. 1] (see also Hasanalizade, Shen, Wong [9])

$$|S(T)| \leqslant 0.110 \log T + 0.290 \log\log T + 2.290, \quad T \geqslant e$$

and, using inequalities,

$$\arctan \frac{1}{t} = \int_0^{1/t} \frac{dx}{1+x^2} \leqslant \frac{1}{t}, \quad t > 0$$

and

$$\log(1+t) \leqslant t, \quad t > -1,$$

we get the desired result. □

LEMMA 8. *If $a, b, \alpha > 0$, then the following inequality holds*

$$\int_\alpha^t \frac{\log \frac{u}{2\pi}\, du}{a^2 + b^2(u-t)^2} \geqslant \frac{1}{ab} \log\left(\frac{t}{2\pi}\right) \arctan\left(\frac{b(t-\alpha)}{a}\right) - \kappa,$$

when $t > t_0 \geqslant \alpha$, *where* t_0 *and constant* $\kappa > 0$ *are both sufficiently large and* κ *is independent of* t.

In particular, if $a = 1/2, b = 1$ *and* $\alpha = \gamma_1 = 14.134725\ldots$, *where* $1/2 + i\gamma_1$ *is the lowest nontrivial zero of* $\zeta(s)$ *in the upper half-plane, then the provided inequality holds if* $t > 23$ *and* $\kappa = 0.135$.

Proof. We set up the function

$$F(t) = \int_\alpha^t \frac{\log \frac{u}{2\pi}\, du}{a^2 + b^2(u-t)^2} - \frac{1}{ab} \log\left(\frac{t}{2\pi}\right) \arctan\left(\frac{b(t-\alpha)}{a}\right) + \kappa$$

and show that t derivative $F'(t) \geqslant 0$ for $t > t_0 \geqslant \alpha$. Indeed, according to the Leibniz integral rule (see, for example, Mackevičius [13] or Spivak [22])

$$F'(t) = 2b^2 \int_\alpha^t \frac{(u-t)\log u/2\pi\, du}{(a^2 + b^2(u-t)^2)^2} + \left(\frac{1}{a^2} - \frac{1}{a^2 + b^2(t-\alpha)^2}\right) \log\frac{t}{2\pi}$$
$$- \frac{\arctan(b(t-\alpha)/a)}{abt}.$$

The last integral is

$$2b^2 \int_\alpha^t \frac{(u-t)\log u/2\pi\, du}{(a^2 + b^2(u-t)^2)^2} = -\int_\alpha^t \log\frac{u}{2\pi}\, d\,\frac{1}{a^2 + b^2(u-t)^2}$$
$$= \frac{\log(\alpha/2\pi)}{a^2 + b^2(t-\alpha)^2} - \frac{\log(t/2\pi)}{a^2} + \int_\alpha^t \frac{du}{u(a^2 + b^2(t-\alpha)^2)},$$

where

$$\int_\alpha^t \frac{du}{u(a^2 + b^2(t-\alpha)^2)} = \frac{b^2}{b^2t^2 + a^2} \int_\alpha^t \left(\frac{1}{b^2 u} + \frac{2t-u}{a^2 + b^2(u-t)^2}\right) du$$
$$= \frac{\log(t/\alpha)}{b^2t^2 + a^2} + \frac{b}{a} \cdot \frac{t}{b^2t^2 + a^2} \arctan\left(\frac{b(t-\alpha)}{a}\right) + \frac{1}{2} \cdot \frac{1}{b^2t^2 + a^2} \log\left(1 + \frac{b^2(t-\alpha)^2}{a^2}\right).$$

Therefore

$$F'(t) = \frac{1/2}{b^2t^2 + a^2} \log\left(\left(\frac{t}{\alpha}\right)^2 + \left(\frac{bt(t-\alpha)}{a\alpha}\right)^2\right) - \frac{\log(t/\alpha)}{a^2 + b^2(t-\alpha)^2}$$
$$- \frac{a}{b} \cdot \frac{1}{t} \cdot \frac{1}{b^2t^2 + a^2} \arctan\left(\frac{b(t-\alpha)}{a}\right).$$

For $t \geqslant \alpha + a/b$, it holds that

$$\frac{bt(t-\alpha)}{a\alpha} \geqslant \frac{t}{\alpha},$$

and

$$F'(t) \geqslant \frac{\log\sqrt{2}}{a^2 + b^2t^2} - \frac{(\alpha(2t-\alpha))\log(t/\alpha)}{(a^2 + b^2t^2)(a^2 + b^2(t-\alpha)^2)} \tag{8}$$
$$- \frac{a}{b} \cdot \frac{1}{t} \cdot \frac{1}{b^2t^2 + a^2} \arctan\left(\frac{b(t-\alpha)}{a}\right).$$

The positive term of the right-hand side of inequality (8) vanishes as t^{-2} while the two negative terms as $t^{-3}\log t$, which means that $F'(t) > 0$ if $t > t_0 \geqslant \alpha$ and t_0 is sufficiently large.

