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SUMUDU TRANSFORM AND THE STABILITY OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a new integral transform, namely, Sumudu transform and we apply the transform to investigate the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of second order linear differential equations.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [46] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [6] for additive mappings and by Th. M. Rassias [42] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. Since then Hyers result has seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution [3, 9, 20, 27, 32]. Furthermore, useful non-stability results for various functional equations have been given by Gajda [15], Bodaghi, Senthil Kumar and Rassias [10] and Alessa *et al.* [2]. For more results on functional equations and applications, there are some published books [1, 12, 13, 19, 25, 30, 43].

The theory of stability is an important branch of the qualitative theory of differential equations. During last decades many interesting results have been investigated on different types differential equation (for more details, see [31, 35, 40, 44]).

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A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\phi\left(f,x,x',x'',\cdots,x^{(n)}\right)=0$ has the Hyers-Ulam stability if for a given $\varepsilon>0$ and a function x such that

$$\left|\phi\left(f,x,x',x'',\cdots,x^{(n)}\right)\right|\leqslant\varepsilon,$$

there exists a solution x_a of the differential equation such that $|x(t)-x_a(t)|\leqslant K(\varepsilon)$ and

$$\lim_{\varepsilon\to 0}K(\varepsilon)=0.$$

If the preceding statement is also true when we replace ε and $K(\varepsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ,φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Alsina and Ger [5] investigated the stability of differential equation $x'(t)-x(t)$. They proved the following celebrated theorem.

THEOREM 1. [5] *Let $f:I\to\mathbb{R}$ be a differentiable function, which is a solution of the following differential inequality $\|x'(t)-x(t)\|\leqslant\varepsilon$, where I is an open interval of $\mathbb{R}$. Then there is a solution $g:I\to\mathbb{R}$ of $x'(t)=x(t)$ such that for any $t\in I$, we have $\|f(t)-g(t)\|\leqslant 3\varepsilon$.*

This result was generalized by Takahasi *et al.* [45], who proved the Hyers-Ulam stability for the Banach space valued differential equation $y'(t)=\lambda y(t)$. Furthermore, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [21, 22, 23, 24, 26, 47].

As well known, many different methods for solving differential equations have been used to study the Hyers-Ulam stability problem for various differential equation. But using initial conditions are have more significant advantage for solving differential equations. In 2011, Gavruta, Jung and Li [17] are studied the Hyers-Ulam stability for second order linear differential equation $y''+\beta(x)y=0$ with initial and boundary conditions using Taylor formula.

Similarly, many different methods for solving differential equations have been used to study the Hyers-Ulam stability problem for various differential equation. But using transform techniques are also have more significant advantage for solving differential equations with initial conditions.

In 2014, Alqifiary and Jung [4] investigated the generalized Hyers-Ulam stability of

$$x^{(n)}(t)+\sum_{k=0}^{n-1}\alpha_k\,x^{(k)}(t) \;=\; f(t),$$

by using the Laplace transform method. In 2020, Murali and Selvan [36] established the different forms of Mittag-Leffler-Hyers-Ulam stability of the first order linear differential equation for both homogeneous and non-homogeneous cases by using Laplace transformation. The Hyers-Ulam stability of differential equations has been given attention and it was established by many authors (see [11, 14, 33, 34]).

In 2020, Murali, Selvan and Park [38] investigated the Hyers-Ulam stability of various differential equations using Fourier transform method (see also [41]). Recently, Jung, Selvan and Murali [28] established the various forms of Hyers-Ulam stability of the first-order linear differential equations with constant coefficients by using Mahgoub integral transform. Very recently, Murali, Selvan, Park and Lee [39] investigated the different forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of second order linear differential equation of the form $u'' + \mu^2 u = q(t)$ by using Abooth transform method (see also [37]).

In this paper, our main goal is to establish the Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the following second order linear differential equations

$$u''(t) + \mu^2 u = 0 \tag{1}$$

and

$$u''(t) + \mu^2 u = q(t) \tag{2}$$

for all $t \in I$, $u(t) \in C^2(I)$ and $q(t) \in C(I)$, $I = [a,b]$, $-\infty < a < b < \infty$, by using a new integral transform, i.e., Sumudu transform method.

2. Preliminaries

In this section, we introduce some standard notations and definitions which will be very useful to obtain our main results.

Throughout this paper, $\mathbb{K}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0,\infty) \to \mathbb{K}$ is said to be of exponential order if there exist constants $A, B \in \mathbb{R}$ such that $|f(t)| \leqslant Ae^{tB}$ for all $t > 0$.

