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ON SEVERAL NEW RESULTS RELATED TO RICHARD'S INEQUALITY

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(*Communicated by M. Krnić*)

Abstract. The main preoccupation of this article is the characterization of Richard's inequality, which it is closely related to Buzano's inequality. Finally, we present a new approach for Richard's inequality, where we use the Selberg operator.

1. Introduction

Lagrange showed the following identity:

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} a_i b_i\right)^2 + \sum_{1\leqslant i<j\leqslant n} (a_i b_j - a_j b_i)^2 . \tag{1}$$

A consequence of Lagrange's identity is the classical Cauchy-Schwarz inequality, in discrete case, which states: if $\mathbf{a} = (a_1,\ldots,a_n)$ and $\mathbf{b} = (b_1,\ldots,b_n)$ are two n-tuples of real numbers, then

$$(a_1b_1+\ldots+a_nb_n)^2 \leqslant \left(a_1^2+\ldots+a_n^2\right)\left(b_1^2+\ldots+b_n^2\right), \tag{2}$$

with the equality holding if and only if $\mathbf{a} = \lambda\mathbf{b}$. This result is called the *Cauchy-Buniakowski-Schwarz inequality*.

Several refinements of the Cauchy-Buniakowski-Schwarz inequality can be found in some papers (see [3], [6], [9] and [17]). We gave one of them: Ostrowski [17], showing the following: if $\mathbf{x} = (x_1,\ldots,x_n), \mathbf{y} = (y_1,\ldots,y_n)$ and $\mathbf{z} = (z_1,\ldots,z_n)$ are n-tuples of real numbers such that $\mathbf{x}$ and $\mathbf{y}$ are not proportional and

$$\sum_{k=1}^{n} y_k z_k = 0 \text{ and } \sum_{k=1}^{n} x_k z_k = 1, \text{ then} \tag{3}$$

$$\sum_{k=1}^{n} y_k^2 \Big/ \sum_{k=1}^{n} z_k^2 \leqslant \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 - \left(\sum_{k=1}^{n} x_k y_k\right)^2 .$$

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In the framework of an inner product space $\mathscr{X}=(\mathscr{X},\langle\cdot,\cdot\rangle)$ over the field of complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$, the Cauchy–Schwarz inequality (C-S), is given by the following:

$$|\langle x,y\rangle|\leqslant\|x\|\cdot\|y\| \tag{4}$$

for all $x,y\in\mathscr{X}$. The equality holds in (4) if and only if the vectors x and y are linearly dependent, i.e., there exists a nonzero constant $\lambda\in\mathbb{C}$ so that $x=\lambda y$.

Buzano [5] proved an extension of the Cauchy-Schwarz inequality, given by the following:

$$|\langle a,x\rangle\langle x,b\rangle|\leqslant\frac{1}{2}\|x\|^2(|\langle a,b\rangle|+\|a\|\cdot\|b\|) \tag{5}$$

for any $x,a,b\in\mathscr{X}$.

For $a=b$ in inequality (5) we deduce the Cauchy-Schwarz inequality.

Precupanu [20] mentioned an inequality related to the Buzano inequality. For any $x,y,a,b\in\mathscr{X}$, $x,y\neq 0$ we have

$$\begin{aligned}\frac{1}{2}(\langle a,b\rangle-\|a\|\cdot\|b\|)&\leqslant\frac{\langle x,a\rangle\langle x,b\rangle}{\|x\|^2}-\frac{\langle y,a\rangle\langle y,b\rangle}{\|y\|^2}-\frac{2\langle x,a\rangle\langle y,b\rangle\langle x,y\rangle}{\|x\|^2\|y\|^2}\\&\leqslant\frac{1}{2}(\langle a,b\rangle+\|a\|\cdot\|b\|).\end{aligned} \tag{6}$$

In [12], Gavrea proved an extention of Buzano's inequality in an inner product space. For a real inner space $\mathscr{X}$, Richard [21], gave the following inequality

$$\left|\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right|\leqslant\frac{1}{2}\|x\|^2\|a\|\cdot\|b\| \tag{7}$$

for any $x,a,b\in\mathscr{X}$.

In [19], Popa and Raşa showed that, for any $x,a,b\in\mathscr{X}$, we have

$$\left|\Re\left(\langle a,x\rangle\langle x,b\rangle-\frac{1}{2}\|x\|^2\langle a,b\rangle\right)\right|\leqslant\frac{1}{2}\|x\|^2\sqrt{\|a\|^2\cdot\|b\|^2-(\Im\langle a,b\rangle)^2}, \tag{8}$$

where $z=\Re(z)+i\Im(z)\in\mathbb{C}$.

In [14], Lupu and Schwarz gave the following inequality:

$$\|a\|^2|\langle b,x\rangle|^2+\|b\|^2|\langle x,a\rangle|^2+\|x\|^2|\langle a,b\rangle|^2\leqslant\|a\|^2\|b\|^2\|x\|^2+2|\langle a,b\rangle\langle b,x\rangle\langle x,a\rangle|, \tag{9}$$

for any vectors $x,a,b\in\mathscr{X}$. This inequality gives us another refinement of the (C-S) inequality

$$0\leqslant\frac{1}{\|x\|^2}(\|a\||\langle b,x\rangle|-\|b\||\langle x,a\rangle|)^2\leqslant\|a\|^2\cdot\|b\|^2-|\langle a,b\rangle|^2,$$

for any vectors $x,a,b\in\mathscr{X}$, $x\neq 0$.

These inequalities, mentioned above, were applied to the theory of Hilbert $\mathbb{C}^*$-modules over non-commutative $\mathbb{C}^*$-algebras, see Aldaz [1], Pečarić and Rajić [18] and Dragomir [8], [9].

The main preoccupation of this article is the characterization of Richard's inequality, in connection with Buzano's inequality. In Section 2 we look at some bounds of the expression $\alpha\langle a,x\rangle\langle x,b\rangle-\beta\|x\|^2\langle a,b\rangle$ which is used in the study of some important inequalities, such as those given by Buzano, Richard, Ostrowski, Dragomir, Khosravi, Drnovšek and Moslehian. In Section 3, we present a new approach for Richard's inequality, where we use the Selberg operator. We also give a result which is the corresponding complex version of Precupanu's inequality.