We next check whether $F(t_0) \geqslant 0$. It is easy to see that

$$\lim_{t\to\alpha^+} F(t) = \kappa > 0.$$

Therefore, due to continuity of $F(t)$, $F(t) > 0$ for at least $t \in (\alpha, t_0]$ if κ is large enough and t_0 is dependent on κ.

For the particular case $a = 1/2, b = 1$ and $\alpha = \gamma_1 = 14.134725\ldots$, where $\zeta(1/2 + i\gamma_1) = 0$, we check that $F'(t) > 0$, when $t > 23$ and $F(23) = 0.00092\ldots$ if $\kappa = 0.135$. □

LEMMA 9. *If $t > 1$, then*

$$\frac{\pi}{2} - \frac{1}{t} < \arctan t < \frac{\pi}{2} - \frac{1}{2t}. \tag{9}$$

Proof. The first inequality of (9) follows from

$$\frac{\pi}{2} = \int_0^\infty \frac{dx}{1+x^2} = \int_0^t \frac{dx}{1+x^2} + \int_t^\infty \frac{dx}{1+x^2} < \arctan t + \int_t^\infty \frac{dx}{x^2} = \arctan t + \frac{1}{t},$$

and the second

$$\frac{\pi}{2} = \int_0^\infty \frac{dx}{1+x^2} = \int_0^t \frac{dx}{1+x^2} + \int_t^\infty \frac{dx}{1+x^2} > \arctan t + \int_t^\infty \frac{dx}{x^2+x^2} = \arctan t + \frac{1}{2t}. \quad \square$$

NOTE 2. The first inequality in (9) holds for $t > 0$ also.

NOTE 3. The function arctan is an odd function and for $t < -1$ the provided estimates (9) are $-\frac{\pi}{2} - \frac{1}{2t} < \arctan(t) < -\frac{\pi}{2} - \frac{1}{t}$.

LEMMA 10. *Let $\alpha > 0$ and $b > a > 0$ be constants. For $t > t_0 \geqslant \alpha + a/b$, let*

$$\tilde{A}(t) := \frac{\pi}{ab}\log\left(\frac{t}{2\pi}\right) - \frac{\log\frac{t}{2\pi}}{b^2(t-\alpha)} - \kappa$$

and

$$\tilde{B}(t) := \left(\frac{\pi}{ab} + \frac{1}{2b^2}\right)\log\frac{t+1}{2\pi} + \frac{\log(t+1)}{b^2 t},$$

where $\kappa > 0$ is a constant from Lemma 8 and t_0 is sufficiently large.

Then

$$\tilde{A}(t) < \int_\alpha^\infty \frac{\log(u/2\pi)\,du}{a^2 + b^2(u-t)^2} < \tilde{B}(t).$$

Proof. For the lower bound, by elementary calculation and Lemmas 8 and 9, we obtain

$$\begin{aligned}
\int_\alpha^\infty \frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2} &= \left(\int_\alpha^t + \int_t^\infty\right)\frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2}\\
&> \int_\alpha^t \frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2} + \log\left(\frac{t}{2\pi}\right)\int_t^\infty \frac{du}{a^2+b^2(u-t)^2}\\
&> \frac{1}{ab}\log\left(\frac{t}{2\pi}\right)\arctan\left(\frac{b(t-\alpha)}{a}\right) - \kappa + \frac{\pi/2}{ab}\log\left(\frac{t}{2\pi}\right)\\
&> \frac{\pi}{ab}\log\left(\frac{t}{2\pi}\right) - \frac{\log\frac{t}{2\pi}}{b^2(t-\alpha)} - \kappa = \tilde{A}(t).
\end{aligned}$$