Consider the set

$$\mathscr{N} = \left\{ f(t) : \exists M,\ \ \eta_1, \eta_2 > 0,\ \ |f(t)| < M\, e^{-\xi/\eta_j} \text{ if } (-1)^j \times [0,\infty) \right\}.$$

For a given function $f(t)$ in the set $\mathscr{N}$, M must be a finite number, η_1 and η_2 may be finite or infinite.

Watugala [48] introduced a new transform and named as Sumudu transform which is defined by the following definition:

DEFINITION 1. [7, 8] The Sumudu integral transform is defined, for a function of exponential order $f(t)$, by

$$\mathscr{S}\{f(t)\} = \int_0^\infty \frac{1}{\xi} f(t)\, e^{-t/\xi}\, dt = F(\xi),$$

or

$$\mathscr{S}\{f(t)\} = F(\xi) = \int_0^\infty f(t\xi)\, e^{-t} dt$$

provided that the integral exists for some ξ, where $\xi \in (\eta_1, \eta_2)$. $\mathscr{S}$ is called the Sumudu transform operator.

THEOREM 2. [7, 8] *Let $F(\xi)$ and $G(\xi)$ be the Sumudu transform of $f(t)$ and $g(t)$ respectively. If*

$$h(t) = f(t) * g(t) = \int_0^t f(x)\, g(t-x)\, dx$$

where $*$ denotes convolution, then the Sumudu transform of $h(t)$ is $\xi F(\xi)G(\xi)$. That is,

$$\mathscr{S}\{h(t)\} = \mathscr{S}\{f(t) * g(t)\} = \xi F(\xi)G(\xi).$$

DEFINITION 2. [29] The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} z^k,$$

where $z, \alpha \in \mathbb{C}$ and $Re(\alpha) > 0$. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k} = e^z.$$

DEFINITION 3. [29] A generalization of $E_\alpha(z)$ is defined as a function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} z^k,$$

where $z, \alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

Let $I, J \subseteq \mathbb{R}$ be intervals. Throughout this paper, we denote the space of k continuously differentiable functions from I to J by $C^k(I,J)$ and denote $C^k(I,I)$ by $C^k(I)$. Furthermore, $C(I,J) = C^0(I,J)$ denotes the space of continuous functions from I to J. In addition, $\mathbb{R}_+ := [0,\infty)$. From now on, we assume that $I = [a,b]$, where $-\infty < a < b < \infty$.

Here, we give some definitions of various forms of Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of differential equations.

DEFINITION 4. We say that the differential equation (2) has the Hyers-Ulam stability if there exists a constant $L > 0$ satisfying the following condition: For every $\varepsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \leqslant \varepsilon$$

for all $t \in I$, there exists some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ and

$$|u(t) - v(t)| \leqslant L\varepsilon$$

for all $t \in I$. We call such L as the Hyers-Ulam stability constant for (2).

DEFINITION 5. We say that the differential equation (2) has the Hyers-Ulam-Rassias stability with respect to $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ if there exists a constant $L_\phi > 0$ such that for every $\varepsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \leqslant \varepsilon\phi(t)$$

for all $t \in I$, there exists some $v \in C^2(I)$ satisfying the differential equation $v''(t) + \mu^2 v = q(t)$ and

$$|u(t) - v(t)| \leqslant L_\phi \varepsilon\phi(t)$$

for all $t \in I$. We call such L as the Hyers-Ulam-Rassias stability constant for (2).

DEFINITION 6. We say that the differential equation (2) has the Mittag-Leffler-Hyers-Ulam stability if there exists a positive constant L satisfying the following condition: For every $\varepsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \leqslant \varepsilon E_\alpha(t)$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying the linear differential equation $v''(t) + \mu^2 v = q(t)$ and

$$|u(t) - v(t)| \leqslant L\varepsilon E_\alpha(t)$$

for all $t \in I$. We call such L as the Mittag-Leffler-Hyers-Ulam stability constant for (2).

DEFINITION 7. We say that the differential equation (2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability with respect to $\phi : (0,\infty) \to (0,\infty)$ if there exists a positive constant L_ϕ satisfying the following condition: For every $\varepsilon > 0$ and some $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \leqslant \phi(t)\varepsilon E_\alpha(t)$$

for all $t \in I$, there exists a solution $v \in C^2(I)$ satisfying the linear differential equation $v''(t) + \mu^2 v = q(t)$ and $|u(t) - v(t)| \leqslant L_\phi \phi(t)\varepsilon E_\alpha(t)$ for all $t \in I$. We call such L_ϕ as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for (2).

3. Hyers-Ulam stability for non-homogeneous differential equation (2) by Sumudu Transforms

Throughout this paper, $\mathbb{K}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

In this section, we investigate the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the non-homogeneous differential equation (2).