2. Main results

First, we look at the expression $\alpha\langle a,x\rangle\langle x,b\rangle-\beta\|x\|^2\langle a,b\rangle$ as the scalar product of two vectors and we give an important identity.

LEMMA 1. *In an inner product space $\mathscr{X}$ over the field of complex numbers $\mathbb{C}$, we have*

$$\|\alpha\langle a,x\rangle x-\beta\|x\|^2a\|^2=\|x\|^2\left(|\langle a,x\rangle|^2|\beta-\alpha|^2+|\beta|^2\left\|\|x\|a-\frac{\langle a,x\rangle}{\|x\|}x\right\|^2\right) \tag{10}$$

for all $a,x\in\mathscr{X}$, $x\neq 0$, and for every $\alpha,\beta\in\mathbb{C}$.

Proof. For $\beta=0$ we obtain the equality in relation to the statement. Next, we consider $\beta\neq 0$. In [16] we found the following identity:

$$\|x+\alpha y\|^2=\left|\alpha\|y\|+\frac{\langle x,y\rangle}{\|y\|}\right|^2+\left\|x-\frac{\langle x,y\rangle}{\|y\|^2}y\right\|^2$$

for all $x,y\in\mathscr{X}$, $y\neq 0$, and for every $\alpha\in\mathbb{C}$.

If we replace α by $-\dfrac{\alpha}{\beta}\dfrac{\langle a,x\rangle}{\|x\|^2}$ (because $x\neq 0$), x by a and y by x in the above identity, then we obtain

$$\begin{aligned}\left\|a-\frac{\alpha}{\beta}\frac{\langle a,x\rangle}{\|x\|^2}x\right\|^2&=\left|-\frac{\alpha}{\beta}\frac{\langle a,x\rangle}{\|x\|^2}\|x\|+\frac{\langle a,x\rangle}{\|x\|}\right|^2+\left\|a-\frac{\langle a,x\rangle}{\|x\|^2}x\right\|^2\\&=\frac{|\langle a,x\rangle|^2|\beta-\alpha|^2}{|\beta|^2\|x\|^2}+\left\|a-\frac{\langle a,x\rangle}{\|x\|^2}x\right\|^2.\end{aligned}$$

Therefore, multiplying by $|\beta|^2\|x\|^4$, in the above relation, we deduce the identity of the statement. $\square$

REMARK 1. Taking into account that $|\beta|^2\left\|\|x\|a-\dfrac{\langle a,x\rangle}{\|x\|}x\right\|^2\geqslant 0$, then, from equality (10), we find

$$\|\alpha\langle a,x\rangle x-\beta\|x\|^2a\|\geqslant\|x\||\langle a,x\rangle||\beta-\alpha| \tag{11}$$

for all $a,x \in \mathscr{X}$, and for every $\alpha \in \mathbb{C}$. The case $x=0$ is checked separately. Since, we have $\left\| \|x\|a - \frac{\langle a,x\rangle}{\|x\|}x \right\|^2 = \|a\|^2\|x\|^2 - |\langle a,x\rangle|^2$, equality (10) becomes

$$\|\alpha\langle a,x\rangle x - \beta\|x\|^2 a\|^2 = \|x\|^2\left(|\alpha-\beta|^2|\langle a,x\rangle|^2 + |\beta|^2\|a\|^2\|x\|^2 - |\beta|^2|\langle a,x\rangle|^2\right) \quad (12)$$

for all $a,x \in \mathscr{X}$, and for every $\alpha,\beta \in \mathbb{C}$, with separate verification for the case $x=0$.

For $\alpha = 2$ and $\beta = 1$ in identity (10) we obtain the following [16]:

$$\|\langle a,x\rangle x - \frac{1}{2}\|x\|^2 a\| = \frac{1}{2}\|x\|^2\|a\| \quad (13)$$

for all $a,x \in \mathscr{X}$.

We know the algebraic inequality $pp_1 + qq_1 \leqslant \max\{p,q\}(p_1+q_1)$ for all $p,p_1,q,q_1 \geqslant 0$. If we take $p = |\alpha-\beta|^2$, $q = |\beta|^2$, $p_1 = |\langle a,x\rangle|^2$ and $q_1 = \|a\|^2\|x\|^2 - |\langle a,x\rangle|^2$, then we have

$$\begin{aligned}\|\alpha\langle a,x\rangle x - \beta\|x\|^2 a\|^2 &\leqslant \|x\|^2 \max\{|\alpha-\beta|^2,|\beta|^2\}\left(|\langle a,x\rangle|^2 + \|a\|^2\|x\|^2 - |\langle a,x\rangle|^2\right)\\ &= \max\{|\alpha-\beta|^2,|\beta|^2\}\|a\|^2\|x\|^4,\end{aligned}$$

which is equivalent to

$$\|\alpha\langle a,x\rangle x - \beta\|x\|^2 a\| \leqslant \max\{|\alpha-\beta|,|\beta|\}\|a\|\|x\|^2 \quad (14)$$

for all $a,x \in \mathscr{X}$ and for every $\alpha,\beta \in \mathbb{C}$.

Also here, it should be mentioned that, combining inequalities (11) and (14), we obtain an improvement of the Cauchy–Schwarz inequality. Thus

$$|\langle a,x\rangle| \leqslant \frac{\|\alpha\langle a,x\rangle x - |\beta|\|x\|^2 a\|}{|\alpha-\beta|\|x\|} \leqslant \|a\|\|x\| \quad (15)$$

for all $a,x \in \mathscr{X}$, $x \neq 0$ and for every $\alpha,\beta \in \mathbb{C}$ with $|\alpha-\beta| \geqslant |\beta| > 0$.

THEOREM 1. *In an inner product space $\mathscr{X}$ over the field of complex numbers $\mathbb{C}$, we have*

$$|\alpha\langle a,x\rangle\langle x,b\rangle - \beta\|x\|^2\langle a,b\rangle| \leqslant \max\{|\beta|,|\alpha-\beta|\}\|x\|^2\|a\|\|b\| \quad (16)$$

for all $a,b,x \in \mathscr{X}$ and for every $\alpha,\beta \in \mathbb{C}$.