By the same thoughts for the upper bound we get

$$\begin{aligned}
\int_\alpha^\infty \frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2} &= \left(\int_\alpha^{t+1} + \int_{t+1}^\infty\right)\frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2}\\
&< \log\left(\frac{t+1}{2\pi}\right)\int_\alpha^{t+1}\frac{du}{a^2+b^2(u-t)^2} + \frac{1}{b^2}\int_{t+1}^\infty \frac{\log(u/2\pi)\,du}{(u-t)^2}\\
&= \frac{1}{ab}\log\left(\frac{t+1}{2\pi}\right)\left(\arctan\left(\frac{b}{a}\right) + \arctan\left(\frac{t-\alpha}{a/b}\right)\right)\\
&\quad + \frac{\left(1+\frac{1}{t}\right)\log(t+1) - \log 2\pi}{b^2}\\
&< \left(\frac{\pi}{ab} - \frac{t-\alpha+1}{2b^2(t-\alpha)}\right)\log\left(\frac{t+1}{2\pi}\right) + \frac{\left(1+\frac{1}{t}\right)\log(t+1) - \log 2\pi}{b^2}\\
&< \left(\frac{\pi}{ab} + \frac{1}{2b^2}\right)\log\frac{t+1}{2\pi} + \frac{\log(t+1)}{b^2 t} = \tilde{B}(t). \qquad \square
\end{aligned}$$

LEMMA 11. *Let $\alpha > 0$ and $b > a \geqslant 0$ be constants. For $t > \alpha + a/b$, let*

$$\tilde{C}(t) := \frac{1}{4b^2 t}\log\left(\frac{t}{2\pi}\right) - \frac{\alpha}{b^2 t^2}\log\left(\frac{\alpha}{2\pi}\right)$$

and

$$\tilde{D}(t) := \frac{1}{2b^2 t}\log\left(\frac{2t^3}{4\pi^3}\right).$$

Then

$$\tilde{C}(t) < \int_\alpha^\infty \frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2} < \tilde{D}(t).$$

Proof. We do the same as in the proof of the previous lemma. For the lower bound

$$\begin{aligned}
\int_\alpha^\infty \frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2} &= \left(\int_\alpha^t + \int_t^\infty\right)\frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2}\\
&> \frac{1}{ab}\log\left(\frac{\alpha}{2\pi}\right)\left(\arctan\frac{2t}{a/b} - \arctan\frac{t+\alpha}{a/b}\right) + \frac{1}{ab}\log\left(\frac{t}{2\pi}\right)\left(\frac{\pi}{2} - \arctan\frac{2t}{a/b}\right)\\
&> \frac{1}{ab}\log\left(\frac{\alpha}{2\pi}\right)\left(\frac{\pi}{2} - \frac{a/b}{2t} - \frac{\pi}{2} + \frac{a/b}{2(t+\alpha)}\right) + \frac{1}{ab}\log\left(\frac{t}{2\pi}\right)\left(\frac{\pi}{2} - \frac{\pi}{2} + \frac{a/b}{4t}\right)\\
&> \frac{1}{4b^2t}\log\left(\frac{t}{2\pi}\right) - \frac{\alpha}{b^2t^2}\log\left(\frac{\alpha}{2\pi}\right) = \tilde{C}(t).
\end{aligned}$$

And for the upper bound

$$\begin{aligned}
\int_\alpha^\infty \frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2} &= \left(\int_\alpha^t + \int_t^\infty\right)\frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2}\\
&< \log\left(\frac{t}{2\pi}\right)\int_\alpha^t \frac{du}{a^2+b^2(u+t)^2} + \int_t^\infty \frac{\log(u/2\pi)\,du}{b^2(u+t)^2}\\
&= \frac{1}{ab}\log\left(\frac{t}{2\pi}\right)\left(\arctan\left(\frac{2t}{a/b}\right) - \arctan\left(\frac{t+\alpha}{a/b}\right)\right) + \frac{1}{2b^2t}\log\left(\frac{2t}{\pi}\right)\\
&< \frac{1}{ab}\log\left(\frac{t}{2\pi}\right)\left(\frac{\pi}{2} - \frac{a/b}{4t} - \frac{\pi}{2} + \frac{a/b}{t+\alpha}\right) + \frac{1}{2b^2t}\log\left(\frac{2t}{\pi}\right)\\
&< \frac{1}{2b^2t}\log\left(\frac{2t}{\pi}\right) + \frac{1}{b^2t}\log\left(\frac{t}{2\pi}\right) = \tilde{D}(t). \qquad \square
\end{aligned}$$

The next lemma we need is well known as a summation by parts.

LEMMA 12. *Let $\{a_n\}_{n=1}^\infty$ be a sequence of complex numbers and $G(u)$ a continuously differentiable function on $[1,x]$. If $A(u) = \sum_{n\leqslant u} a_n$, then*

$$\sum_{n\leqslant x} a_n G(n) = A(x)G(x) - \int_1^x A(u)G'(u)\,du.$$

Proof. See, for example, Murty [16, p. 18] or Apostol [1, p. 54] for the proof. $\square$

In the below met inequalities numbers are rounded up to two or three decimal places.