Firstly, we prove the Hyers-Ulam stability of the linear differential equation (2) by applying Sumudu transforms.

THEOREM 3. *Let $\varepsilon > 0$ be given. If $u(t)$ is a twice continuously differentiable function, then the non-homogeneous linear differential equation (2) has Hyers-Ulam stability.*

Proof. For every $\varepsilon > 0$ and for each solution $u(t) \in C^2(I)$ satisfying the inequality

$$\left|u''(t) + \mu^2 u - q(t)\right| \leqslant \varepsilon \tag{3}$$

for all $t \in I$, we should prove that there exists a real number $\mathscr{K} > 0$ which is independent of ε and u such that $|u(t) - v(t)| \leqslant \mathscr{K}\varepsilon$, for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ for all $t \in I$. Define a function $p : (0,\infty) \to \mathbb{K}$ such that $p(t) =: u''(t) + \mu^2 u(t) - q(t)$ satisfies $|p(t)| \leqslant \varepsilon$. Taking the Sumudu transform to $p(t)$, we have

$$\begin{aligned}
\mathscr{S}\{p(t)\} &= \mathscr{S}\{u''(t)\} + \mathscr{S}\{\mu^2 u(t)\} - \mathscr{S}\{q(t)\} \\
&= \frac{1}{\xi^2}\left[\mathscr{S}\{u\} - u(0) - \xi\, u'(0)\right] + \mu^2 \mathscr{S}\{u\} - \mathscr{S}\{q(t)\} \\
&= \frac{(1+\xi^2\mu^2)\mathscr{S}\{u\} - u(0) - \xi\, u'(0) - \xi^2\, \mathscr{S}\{q(t)\}}{\xi^2}
\end{aligned}$$

and thus we have

$$\mathscr{S}\{u\} = \frac{\xi^2\, \mathscr{S}\{p\} + u(0) + \xi\, u'(0) + \xi^2\, \mathscr{S}\{q\}}{1+\xi^2\mu^2}. \tag{4}$$

The above equality (4) shows that a function $u_0 : (0,\infty) \longrightarrow \mathbb{K}$ is a solution of (2) if and only if

$$(1+\xi^2\mu^2)\mathscr{S}\{u_0\} - u_0(0) - u_0'(0)\,\xi = \xi^2\, \mathscr{S}\{q\}.$$

If there exist constants l and m in $\mathbb{K}$ such that $1+\xi^2\mu^2 = (1-l\xi)(1-m\xi)$ with $l+m=0$ and $lm = \mu^2$, then (4) becomes

$$\mathscr{S}\{u\} = \frac{\xi^2\, \mathscr{S}\{p\} + u(0) + \xi\, u'(0) + \xi^2\, \mathscr{S}\{q\}}{(1-l\xi)(1-m\xi)}. \tag{5}$$

Set $r(t) = \dfrac{e^{lt} - e^{mt}}{l-m}$ and

$$v(t) = u(0)\left(\frac{l\, e^{lt} - m\, e^{mt}}{l-m}\right) + u'(0)\, r(t) + [(r*q)(t)].$$

Then $v(0) = u(0)$ and $u'(0) = v'(0)$. Once more, applying the Sumudu transform to $v(t)$, we have

$$\mathscr{S}\{v\} = \frac{u(0) + \xi\, u'(0) + \xi^2\, \mathscr{S}\{q\}}{(1-l\xi)(1-m\xi)}. \tag{6}$$

On the other hand, we will have

$$\mathscr{S}\{v''(t)+\mu^2 v\} = \frac{(1+\xi^2\mu^2)\mathscr{S}\{v\} - v(0) - \xi\, v'(0)}{\xi^2}.$$

Using (6), the last equality becomes $\mathscr{S}\{v''(t)+\mu^2 v\} = \mathscr{S}\{q\}$. Since $\mathscr{S}$ is a one-to-one operator and linear, $v''(t)+\mu^2 v = q(t)$. It shows that $v(t)$ is a solution of the differential equation (2). Now, the relations (5) and (6) necessitate that

$$\mathscr{S}\{u(t)-v(t)\} = \mathscr{S}\{u\} - \mathscr{S}\{v\} = \frac{\xi^2\,\mathscr{S}\{p\}}{(1-l\xi)(1-m\xi)} = \mathscr{S}\{p(t)*r(t)\}$$

and hence $u(t)-v(t) = p(t)*r(t)$. Taking modulus on both sides of the last equality and using $|p(t)| \leqslant \varepsilon$, we get