Proof. For $\beta = 0$ the inequality of the statement is true. For $\beta \neq 0$, using the Cauchy–Schwarz inequality and inequality (14), we deduce the following:

$$\begin{aligned}|\alpha\langle a,x\rangle\langle x,b\rangle - \beta\|x\|^2\langle a,b\rangle| &= |\langle\alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|\\ &\overset{(C-S)}{\leqslant} \|\langle\alpha\langle a,x\rangle x - \beta\|x\|^2 a\|\|b\|\\ &\overset{(14)}{\leqslant} \max\{|\beta|,|\alpha-\beta|\}\|x\|^2\|a\|\|b\|.\end{aligned}$$

Therefore, the inequality of the statement was proven. □

REMARK 2. We take $\beta = 1$ in inequality (16). Thus we show that

$$|\alpha\langle a,x\rangle\langle x,b\rangle - \|x\|^2\langle a,b\rangle| \leqslant \max\{1,|\alpha-1|\}\|x\|^2\|a\|\|b\| \tag{17}$$

for all $a,b,x \in \mathscr{X}$ and for every $\alpha \in \mathbb{C}$. This inequality is given by Khosravi, Drnovšek and Moslehian [13] as an extension of Buzano's inequality given as a particularization in the context of Hilbert C^*–modules. We mentioned the fact that this inequality was studied by Dragomir in [9], when $|\alpha - 1| = 1$.

For $\alpha = 2$ in relation (17), we obtain Richard's inequality.

THEOREM 2. *In an inner product space $\mathscr{X}$ over the field of complex numbers $\mathbb{C}$, we have*

$$0 \leqslant \frac{\|x\|^2}{\|b\|^2}\left|\alpha\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2} - \beta\langle a,b\rangle\right|^2 \leqslant |\alpha-\beta|^2|\langle a,x\rangle|^2 + |\beta|^2\left(\|a\|^2\|x\|^2 - |\langle a,x\rangle|^2\right) \tag{18}$$

for all $a,b,x \in \mathscr{X}$, $b,x \neq 0$, and for every $\alpha,\beta \in \mathbb{C}$.

Proof. For $x \neq 0$ and $b \neq 0$, we make the following calculations:

$$\begin{aligned}
&|\alpha\langle a,x\rangle\langle x,b\rangle - \beta\|x\|^2\langle a,b\rangle|^2 \\
&= |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|^2 \\
&\overset{(C-S)}{\leqslant} \|\alpha\langle a,x\rangle x - \beta\|x\|^2 a\|^2\|b\|^2 \\
&\overset{(12)}{=} \|x\|^2\|b\|^2\left(|\alpha-\beta|^2|\langle a,x\rangle|^2 + |\beta|^2\left(\|a\|^2\|x\|^2 - |\langle a,x\rangle|^2\right)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x\|^4\left|\alpha\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2} - \beta\langle a,b\rangle\right|^2 \\
&\leqslant \|x\|^2\|b\|^2\left(|\alpha-\beta|^2|\langle a,x\rangle|^2 + |\beta|^2\left(\|a\|^2\|x\|^2 - |\langle a,x\rangle|^2\right)\right).
\end{aligned}$$

But, since $b,x \neq 0$, we divide by $\|x\|^2\|b\|^2$ and we deduce the inequality of the statement. □

COROLLARY 1. *In an inner product space $\mathscr{X}$ over the field of complex numbers $\mathbb{C}$, the following inequality*

$$0 \leqslant \frac{\|x\|^2}{\|b\|^2}\left|\frac{\langle a,x\rangle\langle x,b\rangle}{\|x\|^2} - \langle a,b\rangle\right|^2 \leqslant \|a\|^2\|x\|^2 - |\langle a,x\rangle|^2 \tag{19}$$

holds, for all $a,b,x \in \mathscr{X}, x \neq 0, b \neq 0$.

Proof. In relation (18), we take $\alpha = \beta \neq 0$ and we deduce the inequality of the statement. □

REMARK 3. If we take $\langle x,b\rangle = 0$, in inequality (19), then we obtain

$$\frac{\|x\|^2}{\|b\|^2}\,|\langle a,b\rangle|^2 \leqslant \|a\|^2\,\|x\|^2 - |\langle a,x\rangle|^2 \tag{20}$$

for all $a,b,x\in\mathscr{X}$, $b\neq 0$, $x\neq 0$. This inequality was obtained by Dragomir and Goşa in [10].

In addition, if we consider $\langle a,b\rangle = 1$ (or $|\langle a,b\rangle| = 1$) in inequality (20), then we find the inequality of Ostrowski for inner product spaces over the field of complex numbers,

$$\frac{\|x\|^2}{\|b\|^2} \leqslant \|a\|^2\,\|x\|^2 - |\langle a,x\rangle|^2 \tag{21}$$

for all $a,b,x\in\mathscr{X}$, $b\neq 0$, $x\neq 0$.

The inequality of Ostrowski for inner product spaces over the field of real numbers was studied in [16]. It is easy to see that for $a,b,x\in\mathbb{R}^n$ we obtain inequality (3).

THEOREM 3. *In an inner product space $\mathscr{X}$ over the field of real or complex numbers, for any vectors $x,a,b\in\mathscr{X}$, $a\neq 0$ and $\alpha,\beta\in\mathbb{C}$, $\alpha\neq\beta$, where $|\alpha-\beta|\leqslant|\beta|$ with $\beta\neq 0$, we have*

$$|\beta|\|x\|^2\|a\|\|b\| - |\alpha\langle a,x\rangle\langle x,b\rangle - \beta\|x\|^2\langle a,b\rangle| \geqslant \frac{\left(|\beta|^2-|\alpha-\beta|^2\right)\|b\||\langle a,x\rangle|^2}{2|\beta|\|a\|} \geqslant 0. \tag{22}$$