LEMMA 13. *Let $\rho = \beta + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Let $a, b > 0$ and $\gamma_1 = 14.134725\ldots$, where $\zeta(1/2 + i\gamma_1) = 0$. If $t > \gamma_1$, then*

$$\sum_{\rho=\beta+i\gamma} \frac{1}{a^2+b^2(t-\gamma)^2} = \sum_{\gamma>0} \frac{1}{a^2+b^2(t-\gamma)^2} + \sum_{\gamma>0} \frac{1}{a^2+b^2(t+\gamma)^2} =: S_1 + S_2,$$

where

$$\left|S_1 - \frac{1}{2\pi}\int_{\gamma_1}^{\infty} \frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2}\right| < \frac{0.22\log t + 0.58\log\log t + 4.58}{a^2} + \frac{0.166}{a^2 t}\left(1+\frac{2.411a}{b}\right)$$

and

$$\left|S_2 - \frac{1}{2\pi}\int_{\gamma_1}^{\infty} \frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2}\right| < \frac{3.811}{a^2+b^2(\gamma_1+t)^2} + \frac{0.045}{ab}.$$

Proof. Since $\zeta(\rho) = \zeta(\bar{\rho}) = 0$ we have that

$$\sum_{\rho=\beta+i\gamma} \frac{1}{a^2+b^2(t-\gamma)^2} = \sum_{\gamma>0} \frac{1}{a^2+b^2(t-\gamma)^2} + \sum_{\gamma>0} \frac{1}{a^2+b^2(t+\gamma)^2} = S_1 + S_2.$$

For S_1, by Lemma 12,

$$S_1 = -\int_{\gamma_1}^{\infty} N(u) f'(u)du,$$

where $f(u) := 1/(a^2+b^2(t-u)^2)$ and the step function $N(u)$ is defined in Lemma 7. Let $N_{up}(u)$ and $N_{low}(u)$ be the corresponding continuous upper and lower bounds of $N(u)$. By Lemma 7,

$$N_{up}(u) = \frac{u}{2\pi}\log\frac{u}{2\pi e} + 0.11\log u + 0.29\log\log u + 3.165 + \frac{25}{48\pi u},$$
$$N_{low}(u) = \frac{u}{2\pi}\log\frac{u}{2\pi e} - 0.11\log u - 0.29\log\log u - 1.415 - \frac{25}{48\pi u}.$$

Let us observe that derivative $f'(u)$ is non-negative for $u \leqslant t$ and $f'(u)$ is negative for $u > t$. As $N_{up}(u), N_{low}(u)$ are continuous functions, then

$$\begin{aligned} S_1 &\leqslant -\int_{\gamma_1}^{t} N_{low}(u) f'(u)du - \int_{t}^{\infty} N_{up}(u) f'(u)du \\ &= -\int_{\gamma_1}^{\infty} \frac{u}{2\pi}\log\frac{u}{2\pi e} f'(u)du \\ &\quad + \int_{\gamma_1}^{t}\left(0.11\log u + 0.29\log\log u + 1.415 + \frac{25}{48\pi u}\right) d\,f(u) \\ &\quad - \int_{t}^{\infty}\left(0.11\log u + 0.29\log\log u + 3.165 + \frac{25}{48\pi u}\right) d\,f(u). \end{aligned}$$

Proceeding the upper estimation of S_1, we obtain

$$\begin{aligned} S_1 \leqslant & \frac{1}{2\pi}\int_{\gamma_1}^{\infty} \frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2} + \frac{\gamma_1}{2\pi}\log\left(\frac{\gamma_1}{2\pi e}\right) f(\gamma_1) \\ & + (f(t)-f(\gamma_1))\left(0.11\log t + 0.29\log\log t + 1.415 + \frac{25}{48\pi\gamma_1}\right) \\ & + f(t)\left(3.165+\frac{25}{48\pi t}\right) - 0.11\int_t^{\infty}\log u\, d\,f(u) - 0.29\int_t^{\infty}\log\log u\, d\,f(u). \end{aligned} \tag{10}$$

For the last two integrals in (10) it holds that

$$\begin{aligned} &-\int_t^{\infty}\log u\, d\,f(u) = f(t)\log t + \int_t^{\infty}\frac{f(u)du}{u} < f(t)\log t + \frac{\pi/2}{ab}\cdot\frac{1}{t}, \\ &-\int_t^{\infty}\log\log u\, d\,f(u) < f(t)\log\log t + \frac{\pi/2}{ab}\cdot\frac{1}{t\log t}. \end{aligned}$$