$$|u(t)-v(t)| = |p(t)*r(t)| \leqslant \varepsilon \left| \int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l-m} \right) dt \right| \leqslant \mathscr{K}\,\varepsilon$$

for all $t>0$, where

$$\begin{aligned} \mathscr{K} &= \left| \int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l-m} \right) dx \right| \\ &\leqslant \frac{1}{|l-m|} \left\{ e^{\Re(l)t} \int_0^t e^{-\Re(l)x} dx + e^{\Re(m)t} \int_0^t e^{-\Re(m)x} dx \right\} \\ &\leqslant \frac{\mathscr{L}}{|l-m|}, \end{aligned}$$

the integrals $\int_0^t e^{-\Re(l)x} dx$ and $\int_0^t e^{-\Re(m)x} dx$ exist. Therefore,

$$|u(t)-v(t)| \leqslant \frac{\mathscr{L}}{|l-m|}\,\varepsilon = \mathscr{K}\,\varepsilon.$$

Thus, the linear differential equation (2) has the Hyers-Ulam stability. □

In analogous to Theorem 3, we have the following result which shows the Hyers-Ulam-Rassias stability of the differential equation (2).

THEOREM 4. *The non-homogeneous linear differential equation* (2) *has Hyers-Ulam-Rassias stability.*

Proof. Let $\varepsilon > 0$ and $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $u(t) \in C^2(I)$ satisfies

$$\left| u''(t) + \mu^2 u - q(t) \right| \leqslant \varepsilon \phi(t) \tag{7}$$

for all $t \in I$. We have to show that there exists a real number $\mathscr{K}_\phi > 0$ such that

$$|u(t) - v(t)| \leqslant \mathscr{K}_\phi \varepsilon \phi(t)$$

for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ for all $t \in I$. Define a function $p : (0,\infty) \to \mathbb{K}$ by $p(t) =: u''(t) + \mu^2 u(t) - q(t)$ for all $t > 0$. In view of (7), we have $|p(t)| \leqslant \varepsilon\phi(t)$. Now, taking the Sumudu transform to $p(t)$, we get

$$\mathscr{S}\{u\} = \frac{\xi^2\ \mathscr{S}\{p\} + u(0) + \xi\ u'(0) + \xi^2\ \mathscr{S}\{q\}}{1+\xi^2\mu^2}. \tag{8}$$

In addition, in light of the relation (8), a function $u_0 : (0,\infty) \to \mathbb{K}$ is a solution of (2) if and only if

$$(1+\xi^2\mu^2)\mathscr{S}\{u_0\} - u_0(0) - \xi\ u_0'(0) = \xi^2\ \mathscr{S}\{q\}.$$

Assume that there exist constants l and m in $\mathbb{K}$ such that $1+\xi^2\mu^2 = (1-l\xi)(1-m\xi)$ with $l+m=0$ and $lm=\mu^2$. However, (8) becomes

$$\mathscr{S}\{u\} = \frac{\xi^2\ \mathscr{S}\{p\} + u(0) + \xi\ u'(0) + \xi^2\ \mathscr{S}\{q\}}{(1-l\xi)(1-m\xi)}. \tag{9}$$

Putting $r(t) = \dfrac{e^{lt} - e^{mt}}{l-m}$ and

$$v(t) = u(0)\left(\frac{l\ e^{lt} - m\ e^{mt}}{l-m}\right) + u'(0)r(t) + [(r*q)(t)],$$

one can easily obtain $v(0) = u(0)$ and $u'(0) = v'(0)$. Taking the Sumudu transform to $v(t)$, we have

$$\mathscr{S}\{v\} = \frac{u(0) + \xi\ u'(0) + \xi^2\ \mathscr{S}\{q\}}{(1-l\xi)(1-m\xi)}. \tag{10}$$

Applying now (10), we obtain $\mathscr{S}\{v''(t)+\mu^2 v\} = \mathscr{S}\{q\}$. The last equality implies that

$$v''(t) + \mu^2 v(t) = q(t).$$

This means that $v(t)$ is a solution of the non homogeneous differential equation (2). Hence, by (9) and (10), we obtain

$$\mathscr{S}\{u(t)-v(t)\} = \frac{\xi^2\ \mathscr{S}\{p\}}{(1-l\xi)(1-m\xi)} = \mathscr{S}\{p(t)*r(t)\}.$$