Proof. If we have $x=0$ or $b=0$, then the relation of the statement is true. For $x\neq 0$ and $b\neq 0$, we make the following calculations:

$$\begin{aligned}
&|\beta|\|x\|^2\|a\|\|b\| - |\alpha\langle a,x\rangle\langle x,b\rangle - \beta\|x\|^2\langle a,b\rangle|\\
&= |\beta|\|x\|^2\|a\|\|b\| - |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|\\
&= \frac{|\beta|^2\|x\|^4\|a\|^2\|b\|^2 - |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|^2}{|\beta|\|x\|^2\|a\|\|b\| + |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|}\\
&\overset{(C-S)}{\geqslant} \frac{|\beta|^2\|x\|^4\|a\|^2\|b\|^2 - \|\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a\|^2\|b\|^2}{|\beta|\|x\|^2\|a\|\|b\| + |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|}\\
&\overset{(12)}{=} \frac{|\beta|^2\|x\|^4\|a\|^2\|b\|^2 - |\alpha-\beta|^2\|x\|^2|\langle a,x\rangle|^2\|b\|^2 - |\beta|^2\|a\|^2\|b\|^2\|x\|^4}{|\beta|\|x\|^2\|a\|\|b\| + |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|}\\
&\quad + \frac{|\beta|^2\|x\|^2\|b\|^2|\langle a,x\rangle|^2}{|\beta|\|x\|^2\|a\|\|b\| + |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|}\\
&= \frac{\left(|\beta|^2-|\alpha-\beta|^2\right)\|x\|^2\|b\|^2|\langle a,x\rangle|^2}{|\beta|\|x\|^2\|a\|\|b\| + |\langle \alpha\langle a,x\rangle x - \beta\|x\|^2 a, b\rangle|}\\
&\overset{(14)}{\geqslant} \frac{\left(|\beta|^2-|\alpha-\beta|^2\right)\|x\|^2\|b\|^2|\langle a,x\rangle|^2}{|\beta|\|x\|^2\|a\|\|b\| + \max\{|\beta|,|\alpha-\beta|\}\|x\|^2\|a\|\|b\|}
\end{aligned}$$

$$= \frac{\left(|\beta|^2 - |\alpha-\beta|^2\right) \|x\|^2\|b\|^2|\langle a,x\rangle|^2}{2|\beta|\|x\|^2\|a\|\|b\|}$$
$$= \frac{\left(|\beta|^2 - |\alpha-\beta|^2\right) \|b\||\langle a,x\rangle|^2}{2|\beta|\|a\|}.$$

Consequently, the inequality of the statement is true. □

THEOREM 4. *In an inner product space $\mathscr{X}$ over the field of real or complex numbers, for any nonzero vectors $x,a,b \in \mathscr{X}$ and $\alpha,\beta \in \mathbb{C}$, $\max\{|\alpha-\beta|,|\beta|\} \neq 0$, we have*

$$\max\{|\alpha-\beta|,|\beta|\} \|x\|^2 \|a\| \cdot \|b\| - \left|\alpha \langle a,x\rangle \langle x,b\rangle - \beta \|x\|^2 \langle a,b\rangle\right|$$
$$\geqslant \frac{A(\alpha,\beta)}{2\max\{|\alpha-\beta|,|\beta|\} \|x\|^2 \|a\| \cdot \|b\|} \geqslant 0, \tag{23}$$

where

$$A(\alpha,\beta) = \left(|\alpha|\,|\langle a,x\rangle| \left(\|x\|^2\|b\|^2 - |\langle x,b\rangle|^2\right)^{\frac{1}{2}} - |\beta|\,\|x\|^2 \left(\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2\right)^{\frac{1}{2}}\right)^2.$$

Proof. For all $x,y \in \mathscr{X}$, and $y \neq 0$, we have the following equality:

$$\left\| \|y\|x - \frac{\langle x,y\rangle}{\|y\|} y \right\|^2 = \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2.$$

Hence, using the Cauchy-Schwarz inequality to the denominator, we have the relation

$$\|x\| \cdot \|y\| - |\langle x,y\rangle| = \frac{\left\| \|y\|x - \frac{\langle x,y\rangle}{\|y\|} y \right\|^2}{\|x\| \cdot \|y\| + |\langle x,y\rangle|} \geqslant \frac{\left\| \|y\|x - \frac{\langle x,y\rangle}{\|y\|} y \right\|^2}{2\|x\| \cdot \|y\|}.$$

In this inequality, we replace x and y by $\alpha\langle a,x\rangle x - \beta\|x\|^2 a$ and b in the above inequality and using inequality (14), i.e. $\|\alpha\langle a,x\rangle x - \beta\|x\|^2 a\| \leqslant \max\{|\alpha-\beta|,|\beta|\}\|a\|\|x\|^2$, implies

$$\left\|\alpha\langle a,x\rangle x - \beta\|x\|^2 a\right\| \|b\| - \left|\alpha\langle a,x\rangle\langle x,b\rangle - \beta\|x\|^2\langle a,b\rangle\right|$$
$$\geqslant \frac{\|u-v\|^2}{2\left\|\alpha\langle a,x\rangle x - \beta\|x\|^2 a\right\| \|b\|} \geqslant \frac{\|u-v\|^2}{2\max\{|\alpha-\beta|,|\beta|\}\|a\|\|b\|\|x\|^2},$$

where $u = \alpha\langle a,x\rangle \left(\|b\|x - \frac{\langle x,b\rangle}{\|b\|} b\right)$ and $v = \beta\|x\|^2 \left(\|b\|a - \frac{\langle a,b\rangle}{\|b\|} b\right)$.

But, we have that $\|u\| = |\alpha|\,|\langle a,x\rangle| \left(\|x\|^2\|b\|^2 - |\langle x,b\rangle|^2\right)^{\frac{1}{2}}$ and

$$\|v\| = |\beta|\,\|x\|^2 \left(\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2\right)^{\frac{1}{2}}.$$

Therefore, since $\|u-v\|^2 \geqslant (\|u\|-\|v\|)^2$, $u,v \in X$, we obtain the inequality of the statement. $\square$

REMARK 4. For $\alpha = 1$ and $\beta = \frac{1}{2}$ we obtain an important inequality given in [16]. Thus

$$\begin{aligned} &\frac{1}{2}\|x\|^2\|a\|\cdot\|b\| - \left|\langle a,x\rangle\langle x,b\rangle - \frac{1}{2}\|x\|^2\langle a,b\rangle\right| \\ &\geqslant \frac{A}{\|x\|^2\|a\|\cdot\|b\|} \geqslant 0, \end{aligned} \tag{24}$$

where

$$\begin{aligned} A &= A\left(1,\frac{1}{2}\right) \\ &= \left(|\langle a,x\rangle|\left(\|x\|^2\|b\|^2 - |\langle x,b\rangle|^2\right)^{\frac{1}{2}} - \frac{1}{2}\|x\|^2\left(\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2\right)^{\frac{1}{2}}\right)^2. \end{aligned}$$