Therefore

$$\begin{aligned} S_1 < & \frac{1}{2\pi}\int_{\gamma_1}^{\infty}\frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2} + \frac{0.220\log t + 0.580\log\log t + 4.580}{a^2} \\ & + \frac{0.166}{a^2 t}\left(1+\frac{2.413a}{b}\right). \end{aligned}$$

By similar arguments, the lower bound of S_1 is

$$\begin{aligned} S_1 > & \frac{1}{2\pi}\int_{\gamma_1}^{\infty}\frac{\log(u/2\pi)\,du}{a^2+b^2(u-t)^2} - \frac{0.220\log t + 0.580\log\log t + 4.580}{a^2} \\ & - \frac{0.166}{a^2 t}\left(1+\frac{2.413a}{b}\right). \end{aligned}$$

The upper bound of

$$S_2 = -\int_{\gamma_1}^{\infty} N(u)g'(u)du,\; g(u) := 1/(a^2+b^2(t+u)^2),$$

observing that $g(u)$ is decreasing for $u \geqslant 0$, is

$$\begin{aligned} S_2 < & -\int_{\gamma_1}^{\infty} N_{up}(u)g'(u)\,du = -\int_{\gamma_1}^{\infty}\frac{u}{2\pi}\log\frac{u}{2\pi e}\,d\,g(u) \\ & -\int_{\gamma_1}^{\infty}\left(0.11\log u + 0.29\log\log t + 3.165 + \frac{25}{48\pi u}\right) g'(u)\,du \\ = & \frac{1}{2\pi}\int_{\gamma_1}^{\infty}\frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2} + \frac{\gamma_1}{2\pi}\log\left(\frac{\gamma_1}{2\pi e}\right) g(\gamma_1) \\ & -0.11\int_{\gamma_1}^{\infty}\log u\, g'(u)\,du - 0.29\int_{\gamma_1}^{\infty}\log\log u\, g'(u)\,du \qquad (11) \\ & -\int_{\gamma_1}^{\infty}\left(3.165+\frac{25}{48\pi u}\right) g'(u)\,du. \qquad (12) \end{aligned}$$

The integrals in (11) and (12) evaluate to

$$-\int_{\gamma_1}^{\infty}\log u\, g'(u)\,du = g(\gamma_1)\log\gamma_1+\int_{\gamma_1}^{\infty}\frac{du}{u(a^2+b^2(t+u)^2)}<g(\gamma_1)\log\gamma_1+\frac{\pi/2}{\gamma_1 ab},$$
$$-\int_{\gamma_1}^{\infty}\log\log u\, g'(u)\,du<g(\gamma_1)\log\gamma_1+\frac{\pi/2}{\gamma_1 ab},$$
$$-\int_{\gamma_1}^{\infty}\left(3.165+\frac{25}{48\pi u}\right)g'(u)\,du<g(\gamma_1)\left(3.165+\frac{25}{48\pi\gamma_1}\right).$$

Therefore

$$S_2<\frac{1}{2\pi}\int_{\gamma_1}^{\infty}\frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2}+3.811g(\gamma_1)+\frac{0.045}{ab}.$$

Arguing the same, the lower bound of S_2 is

$$S_2>\frac{1}{2\pi}\int_{\gamma_1}^{\infty}\frac{\log(u/2\pi)\,du}{a^2+b^2(u+t)^2}-3.811g(\gamma_1)-\frac{0.045}{ab}.$$

The proof follows by collecting the upper and lower bounds of S_1 and S_2. □

3. Proof of Theorem 4

In this section, we prove the Theorem 4.

Proof. [Theorem 4] Let $1/2<\sigma<1$. Since $0<(\sigma-1/2)^2<1/4$, we have that

$$\sum_{\rho=1/2+i\gamma}\frac{\sigma-1/2}{1/4+(t-\gamma)^2}<\Re\sum_{\rho=1/2+i\gamma}\frac{1}{s-\rho}<\sum_{\rho=1/2+i\gamma}\frac{(\sigma-1/2)^{-1}}{1+4(t-\gamma)^2}. \tag{13}$$