Thus $u(t) - v(t) = p(t)*r(t)$. Then by using $|p(t)| \leqslant \varepsilon\phi(t)$, we get

$$|u(t)-v(t)| \leqslant \varepsilon \left| \int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l-m}\right)\ \phi(t)\ dx \right| \leqslant \mathscr{K}_\phi\ \varepsilon\phi(t)$$

for all $t > 0$, where

$$\begin{aligned}
\mathscr{K}_\phi &= \left| \int_0^t \left(\frac{e^{l(t-x)} - e^{m(t-x)}}{l-m} \right) \phi(x)\, dx \right| \\
&\leqslant \frac{1}{|l-m|} \left\{ e^{\Re(l)t} \int_0^t e^{-\Re(l)x}\, \phi(x)\, dx + e^{\Re(m)t} \int_0^t e^{-\Re(m)x}\, \phi(x)\, dx \right\} \\
&\leqslant \frac{\mathscr{L}_\phi\, \phi(t)}{|l-m|},
\end{aligned}$$

and the integrals $\int_0^t e^{-\Re(l)x}\, \phi(x)\, dx$ and $\int_0^t e^{-\Re(m)x}\, \phi(x)\, dx$ exist for all $t > 0$ and an integrable function ϕ. Hence

$$|u(t) - v(t)| \leqslant \frac{\mathscr{L}_\phi\, \phi(t)}{|l-m|}\, \varepsilon = \mathscr{K}_\phi\, \varepsilon\phi(t).$$

This finishes the proof. □

4. Application of Theorem 3

By using the same approach as applied in Theorem 3, we can also prove that the following theorem which shows the Mittag-Leffler-Hyers-Ulam stability of the differential equation (2). The method of the proof is similar, but we include it for the sake of completeness.

THEOREM 5. *The differential equation* (2) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. For every $\varepsilon > 0$, and for each solution $u(t) \in C^2(I)$ satisfying the inequality

$$\left| u''(t) + \mu^2 u - q(t) \right| \leqslant \varepsilon E_\alpha(t) \tag{11}$$

for all $t \in I$, we prove that there exists a real number $\mathscr{K} > 0$ which is independent of ε and u such that $|u(t) - v(t)| \leqslant \mathscr{K}\, \varepsilon E_\alpha(t)$ for some $v \in C^2(I)$ satisfying $v''(t) + \mu^2 v = q(t)$ for all $t \in I$. Then the function $p : (0,\infty) \to \mathbb{K}$ defined by $p(t) =: u''(t) + \mu^2 u(t) - q(t)$ satisfies $|p(t)| \leqslant \varepsilon E_\alpha(t)$. Then by using Theorem 3 and using $|p(t)| \leqslant \varepsilon E_\alpha(t)$, we get

$$|u(t) - v(t)| \leqslant \mathscr{K}\ \varepsilon E_\alpha(t),$$

for all $t > 0$, where

$$\mathscr{K} = \frac{1}{|l-m|}\left\{ e^{\Re(l)t}\int_0^t e^{-\Re(l)x}dx + e^{\Re(m)t}\int_0^t e^{-\Re(m)x}dx \right\}$$
$$\leqslant \frac{\mathscr{L}}{|l-m|},$$

and the integrals $\int_0^t e^{-\Re(l)x}dx$ and $\int_0^t e^{-\Re(m)x}dx$ exist. Thus

$$|u(t)-v(t)| \leqslant \frac{\mathscr{L}\varepsilon E_\alpha(t)}{|l-m|}\,\varepsilon = \mathscr{K}\ \varepsilon E_\alpha(t).$$

So the linear differential equation (2) has the Mittag-Leffler-Hyers-Ulam stability. □

Next, we prove the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the homogeneous linear differential equation (1) by using the Sumudu transform.

If we set $q(t) \equiv 0$ in Theorem 3 and use the inequality

$$\left|u''(t)+\mu^2 u(t)\right| \leqslant \varepsilon \quad (t \geqslant 0)$$

then by applying Sumudu transforms, we can easily prove the Hyers-Ulam stability of the homogeneous linear differential equation (1).

THEOREM 6. *The differential equation* (1) *is Hyers-Ulam stable.*

Proof. Let $\varepsilon > 0$. Suppose that $u(t) \in C^2(I)$ satisfies

$$\left|u''(t)+\mu^2 u(t)\right| \leqslant \varepsilon \tag{12}$$

for all $t \in I$. We prove that there exists a real number $\mathscr{K} > 0$ which is independent of ε and u such that $|u(t)-v(t)| \leqslant \mathscr{K}\varepsilon$ for some $v \in C^2(I)$ satisfying $v''(t)+\mu^2 v = 0$ for all $t \in I$.