This inequality represents an improvement of Richard's inequality, given thus:

$$\left|\langle a,x\rangle\langle x,b\rangle - \frac{1}{2}\|x\|^2\langle a,b\rangle\right| \leqslant \frac{1}{2}\|x\|^2\|a\|\cdot\|b\| - \frac{A}{\|x\|^2\|a\|\cdot\|b\|}.$$

COROLLARY 2. *In an inner product space $\mathscr{X}$ over the field of real or complex numbers, for any nonzero vectors $x,a,b \in \mathscr{X}$ and $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, we have*

$$\begin{aligned} &\|x\|^2\left(|\beta||\langle a,b\rangle| - \max\{|\alpha-\beta|,|\beta|\}\|a\|\cdot\|b\|\right) + \frac{A(\alpha,\beta)}{2\max\{|\alpha-\beta|,|\beta|\}\|x\|^2\|a\|\cdot\|b\|} \\ &\leqslant |\alpha||\langle a,x\rangle\langle x,b\rangle| \\ &\leqslant \|x\|^2\left(|\beta||\langle a,b\rangle| + \max\{|\alpha-\beta|,|\beta|\}\|a\|\cdot\|b\|\right) - \frac{A(\alpha,\beta)}{2\max\{|\alpha-\beta|,|\beta|\}\|x\|^2\|a\|\cdot\|b\|}. \end{aligned} \tag{25}$$

Proof. By using inequality (23), and from the continuity property of the modulus, i.e., $|u-v| \geqslant ||u|-|v||$, $u,v \in \mathbb{C}$, we easily deduce the desired inequality. $\square$

We obtain from inequality (25) a refinement of Buzano's inequality, as follows.

PROPOSITION 1. *In an inner product space $\mathscr{X}$ over the field of real or complex numbers, for any nonzero vectors $x,a,b \in \mathscr{X}$, we have*

$$\begin{aligned} &|\langle a,x\rangle\langle x,b\rangle| \\ &\leqslant \|x\|^2\left(\frac{1}{2}|\langle a,b\rangle| + \frac{1}{2}\|a\|\cdot\|b\|\right) - \frac{1}{\|x\|^2\|a\|\cdot\|b\|}\max\left\{A\left(1,\frac{1}{2}\right),\frac{1}{4}A(2,1)\right\}. \end{aligned} \tag{26}$$

Proof. The refinement of Buzano's inequality follows from inequality (25), when $\alpha = 1$ and $\beta = \frac{1}{2}$ or $\alpha = 2$ and $\beta = 1$. $\square$

3. A new approach for Richard's inequality

Throughout this section, we denote by $\mathscr{X}$ a complex Hilbert space, i.e. a complete and inner product space, where the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$ are defined. We denote the C^*-algebra of all bounded linear operators acting on $\mathscr{X}$ as $\mathscr{B}(\mathscr{X})$ and the identity operator is represented by I.

For an operator $T\in\mathscr{B}(\mathscr{X})$, the nullspace of T is denoted as $\mathscr{N}(T)$, and T^* represents its adjoint. We define a positive operator, denoted as $T\geqslant 0$, as an operator that satisfies $\langle Tx,x\rangle\geqslant 0$ for all $x\in\mathscr{H}$. Moreover, the order relation $T\geqslant S$ is introduced for self-adjoint operators, which holds when $T-S\geqslant 0$.

Given a bounded linear operator T defined on $\mathscr{X}$, recall that the numerical radius, denoted as $\omega(T)$, is defined as the supremum (or maximum) of the absolute values of the numbers in the numerical range $W(T)$, more precisely

$$\omega(T)=\sup\{|\lambda|:\lambda\in W(T)\},$$

where $W(T)=\{\langle Tx,x\rangle : x\in\mathscr{H}, \|x\|=1\}$. The operator norm of T is given by

$$\|T\|=\sup\{\|Tx\|:\|x\|=1,x\in\mathscr{H}\}=\sup\{|\langle Tx,y\rangle|:\|x\|=\|y\|=1,x,y\in\mathscr{H}\}.$$

For the subsequent discussion, it is important to recall that expression $x\otimes y$ represents a rank one operator defined by $x\otimes y(z)=\langle z,y\rangle x$, where x, y, and z are vectors in space $\mathscr{X}$. So, we can rewrite inequality (16), as follows:

$$|\langle[\alpha(x\otimes x)-\beta\|x\|^2I]a,b\rangle|\leqslant\max\{|\beta|,|\alpha-\beta|\}\|x\|^2\|a\|\|b\|, \tag{27}$$

where $a,b,x\in\mathscr{X}$ and $\alpha,\beta\in\mathbb{C}$.

Taking the supremum in relation (27) for $\|a\|=\|b\|=1$, we deduce

$$\|\alpha(x\otimes x)-\beta\|x\|^2I\|\leqslant\max\{|\beta|,|\alpha-\beta|\}\|x\|^2.$$

It is well-known that if $x\in\mathscr{X}$ with $\|x\|=1$, then $P_x=x\otimes x$ is the orthogonal projection on $\operatorname{span}\{x\}$.

REMARK 5. If in inequality (27), we assume that $\alpha=2,\beta=1$ and $x\in\mathscr{X}$ is a norm one vector, then

$$\|2P_x-I\|=\|2(x\otimes x)-I\|\leqslant\max\{|1|,|2-1|\}=1.$$

Fuji and Kubo [11], used this inequality to give a simpler proof of Buzano's inequality.