Recall that c denotes the lower bound of the part of all nontrivial zeros of $\zeta(s)$ on the line $1/2+it$, $0<t<T$, i.e. $c\leqslant N_{1/2}(T)/N(T)$, $T\geqslant\gamma_1=14.134725\ldots$, where $\zeta(1/2+i\gamma_1)=0$. Then, in view of (5), Lemma 7 gives the continuous lower and upper bounds of $N_{1/2}(T)$, and by Lemma 7 we get

$$\begin{aligned}\Re\sum_{\rho=1/2+i\gamma}\frac{1}{s-\rho}&=\sum_{\rho=1/2+i\gamma}\frac{\sigma-1/2}{(\sigma-1/2)^2+(t-\gamma)^2} \\ &>\frac{c(\sigma-1/2)}{2\pi}\int_{\gamma_1}^{\infty}\left(\frac{\log(u/2\pi)}{(\sigma-1/2)^2+(u-t)^2}+\frac{\log(u/2\pi)}{(\sigma-1/2)^2+(u+t)^2}\right)du \\ &\quad+c(\sigma-1/2)M(t),\end{aligned} \tag{14}$$

where $M(t)=O(\log t)$ as $t\to\infty$ and the explicit lower and upper bounds of $M(t)$ for $t>\gamma_1=14.134725\ldots$, where $\zeta(1/2+i\gamma_1)=0$, are given in Lemma 13.

Combining (13) and (14), applying Lemmas 10, 11 and 13 with $a = 1/2$, $b = 1$ and choosing $\alpha = \gamma_1 = 14.134725\ldots$, where γ_1 is the imaginary part of lowest non-trivial zero of $\zeta(s)$ on the critical line $1/2 + it$, $t > 0$, for the lower bound we get

$$\begin{aligned}\Re \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} &> \frac{c(\sigma-1/2)}{2\pi} \int_{\gamma_1}^{\infty} \left(\frac{\log(u/2\pi)}{1/4+(u-t)^2} + \frac{\log(u/2\pi)}{1/4+(u+t)^2} \right) du \\ &+ c(\sigma-1/2) \left(-0.88 \log t - 2.32 \log\log t - 18.41 - \frac{1.465}{t} - \frac{3.811}{0.25+(\gamma_1+t)^2} \right) \\ &> c(\sigma-1/2) \left(0.12 \log \frac{t}{2\pi} - 2.32 \log\log t - 18.432 - \varepsilon_1(t) \right),\end{aligned}$$

where

$$\varepsilon_1(t) = \left(\frac{1}{8\pi t} - \frac{1}{2\pi(t-\gamma_1)} \right) \log \frac{t}{2\pi} - \frac{1.465}{t} - \frac{\gamma_1 \log \frac{\gamma_1}{(2\pi)}}{2\pi t^2} - \frac{3.811}{0.25+(\gamma_1+t)^2}. \tag{15}$$

We check that

$$0.12 \log \frac{t}{2\pi} - 2.32 \log\log t - 18.432 \geqslant 49 \times 10^{-6}, \quad |\varepsilon_1(t)| \leqslant 1.65 \times 10^{-113},$$

when $t \geqslant 1.984 \times 10^{114}$.

By the same arguments and Lemma 13 with $a = 1$ and $b = 2$, for the upper bound, we get

$$\begin{aligned}\Re \sum_{\rho=1/2+i\gamma} \frac{1}{s-\rho} &< \frac{(\sigma-1/2)^{-1}}{2\pi} \int_{\gamma_1}^{\infty} \left(\frac{\log(u/2\pi)}{1+4(u-t)^2} + \frac{\log(u/2\pi)}{1+4(u+t)^2} \right) du \\ &+ (\sigma-1/2)^{-1} \left(0.22 \log t + 0.58 \log\log t + 4.603 + \frac{0.367}{t} + \frac{3.811}{1+4(\gamma_1+t)^2} \right) \\ &< (\sigma-1/2)^{-1} \left(0.49 \log \frac{t}{2\pi} + 0.58 \log\log t + 4.603 + \varepsilon_2(t) \right),\end{aligned}$$

where

$$\varepsilon_2(t) = \frac{0.637}{t} + \frac{3.811}{1+4(t+\gamma_1)^2} + \frac{\log(t+1) + \frac{1}{2} \log \frac{2t^3}{4\pi^3}}{8\pi t}. \quad \square \tag{16}$$

4. The positivity area of $\Re \xi'/\xi(s)$ if there are zeros off the critical line

In this section, we assume that the Riemann hypothesis fails by three different scenarios:

I. There is only one zero in the region $1/2 < \sigma < 1$, $t > 0$

II. There is a finite number $n \geqslant 2$ of zeros off the critical line

III. There are infinitely many zeros off the critical line.

I. Assume that there is one point $\tilde{\beta}+i\tilde{\gamma}$ such that $\zeta(\tilde{\beta}+i\tilde{\gamma})=0$ when $1/2<\tilde{\beta}<1$, $\tilde{\gamma}>0$. Then, by Theorem 4 with $0<c\leqslant 1$ and estimation,

$$\begin{aligned}\Re\frac{\xi'}{\xi}(s) &= \left(\sigma-\frac{1}{2}\right)\sum_{\rho=1/2+i\gamma}\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2}\\ &\quad+\frac{\sigma-\tilde{\beta}}{(\sigma-\tilde{\beta})^2+(t-\tilde{\gamma})^2}+\frac{\sigma-\tilde{\beta}}{(\sigma-\tilde{\beta})^2+(t+\tilde{\gamma})^2}\\ &\quad+\frac{\sigma-(1-\tilde{\beta})}{(\sigma-(1-\tilde{\beta}))^2+(t-\tilde{\gamma})^2}+\frac{\sigma-(1-\tilde{\beta})}{(\sigma-(1-\tilde{\beta}))^2+(t+\tilde{\gamma})^2}\\ &> c\cdot 0.11\left(\sigma-\frac{1}{2}\right)\log\frac{t}{2\pi}+\frac{\sigma-\tilde{\beta}}{(\sigma-\tilde{\beta})^2+(t-\tilde{\gamma})^2}+O\left(\frac{\log\log t}{\log t}\right)>0\end{aligned}$$

if

$$(\sigma,t)\in\left\{\frac{\sigma-\tilde{\beta}}{(\sigma-\tilde{\beta})^2+(t-\tilde{\gamma})^2}>-c\cdot 0.11\left(\sigma-\frac{1}{2}\right)\log\frac{t}{2\pi}\right\} \tag{17}$$

and t is sufficiently large that $\log\log t/\log t$ is negligible. The region of (σ,t) given by (17) might have the following gray view given in Figure 1. Figure 1 was obtained with some chosen point $\tilde{\beta}+i\tilde{\gamma}$ and $c=1-1/(N+1)\approx 1$, where $N=1.236\times 10^{13}$, see the description of N before Theorem 4.

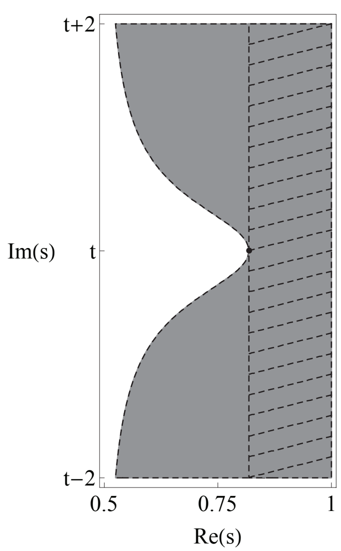

Figure 1: *The entire gray region satisfies the hypothetical inequality (17). Theorem 2 or 3 would provide a dashed gray strip only, where* $\Re\xi'/\xi(s)>0$ *if there is zero off the critical line.*

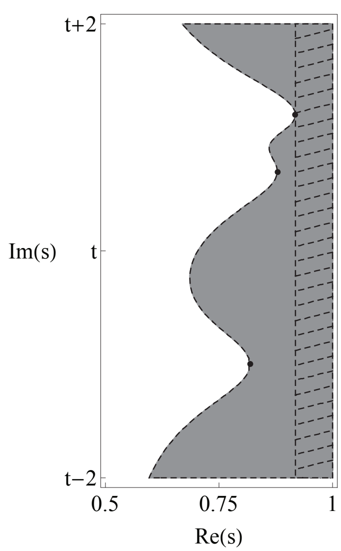

Figure 2: *The entire gray region satisfies the hypothetical inequality (18). Theorem 2 or 3 would provide a dashed gray strip only, where* $\Re\xi'/\xi(s)>0$ *if there are zeros off the critical line.*

II. Assume that there is a finite number $n \geqslant 2$ of points $\tilde{\beta}_k + i\tilde{\gamma}_k$, $k = 1, 2, \ldots, n$ such that $\zeta(\tilde{\beta}_k + i\tilde{\gamma}_k) = 0$ for $1/2 < \tilde{\beta}_k < 1$, $t > 0$. Then, by Theorem 4 with $0 < c \leqslant 1$ and previous means,

$$\Re \frac{\xi'}{\xi}(s) > c \cdot 0.11 \left(\sigma - \frac{1}{2}\right) \log \frac{t}{2\pi} + \sum_{k=1}^{n} \frac{\sigma - \tilde{\beta}_k}{(\sigma - \tilde{\beta}_k)^2 + (t - \tilde{\gamma}_k)^2} + O\left(\frac{\log\log t}{\log t}\right) > 0$$