Define a function $p : (0,\infty) \to \mathbb{K}$ such that $p(t) =: u''(t)+\mu^2 u(t)$ for all $t > 0$. In view of (12), we have $|p(t)| \leqslant \varepsilon$. Taking the Sumudu transform to $p(t)$, we have

$$\mathscr{S}\{p\} = \frac{(1+\xi^2\mu^2)\mathscr{S}\{u\} - u(0) - \xi\ u'(0)}{\xi^2} \tag{13}$$

and thus

$$\mathscr{S}\{u\} = \frac{\xi^2\ \mathscr{S}\{p\} + u(0) + \xi\ u'(0)}{1+\xi^2\mu^2}.$$

In view of (13), a function $u_0 : (0,\infty) \longrightarrow \mathbb{K}$ is a solution of (1) if and only if

$$(1+\xi^2\mu^2)\mathscr{S}\{u_0\} - u_0(0) - u_0'(0)\ \xi = 0.$$

If there exist constants l and m in $\mathbb{K}$ such that $1+\xi^2\mu^2=(1-l\xi)(1-m\xi)$ with $l+m=0$ and $lm=\mu^2$, then (13) becomes

$$\mathscr{S}\{u\}=\frac{\xi^2\ \mathscr{S}\{p\}+u(0)+\xi\ u'(0)}{(1-l\xi)(1-m\xi)}. \tag{14}$$

Set

$$v(t)=u(0)\left(\frac{l\ e^{lt}-m\ e^{mt}}{l-m}\right)+u'(0)\left(\frac{e^{lt}-e^{mt}}{l-m}\right).$$

We have $v(0)=u(0)$ and $u'(0)=v'(0)$. Taking the Sumudu transform to $v(t)$, we obtain

$$\mathscr{S}\{v\}=\frac{u(0)+\xi\ u'(0)}{(1-l\xi)(1-m\xi)}. \tag{15}$$

On the other hand, using (15), we get $\mathscr{S}\{v''(t)+\mu^2 v\}=0$. Since $\mathscr{S}$ is a one-to-one operator and linear, $v''(t)+\mu^2 v=0$. This means that $v(t)$ is a solution of (1). It follows from (14) and (15) that

$$\begin{aligned}\mathscr{S}\{u\}-\mathscr{S}\{v\}&=\frac{\xi^2\ \mathscr{S}\{p\}+u(0)+\xi\ u'(0)}{(1-l\xi)(1-m\xi)}-\frac{u(0)+\xi\ u'(0)}{(1-l\xi)(1-m\xi)},\\ \mathscr{S}\{u(t)-v(t)\}&=\mathscr{S}\left\{p(t)*\left(\frac{e^{lt}-e^{mt}}{l-m}\right)\right\}.\end{aligned}$$

The above equalities show that

$$u(t)-v(t)=p(t)*\left(\frac{e^{lt}-e^{mt}}{l-m}\right).$$

Taking modulus on both sides and using $|p(t)|\leqslant\varepsilon$, we get

$$\begin{aligned}|u(t)-v(t)|&=\left|p(t)*\left(\frac{e^{lt}-e^{mt}}{l-m}\right)\right|\\ &\leqslant\left|\int_0^t p(x)\left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)\,dx\right|\\ &\leqslant\varepsilon\left|\int_0^t\left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)\,dx\right|\leqslant\mathscr{K}\ \varepsilon\end{aligned}$$

for all $t>0$, where

$$\begin{aligned}\mathscr{K}&=\left|\int_0^t\left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)\,dx\right|\\ &\leqslant\frac{1}{|l-m|}\left\{e^{\Re(l)t}\int_0^t e^{-\Re(l)x}dx+e^{\Re(m)t}\int_0^t e^{-\Re(m)x}dx\right\}\\ &\leqslant\frac{\mathscr{L}}{|l-m|},\end{aligned}$$

and $\int_0^t e^{-\Re(l)x}dx$ and $\int_0^t e^{-\Re(m)x}dx$ exist. Hence $|u(t)-v(t)| \leqslant \frac{\mathscr{L}}{|l-m|}\,\varepsilon = \mathscr{K}\,\varepsilon$. Thus the linear differential equation (1) has the Hyers-Ulam stability. This finishes the proof. $\square$

THEOREM 7. *The differential equation* (1) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let $\varepsilon > 0$. Suppose that $u(t) \in C^2(I)$ satisfies

$$\left|u''(t) + \mu^2 u\right| \leqslant \varepsilon E_\alpha(t) \tag{16}$$

for all $t \in I$. We prove that there exists a real number $\mathscr{K} > 0$ which is independent of ε and u such that $|u(t)-v(t)| \leqslant \mathscr{K}\varepsilon E_\alpha(t)$ for some $v \in C^2(I)$ satisfying $v''(t)+\mu^2 v = 0$ for all $t \in I$.