A significant inequality was discovered by A. Selberg ([15, p. 394]). If we consider vectors $x,z_1,\dots,z_n$ in $\mathscr{X}$, where $z_i\neq 0$ for all $i\in\{1,\dots,n\}$, we can consider Selberg's inequality, which asserts that:

$$\sum_{i=1}^{n}\frac{|\langle x,z_i\rangle|^2}{\sum_{j=1}^{n}|\langle z_i,z_j\rangle|}\leqslant\|x\|^2. \tag{28}$$

In [2], was introduced the Selberg operator defined as follows: given a subset $\mathscr{Z} = \{z_i : i = 1, \cdots, n\}$ of nonzero vectors in space $\mathscr{X}$, the Selberg operator $S_{\mathscr{Z}}$ is defined by

$$S_{\mathscr{Z}} = \sum_{i=1}^{n} \frac{z_i \otimes z_i}{\sum_{j=1}^{n} |\langle z_i, z_j \rangle|} \in \mathscr{B}(\mathscr{X}).$$

REMARK 6. Selberg's inequality gives us another refinement of the (C-S) inequality, since if $a, b \in \mathscr{X}$ with a and b nonzero vectors in $\mathscr{X}$, then

$$0 \leqslant \left(\|a\|^2 - \langle S_{\{b\}} a, a \rangle\right)\left(\|b\|^2 - \langle S_{\{a\}} b, b \rangle\right) \leqslant \|b\|^2 \left(\|a\|^2 - \langle S_{\{b\}} a, a \rangle\right),$$

or equivalently,

$$0 \leqslant \left(\|a\|^2 - \langle S_{\{b\}} a, a \rangle\right)\left(\|b\|^2 - \langle S_{\{a\}} b, b \rangle\right) \leqslant \|b\|^2 \|a\|^2 - |\langle a, b \rangle|^2.$$

Now, we will express Richard's inequality using an appropiate Selberg operator, more precisely

$$\left| \langle S_{\mathscr{Z}} a, b \rangle - \frac{1}{2} \langle a, b \rangle \right| \leqslant \frac{1}{2} \|a\| \|b\|, \tag{29}$$

where $\mathscr{Z} = \{x\}$, $x, a, b \in \mathscr{X}$ and $\|x\| = 1$.

Before we point out some generalization of (29), we collect some results recently obtained by one of the authors in [2, 4].

LEMMA 2. *Let $\mathscr{Z} = \{z_i : i = 1, \cdots, n\}$ be a subset of nonzero vectors in $\mathscr{X}$, then*

1. *$S_{\mathscr{Z}}$ is a positve operator and $\|S_{\mathscr{Z}}\| \leqslant 1$.*
2. *$\|2S_{\mathscr{Z}} - I\| \leqslant 1$.*
3. *For any $a, b \in \mathscr{X}$, we have*

$$\begin{aligned} \|a\| \|b\| &\geqslant |\langle a, b \rangle - \langle S_{\mathscr{Z}} a, b \rangle| + \langle S_{\mathscr{Z}} a, a \rangle^{1/2} \langle S_{\mathscr{Z}} b, b \rangle^{1/2} \\ &\geqslant |\langle a, b \rangle| - |\langle S_{\mathscr{Z}} a, b \rangle| + \langle S_{\mathscr{Z}} a, a \rangle^{1/2} \langle S_{\mathscr{Z}} b, b \rangle^{1/2} \\ &\geqslant |\langle a, b \rangle|. \end{aligned}$$

Now, we generalize Richard's inequality for any subset $\mathscr{Z}$ contained in $\mathscr{X}$, and we characterize when the equality holds.

THEOREM 5. *For any $a, b \in \mathscr{X}$ and $\mathscr{Z} = \{z_i : i = 1, \cdots, n\}$ a subset of nonzero vectors in $\mathscr{X}$, it holds*

$$\left| \langle S_{\mathscr{Z}} a, b \rangle - \frac{1}{2} \langle a, b \rangle \right| \leqslant \frac{1}{2} \|a\| \|b\|. \tag{30}$$

The case of equality holds in (30) if and only if

$$S_{\mathscr{Z}} a = \frac{1}{2} a + \frac{1}{2} \frac{\|a\|}{\|b\|} e^{i\theta} b, \tag{31}$$

for some $\theta \in [0, 2\pi)$.

Proof. The first inequality is a simple consequence of Lemma 2, but to make this article complete, we have included the proof.

Let $a,b \in \mathscr{X}$, then from Cauchy-Schwarz's inequality and Lemma 2, we have

$$\left|\langle S_{\mathscr{Z}}a,b\rangle - \frac{1}{2}\langle a,b\rangle\right| = \left|\left\langle \left(S_{\mathscr{Z}} - \frac{1}{2}I\right)a,b\right\rangle\right| \leqslant \frac{1}{2}\,\|2S_{\mathscr{Z}} - I\|\,\|a\|\|b\|$$
$$\leqslant \frac{1}{2}\|a\|\|b\|.$$

Then, the equality holds, for $b \neq 0$, if and only if

$$S_{\mathscr{Z}}a = \frac{1}{2}a + \delta b, \tag{32}$$

for some $\delta \in \mathbb{C}$. Using expression (32), we deduce that $\delta = \frac{\|a\|}{\|b\|}e^{i\theta}$ for some $\theta \in [0,2\pi)$. $\square$

Incidentally, if $\{a,b\}$ is linearly dependent, then the equality in (30), holds for any subset $\mathscr{Z}$, if and only if $S_{\mathscr{Z}}b = \frac{1}{2}(1+e^{i\beta})b$ for some $\beta \in [0,2\pi)$.

PROPOSITION 2. *Let $\mathscr{Z}$ be a finite subset of nonzero vectors in $\mathscr{X}$. If there exists $e \in \mathscr{Z}^{\perp}$ with $\|e\| = 1$, then for any $x,y \in \mathscr{X}$ it holds that*

$$\left|\langle S_{\mathscr{Z}}a,b\rangle - \frac{1}{2}\langle a,b\rangle\right| \leqslant \left|\langle S_{\mathscr{Z}}a,b\rangle - \frac{1}{2}\langle a,b\rangle + \frac{1}{2}\langle a,e\rangle\langle e,b\rangle\right| + \frac{1}{2}\,|\langle a,e\rangle\langle e,b\rangle|$$
$$\leqslant \frac{1}{2}\|a\|\|b\|. \tag{33}$$

Proof. Let $\mathscr{Z}_1 = \{e\}$, then $S_{\mathscr{Z}_1} = e \otimes e$ and by Lemma 2 we have

$$|\langle a,b\rangle| \leqslant |\langle a,b\rangle - \langle a,e\rangle\langle e,b\rangle| + |\langle a,e\rangle\langle e,b\rangle| \leqslant \|a\|\|b\|, \tag{34}$$

for any $a,b \in \mathscr{X}$. Now, if in the last inequality, which is a refinement of the Cauchy-Schwarz's inequality, we replace a by $(S_{\mathscr{Z}} - \frac{1}{2}I)a$ and we use that $S_{\mathscr{Z}}e = 0$, then we obtain

$$\left|\left\langle \left(S_{\mathscr{Z}} - \frac{1}{2}I\right)a,b\right\rangle\right| \leqslant \left|\langle S_{\mathscr{Z}}a,b\rangle - \frac{1}{2}\langle a,b\rangle + \frac{1}{2}\langle a,e\rangle\,\langle e,b\rangle\right| + \frac{1}{2}\,|\langle a,e\rangle\,\langle e,b\rangle|$$
$$\leqslant \frac{1}{2}\,\|2S_{\mathscr{Z}} - I\|\,\|a\|\|b\| \leqslant \frac{1}{2}\|a\|\|b\|$$

for any $a,b \in \mathscr{X}$. $\square$

REMARK 7. Notice that (34) is also established by Dragomir in [7]. However, our approach here is different from his.