if

$$(\sigma, t) \in \left\{ \sum_{k=1}^{n} \frac{\sigma - \tilde{\beta}_k}{(\sigma - \tilde{\beta}_k)^2 + (t - \tilde{\gamma}_k)^2} > -c \cdot 0.11 \left(\sigma - \frac{1}{2}\right) \log \frac{t}{2\pi} \right\} \tag{18}$$

and t is sufficiently large that $\log\log t / \log t$ is negligible. The region of (σ, t) given by (18) might have the following gray view given in Figure 2. Figure 2 was obtained with some chosen $\tilde{\beta}_k$ and $\tilde{\gamma}_k$ (the black points in Figure 2 are $\tilde{\beta}_k + i\tilde{\gamma}_k$), and $c = 1 - n/(N + n) \approx 1$ assuming that the size of n in (18) is negligible comparing it to N described before Theorem 4.

III. Assume that there are infinitely many points $\tilde{\beta}_k + i\tilde{\gamma}_k$, $k = 1, 2, \ldots$ such that $\zeta(\tilde{\beta}_k + i\tilde{\gamma}_k) = 0$ for $1/2 < \tilde{\beta}_k < 1$, $t > 0$.

Then, by the same arguments as under scenarios I. and II.,

$$\begin{aligned} \Re \frac{\xi'}{\xi}(s) &> c \cdot 0.11 \left(\sigma - \frac{1}{2}\right) \log \frac{t}{2\pi} \\ &+ \sum_{\substack{\tilde{\rho} = \tilde{\beta}_k + i\tilde{\gamma}_k \\ \tilde{\gamma}_k > 0}} \frac{\sigma - \tilde{\beta}_k}{(\sigma - \tilde{\beta}_k)^2 + (t - \tilde{\gamma}_k)^2} - \sum_{\tilde{\gamma}_k > 0} \frac{1/2}{(t + \tilde{\gamma}_k)^2} + O\left(\frac{\log\log t}{\log t}\right) > 0 \end{aligned} \tag{19}$$

if

$$(\sigma, t) \in \left\{ \sum_{\substack{\tilde{\rho} = \tilde{\beta}_k + i\tilde{\gamma}_k \\ \tilde{\gamma}_k > 0}} \frac{(\sigma - \tilde{\beta}_k)}{(\sigma - \tilde{\beta}_k)^2 + (t - \tilde{\gamma}_k)^2} > -c \cdot 0.11 \left(\sigma - \frac{1}{2}\right) \log \frac{t}{2\pi} \right\}$$

and t is sufficiently large. We note that $\frac{1}{2} \sum_{\tilde{\gamma}_k > 0} (t + \tilde{\gamma}_k)^{-2} = O(\log t / t)$, $t \to \infty$ in (19), see Lemmas 11 and 13.

The lower bound of $\Re \sum_{\rho = 1/2 + i\gamma} (s - \rho)^{-1}$ in Theorem 4 might be interpreted as an "explicit inertia of positivity" of $\Re \xi'/\xi(s)$. This lower bound, together with the pictures in Figure 1 and 2, basically states that the positivity of $\Re \xi'/\xi(s)$ recovers asymptotically near the critical line for some t which is vertically far enough from the hypothetical zero of $\zeta(s)$ lying off the critical line. This effect can also be intuitively echoed by the equality

$$\begin{aligned} \Re \sum_{\rho = 1/2 + i\gamma} \frac{1}{s - \rho} = \left(\sigma - \frac{1}{2}\right) \Bigg(&\frac{1}{(\sigma - 1/2)^2 + (t - \gamma_1)^2} + \frac{1}{(\sigma - 1/2)^2 + (t + \gamma_1)^2} \\ &+ \frac{1}{(\sigma - 1/2)^2 + (t - \gamma_2)^2} + \frac{1}{(\sigma - 1/2)^2 + (t + \gamma_2)^2} + \ldots \Bigg), \end{aligned} \tag{20}$$

where $\gamma_1, \gamma_2, \ldots$ denote the imaginary parts of the non-trivial zeros of $\zeta(s)$ on the critical line. By taking such s $(\sigma > 1/2)$ which is close enough to some zero $\rho_1 = 1/2 + i\gamma_1$, $\rho_2 = 1/2 + i\gamma_2$, $\ldots$ by (20) we see that $\Re\xi'/\xi(s)$ must be positive at least in some small environment to the right of $\rho_1, \rho_2, \ldots$ despite if there are zeros of $\zeta(s)$ off the critical line.

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