Let us define a function $p : (0,\infty) \to \mathbb{K}$ such that $p(t) =: u''(t) + \mu^2 u(t)$ for all $t > 0$. In view of (16), we have $|p(t)| \leqslant \varepsilon E_\alpha(t)$. Then by applying Theorem 3, one can obtain that

$$\begin{aligned}
\mathscr{S}\{u(t)-v(t)\} &= \frac{\xi^2\, \mathscr{S}\{p\}}{(1-l\xi)(1-m\xi)} \\
&= \mathscr{S}\left\{p(t) * \left(\frac{e^{lt}-e^{mt}}{l-m}\right)\right\}.
\end{aligned}$$

The above equalities show that

$$u(t)-v(t) = p(t) * \left(\frac{e^{lt}-e^{mt}}{l-m}\right)$$

and by using $|p(t)| \leqslant \varepsilon E_\alpha(t)$, we get

$$|u(t)-v(t)| \leqslant \varepsilon E_\alpha(t) \left|\int_0^t \left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right) dx\right|$$

for all $t > 0$, where

$$\begin{aligned}
\mathscr{K} &= \left|\int_0^t \left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right) dx\right| \\
&\leqslant \frac{1}{|l-m|}\left\{e^{\Re(l)t}\int_0^t e^{-\Re(l)x}dx + e^{\Re(m)t}\int_0^t e^{-\Re(m)x}dx\right\} \\
&\leqslant \frac{\mathscr{L}}{|l-m|},
\end{aligned}$$

and $\int_0^t e^{-\Re(l)x}dx$ and $\int_0^t e^{-\Re(m)x}dx$ exist. Hence $|u(t)-v(t)| \leqslant \mathscr{K}\,\varepsilon E_\alpha(t)$. Therefore, the linear differential equation (1) has the Hyers-Ulam stability. This completes the proof. $\square$

5. Application of Theorem 4

By using the same ideas as used in Theorem 4, we can also prove that the following theorems which shows the Hyers-Ulam-Rassias stability of the differential equation (1). The method of the proof is similar, but we include it for the sake of completeness.

If we set $q(t) \equiv 0$ in Theorem 4 and use the inequality

$$\left|u''(t)+\mu^2 u(t)\right| \leqslant \varepsilon\phi(t) \quad (t \geqslant 0)$$

then we can prove the Hyers-Ulam-Rassias stability of the homogeneous linear differential equation (1).

THEOREM 8. *The homogeneous linear differential equation* (1) *has the Hyers-Ulam-Rassias stability.*

Proof. Assume that $u(t) \in C^2(I)$ satisfies

$$\left|u''(t)+\mu^2 u\right| \leqslant \varepsilon\phi(t) \tag{17}$$

for all $t \in I$, $\varepsilon > 0$ and an integrable function $\phi \in C(\mathbb{R}_+,\mathbb{R}_+)$. We show that there exists a real number $\mathscr{K}_\phi > 0$ such that $|u(t)-v(t)| \leqslant \mathscr{K}_\phi \varepsilon\phi(t)$ for some $v \in C^2(I)$ satisfying $v''(t)+\mu^2 v = 0$ for all $t \in I$.

Define a function $p:(0,\infty) \to \mathbb{K}$ such that $p(t) =: u''(t)+\mu^2 u(t)$ for all $t > 0$. By (17), we have $|p(t)| \leqslant \varepsilon\phi(t)$. Now, taking the Sumudu transform to $p(t)$, we have

$$\mathscr{S}\{p\} = \frac{(1+\xi^2\mu^2)\mathscr{S}\{u\}-u(0)-\xi\, u'(0)}{\xi^2}. \tag{18}$$

We know a function $u_0:(0,\infty) \longrightarrow \mathbb{K}$ is a solution of (1) if and only if

$$(1+\xi^2\mu^2)\mathscr{S}\{u_0\}-u_0(0)-u_0'(0)\,\xi = 0.$$

If there exist constants l and m in $\mathbb{K}$ such that $1+\xi^2\mu^2 = (1-l\xi)(1-m\xi)$ with $l+m=0$ and $lm=\mu^2$, then (18) becomes

$$\mathscr{S}\{u\} = \frac{\xi^2\,\mathscr{S}\{p\}+u(0)+\xi\, u'(0)}{(1-l\xi)(1-m\xi)}. \tag{19}$$