We can obtain a refinement of Richard's inequality, from the previous statement, by considering the positivity of $S_{\mathscr{Z}}$ and appropiate set $\mathscr{Z}$.

COROLLARY 3. *For any* $x, a, b \in \mathscr{X}$ *with* $\|x\| = 1$, *we have*

$$\left|\langle a,x\rangle\langle x,b\rangle - \frac{1}{2}\langle a,b\rangle\right| \leqslant \left|\langle a,x\rangle\langle x,b\rangle - \frac{1}{2}\langle a,b\rangle + \frac{1}{2}\langle a,e\rangle\langle e,b\rangle\right| + \frac{1}{2}|\langle a,e\rangle\langle e,b\rangle| \leqslant \frac{1}{2}\|a\|\cdot\|b\|,$$

where $e \in \mathscr{X}$ *with* $\langle x,e\rangle = 0$.

Proof. We consider $\mathscr{Z} = \{x\}$, then $e \in \mathscr{Z}^{\perp}$ and by Proposition 2, we conclude

$$\begin{aligned}\left|\langle a,x\rangle\langle x,b\rangle - \frac{1}{2}\langle a,b\rangle\right| &= \left|\langle S_{\mathscr{Z}}a,b\rangle - \frac{1}{2}\langle a,b\rangle\right| \\ &\leqslant \left|\langle a,x\rangle\langle x,b\rangle - \frac{1}{2}\langle a,b\rangle + \frac{1}{2}\langle a,e\rangle\langle e,b\rangle\right| + \frac{1}{2}|\langle a,e\rangle\langle e,b\rangle| \\ &\leqslant \frac{1}{2}\|a\|\cdot\|b\|. \quad \square\end{aligned}$$

From Proposition 2, we also obtain the following refinement of the Buzano type inequality.

COROLLARY 4. *Let* $\mathscr{Z}$ *be a finite subset of nonzero vectors in* $\mathscr{X}$. *If there exists* $e \in \mathscr{Z}^{\perp}$ *with* $\|e\| = 1$, *then for any* $x, y \in \mathscr{X}$ *it holds that*

$$\begin{aligned}|\langle S_{\mathscr{Z}}a,b\rangle| &\leqslant \left|\langle S_{\mathscr{Z}}a,b\rangle - \frac{1}{2}\langle a,b\rangle + \frac{1}{2}\langle a,e\rangle\langle e,b\rangle\right| + \left|\frac{1}{2}\langle a,e\rangle\langle e,b\rangle\right| + \frac{1}{2}|\langle a,b\rangle| \\ &\leqslant \frac{1}{2}\left(|\langle a,b\rangle| + \|a\|\|b\|\right). \end{aligned} \tag{35}$$

Using the argument of the proof of Theorem 5, with different subsets $\mathscr{Z}$ contained in $\mathscr{X}$, we get the following result, which is the corresponding complex version of Precupanu's inequality.

PROPOSITION 3. *Let* $a, b, w, z \in \mathscr{X}$ *with* w *and* z *nonzero vectors. Then*

$$\left|\frac{\langle a,w\rangle\langle w,b\rangle}{\|w\|^2} + \frac{\langle a,z\rangle\langle z,b\rangle}{\|z\|^2} - 2\frac{\langle a,w\rangle\langle w,z\rangle\langle z,b\rangle}{\|w\|^2\|z\|^2} - \frac{1}{2}\langle a,b\rangle\right| \leqslant \frac{1}{2}\|a\|\|b\|. \tag{36}$$

Proof. We consider the following sets $\mathscr{Z}_1 = \{w\}$ and $\mathscr{Z}_2 = \{z\}$ contained in $\mathscr{X}$. Thus,

$$\frac{\langle a,w\rangle\langle w,b\rangle}{\|w\|^2} + \frac{\langle a,z\rangle\langle z,b\rangle}{\|z\|^2} - 2\frac{\langle a,w\rangle\langle w,z\rangle\langle z,b\rangle}{\|w\|^2\|z\|^2} = \langle S_{\mathscr{Z}_1}a,b\rangle + \langle S_{\mathscr{Z}_2}a,b\rangle - 2\langle S_{\mathscr{Z}_1}a, S_{\mathscr{Z}_2}b\rangle.$$

As $S_{\mathscr{Z}_2}$ is a positive operator and, in particular a selfadjoint operator, we get that

$$\frac{\langle a,w\rangle\langle w,b\rangle}{\|w\|^2} + \frac{\langle a,z\rangle\langle z,b\rangle}{\|z\|^2} - 2\frac{\langle a,w\rangle\langle w,z\rangle\langle z,b\rangle}{\|w\|^2\|z\|^2} = \langle (S_{\mathscr{Z}_1} + S_{\mathscr{Z}_2} - 2S_{\mathscr{Z}_2}S_{\mathscr{Z}_1})a,b\rangle.$$

We remark that

$$S_{\mathscr{Z}_1}+S_{\mathscr{Z}_2}-2S_{\mathscr{Z}_2}S_{\mathscr{Z}_1}-\frac{1}{2}I=(-2)\left(S_{\mathscr{Z}_2}-\frac{1}{2}I\right)\left(S_{\mathscr{Z}_1}-\frac{1}{2}I\right).$$

Therefore, we deduce

$$\begin{aligned}\left|\left\langle\left(S_{\mathscr{Z}_1}+S_{\mathscr{Z}_2}-2S_{\mathscr{Z}_2}S_{\mathscr{Z}_1}-\frac{1}{2}I\right)a,b\right\rangle\right| &= \left|\left\langle(-2)\left(S_{\mathscr{Z}_1}-\frac{1}{2}I\right)\left(S_{\mathscr{Z}_2}-\frac{1}{2}I\right)a,b\right\rangle\right|\\ &\leqslant 2\left\|S_{\mathscr{Z}_1}-\frac{1}{2}I\right\|\left\|S_{\mathscr{Z}_2}-\frac{1}{2}I\right\|\|a\|\|b\|\\ &\leqslant \frac{1}{2}\|a\|\|b\|. \quad \square\end{aligned}$$

From (36), we get the following generalization of Buzano's inequality.