Let $v(t) = u(0)\left(\dfrac{l\,e^{lt}-m\,e^{mt}}{l-m}\right)+u'(0)\left(\dfrac{e^{lt}-e^{mt}}{l-m}\right)$. Then $v(0)=u(0)$ and $u'(0) = v'(0)$. Taking again the Sumudu transform to $v(t)$, we have

$$\mathscr{S}\{v\} = \frac{u(0)+\xi\, u'(0)}{(1-l\xi)(1-m\xi)}. \tag{20}$$

Furthermore, using (20), we get $\mathscr{S}\{v''(t)+\mu^2 v\} = 0$ and so $v''(t)+\mu^2 v = 0$. Applying (19) and (20), we get

$$\mathscr{S}\{u(t)-v(t)\} = \mathscr{S}\left\{p(t)*\left(\frac{e^{lt}-e^{mt}}{l-m}\right)\right\}.$$

Therefore, $u(t)-v(t)=p(t)*\left(\dfrac{e^{lt}-e^{mt}}{l-m}\right)$. Thus by using $|p(t)|\leqslant \varepsilon\phi(t)$, we get

$$|u(t)-v(t)|\leqslant \varepsilon\left|\int_0^t\left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)\phi(t)\,dx\right|$$

for all $t>0$, where

$$\begin{aligned}\mathscr{K}_\phi &= \left|\int_0^t\left(\frac{e^{l(t-x)}-e^{m(t-x)}}{l-m}\right)\phi(x)\,dx\right|\\ &\leqslant \frac{1}{|l-m|}\left\{e^{\Re(l)t}\int_0^t e^{-\Re(l)x}\,\phi(x)\,dx+e^{\Re(m)t}\int_0^t e^{-\Re(m)x}\,\phi(x)\,dx\right\}\\ &\leqslant \frac{\mathscr{L}_\phi\,\phi(t)}{|l-m|},\end{aligned}$$

and $\int_0^t e^{-\Re(l)x}\,\phi(x)\,dx$ and $\int_0^t e^{-\Re(m)x}\,\phi(x)\,dx$ exist for all $t>0$ and an integrable function ϕ. Hence $|u(t)-v(t)|\leqslant \dfrac{\mathscr{L}_\phi\,\phi(t)}{|l-m|}\,\varepsilon=\mathscr{K}_\phi\,\varepsilon\phi(t)$. $\square$

In analogous to Theorem 4, we have the following corollary which shows the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (2).

COROLLARY 1. *For every $\varepsilon>0$, let $u(t)$ be a twice continuously differentiable function on I which satisfies the inequality*

$$\left|u''(t)+\mu^2u-q(t)\right|\leqslant \varepsilon\phi(t)E_\alpha(t)$$

for all $t\in I$. Then there exists a real number $\mathscr{K}_\phi>0$ which is independent of ε and u such that

$$|u(t)-v(t)|\leqslant \mathscr{K}_\phi\varepsilon\phi(t)E_\alpha(t)$$

for some $v\in C^2(I)$ satisfying $v''(t)+\mu^2v=q(t)$ for all $t\in I$.

The following corollary proves the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (1). The method of proof is similar to the proof of Theorem 7 and if we let $q(t)\equiv 0$ in Corollary 1 and use the inequality

$$\left|u''(t)+\mu^2u(t)\right|\leqslant \varepsilon\phi(t)E_\alpha(t)\quad (t\geqslant 0)$$

then we easily establish the Mittag-Leffler-Hyers-Ulam-Rassias stability of the homogeneous linear differential equation (1).

COROLLARY 2. *For every* $\varepsilon > 0$, *let* $u(t)$ *be a twice continuously differentiable function on* I *which satisfies the inequality*

$$\left|u''(t) + \mu^2 u\right| \leqslant \varepsilon \phi(t) E_\alpha(t)$$

for all $t \in I$. *Then there exists a real number* $\mathscr{K}_\phi > 0$ *which is independent of* ε *and* u *such that*

$$|u(t) - v(t)| \leqslant \mathscr{K}_\phi \varepsilon \phi(t) E_\alpha(t)$$

for some $v \in C^2(I)$ *satisfying* $v''(t) + \mu^2 v = 0$ *for all* $t \in I$.

6. Conclusion

In this paper, we introduced a new integral transform, namely, Sumudu transform and we applied the transform to investigate the Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of second order linear differential equations with constant coefficients.

In other words, we established sufficient criteria for the Hyers-Ulam stability of second-order linear differential equations with constant coefficients using the Sumudu transform method. Moreover, this paper provides a new method to investigate the Hyers-Ulam stability of differential equations. This is the first attempt to use the Sumudu transformation to prove the Hyers-Ulam stability for linear differential equations of the second order. Furthermore, this paper shows that the Sumudu transform method is more convenient for investigating the stability problems for linear differential equations with constant coefficients. Readers can also apply this terminology to various problems on differential equations.

Declarations

Availablity of data and materials. Not applicable.

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