COROLLARY 5. *For any $a,b,w,z\in\mathscr{X}$ with $w\neq 0$ and $z\neq 0$, it holds that*

$$\left|\frac{\langle a,w\rangle\langle w,b\rangle}{\|w\|^2}+\frac{\langle a,z\rangle\langle z,b\rangle}{\|z\|^2}-2\frac{\langle a,w\rangle\langle w,z\rangle\langle z,b\rangle}{\|w\|^2\|z\|^2}\right|\leqslant\frac{1}{2}(|\langle a,b\rangle|+\|a\|\|b\|). \tag{37}$$

In particular, if $\langle z,b\rangle=0$ in (37), then we obtain Buzano's inequality.

Motivated by the proof of Proposition 3, we obtain the following statement.

THEOREM 6. *Let $\mathscr{Z}_1,\cdots,\mathscr{Z}_n$ be finite subsets of nonzero vectors in $\mathscr{X}$, then for any $a,b\in\mathscr{X}$ and $z_k\in\mathbb{C}$, $k=1,\ldots,n$, it holds that*

$$\left|\left\langle\sum_{k=1}^{n}z_k\left(S_{\mathscr{Z}_k}-\frac{1}{2}I\right)a,b\right\rangle\right|\leqslant\frac{\sum_{k=1}^{n}|z_k|}{2}\|a\|\|b\|, \tag{38}$$

and

$$\left|\left\langle\prod_{k=1}^{n}z_k\left(S_{\mathscr{Z}_k}-\frac{1}{2}I\right)a,b\right\rangle\right|\leqslant\frac{\prod_{k=1}^{n}|z_k|}{2^n}\|a\|\|b\|. \tag{39}$$

Proof. It is consequence of the fact that $\|\cdot\|$ is a submultiplicative norm on $\mathscr{B}(\mathscr{X})$, the triangle inequality and Lemma 2. $\square$

The inequality (39) is a generalization of Precupanu's inequality.

In particular, if $(-1)^n\prod_{k=1}^{n}z_k=2^{n-1}$, then

$$\left|\left\langle\left(\prod_{k=1}^{n}z_kS_{\mathscr{Z}_k}\right)a,b-\frac{1}{2}\langle a,b\rangle\right\rangle\right|\leqslant\frac{1}{2}\|a\|\|b\|. \tag{40}$$

COROLLARY 6. *Let $\mathscr{Z}_1,\cdots,\mathscr{Z}_n$ be finite subsets of nonzero vectors in $\mathscr{X}$ and $z_1,\cdots,z_n$ complex numbers such that $\sum_{k=1}^{n}|z_k|=\sum_{k=1}^{n}z_k=1$. Then, for any $a,b\in\mathscr{X}$ it holds that*

$$\left|\left\langle\left(\sum_{k=1}^{n}z_kS_{\mathscr{Z}_k}\right)a,b\right\rangle-\frac{1}{2}\langle a,b\rangle\right|\leqslant\frac{1}{2}\|a\|\|b\|. \tag{41}$$

Proof. By hypothesis $\sum_{k=1}^{n} z_k = 1$ we get

$$\left(\sum_{k=1}^{n} z_k S_{\mathscr{Z}_k}\right) - \frac{1}{2}I = \sum_{k=1}^{n} z_k \left(S_{\mathscr{Z}_k} - \frac{1}{2}I\right). \tag{42}$$

Then, for any $a, b \in \mathscr{X}$, we have as consequence of (38)

$$\begin{aligned}\left|\left\langle \left(\sum_{k=1}^{n} z_k S_{\mathscr{Z}_k}\right) a, b\right\rangle - \frac{1}{2}\langle a, b\rangle\right| &= \left|\left\langle \sum_{k=1}^{n} z_k \left(S_{\mathscr{Z}_k} - \frac{1}{2}I\right) a, b\right\rangle\right| \\ &\leqslant \frac{\sum_{k=1}^{n} |z_k|}{2} \|a\|\|b\| \\ &= \frac{1}{2}\|a\|\|b\|. \end{aligned} \tag{43}$$

□

As consequence of the previous result, we attain a generalization of Buzano's inequality.

PROPOSITION 4. *Let $\mathscr{Z}_1, \cdots, \mathscr{Z}_n$ be finite subsets of nonzero vectors in $\mathscr{X}$ and $z_1, \cdots, z_n$ complex numbers such that $\sum_{k=1}^{n} |z_k| = \sum_{k=1}^{n} z_k = 1$. Then, for any $x, y \in \mathscr{X}$ it holds that*

$$\left|\left\langle \left(\sum_{k=1}^{n} z_k S_{\mathscr{Z}_k}\right) a, b\right\rangle\right| \leqslant \frac{1}{2}(|\langle a, b\rangle| + \|a\|\|b\|). \tag{44}$$

THEOREM 7. *In an Hilbert space $\mathscr{X}$ over the field of complex numbers $\mathbb{C}$, we have*

$$\|\alpha S_{\mathscr{Z}} - \beta I\| \leqslant \max\{|\beta|, |\alpha - \beta|\} \tag{45}$$

for all $x \in \mathscr{X}$, $\mathscr{Z} = \{x\}$, $\|x\| = 1$, and for every $\alpha, \beta \in \mathbb{C}$.

Proof. For $a, b \in \mathscr{X}$ and from inequality (16) we deduce

$$|\langle (\alpha S_{\mathscr{Z}} - \beta I) a, b\rangle| \leqslant \max\{|\beta|, |\alpha - \beta|\}\|a\|\|b\|. \tag{46}$$

Taking the supremum in the above relation for $\|a\| = \|b\| = 1$, we find the inequality of the statement. □

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