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# ON CONVEXITY AND POWER SERIES EXPANSION FOR LOGARITHM OF NORMALIZED TAIL OF POWER SERIES EXPANSION FOR SQUARE OF TANGENT

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*This paper is dedicated to Professor Bai-Ni Guo for her retirement in August 2024.*

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*Abstract.* In the paper, the authors introduce the normalized tail of the Maclaurin power series expansion of the square of the tangent function, find out the logarithmic convexity of the normalized tail in light of the monotonicity rule for the ratio of two series, and expand the logarithm of the normalized tail into a Maclaurin power series with the help of a formula for higher order derivatives of the ratio of two differentiable functions.

## 1. Motivations

In April 2023, Qi and several mathematicians considered the decreasing property of the ratio $\frac{F(x)}{G(x)}$ on $\left(0,\frac{\pi}{2}\right)$, where

$$F(x)=\begin{cases}\ln\dfrac{3(\tan x-x)}{x^3}, & 0<|x|<\dfrac{\pi}{2}\\ 0, & x=0\end{cases} \tag{1}$$

and

$$G(x)=\begin{cases}\ln\dfrac{\tan x}{x}, & 0<|x|<\dfrac{\pi}{2}\\ 0, & x=0.\end{cases} \tag{2}$$

are even functions on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. The reason why we investigated the ratio $\frac{F(x)}{G(x)}$ and its monotonicity on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ was stated in [9, Remark 10].

Qi observed that the functions

$$\frac{\tan x}{x} \quad\text{and}\quad \frac{3(\tan x-x)}{x^3}=\frac{\tan x-x}{x^3/3} \tag{3}$$

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are related to the first two terms in the Maclaurin power series expansion

$$\begin{aligned}\tan x &= \sum_{j=1}^{\infty} \frac{2^{2j}\left(2^{2j}-1\right)}{(2j)!}|B_{2j}|x^{2j-1} \\ &= x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\cdots, \quad |x|<\frac{\pi}{2},\end{aligned} \tag{4}$$

where the Bernoulli numbers $B_j$ are generalized by

$$\frac{x}{e^x-1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!} = 1-\frac{x}{2}+\sum_{j=1}^{\infty} B_{2j}\frac{x^{2j}}{(2j)!}, \quad 0<|x|<2\pi. \tag{5}$$

Motivated by the above observation, Qi further constructed the functions

$$\begin{cases} \ln \dfrac{2(1-\cos x)}{x^2}, & 0<|x|<2\pi; \\ 0, & x=0, \end{cases} \tag{6}$$

$$\begin{cases} \dfrac{\ln \frac{2(1-\cos x)}{x^2}}{\ln \cos x}, & 0<|x|<\dfrac{\pi}{2}; \\ \dfrac{1}{6}, & x=0; \\ 0, & x=\pm\dfrac{\pi}{2}, \end{cases} \tag{7}$$

$$\begin{cases} \ln \dfrac{6(x-\sin x)}{x^3}, & 0<|x|<\infty; \\ 0, & x=0, \end{cases} \tag{8}$$

and

$$\begin{cases} \dfrac{\ln \frac{6(x-\sin x)}{x^3}}{\ln \frac{\sin x}{x}}, & |x|\in(0,\pi); \\ \dfrac{3}{10}, & x=0; \\ 0, & x=\pm\pi \end{cases} \tag{9}$$

in the papers [7, 10], respectively, basing on the first two terms in the Maclaurin power series expansions

$$\begin{aligned}\cos x &= \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} \\ &= 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\cdots, \quad x\in\mathbb{R}\end{aligned} \tag{10}$$

and

$$\begin{aligned}\sin x &= \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \\ &= x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\frac{x^9}{362880}-\cdots, \quad x\in\mathbb{R}.\end{aligned} \tag{11}$$

Generally, for generalizing the above observations, in the papers [16, 22, 23, 27] and [8, Remark 7], Qi posed the concept of the normalized tails (also known as the normalized remainders) of the Maclaurin power series expansions (11) and (10) by

$$\operatorname{SinR}_n(x)=\begin{cases}(-1)^n\dfrac{(2n+1)!}{x^{2n+1}}\left[\sin x-\displaystyle\sum_{j=0}^{n-1}(-1)^j\dfrac{x^{2j+1}}{(2j+1)!}\right], & x\ne0;\\ 1, & x=0\end{cases}$$

and

$$\operatorname{CosR}_n(x)=\begin{cases}(-1)^n\dfrac{(2n)!}{x^{2n}}\left[\cos x-\displaystyle\sum_{j=0}^{n-1}(-1)^j\dfrac{x^{2j}}{(2j)!}\right], & x\ne0;\\ 1, & x=0.\end{cases}$$

These normalized tails are generalizations of the functions

$$\cos x=\frac{\cos x}{1},\quad \frac{2(1-\cos x)}{x^2}=\frac{\cos x-1}{-x^2/2},\quad \frac{\sin x}{x},\quad \frac{6(x-\sin x)}{x^3}=\frac{\sin x-x}{-x^3/6}$$

appeared in (6), (7), (8), and (9), respectively.

In [26], basing on the Maclaurin power series expansion (5), Qi invented the normalized tail

$$\begin{cases}\dfrac{1}{B_{2n+2}}\dfrac{(2n+2)!}{x^{2n+2}}\left[\dfrac{x}{e^x-1}-1+\dfrac{x}{2}-\displaystyle\sum_{j=1}^{n}B_{2j}\dfrac{x^{2j}}{(2j)!}\right], & x\ne0;\\ 1, & x=0.\end{cases}$$

Through studying this normalized tail, some new knowledge about the Bernoulli polynomials were created in [24, 26] and closely related references therein.

In the paper [1, 18], Qi constructed the normalized tail

$$\begin{cases}\dfrac{n!}{x^n}\left(e^x-\displaystyle\sum_{j=0}^{n-1}\dfrac{x^j}{j!}\right), & x\ne0\\ 1, & x=0\end{cases}$$

basing on the Mauclaurin power series expansion

$$e^x=\sum_{j=0}^{\infty}\frac{x^j}{j!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots,\quad |x|\in\mathbb{R}.$$

In [1, Section 5], Qi summed up his idea and thought to novelly design the normalized tails as follows:

> Suppose that a real function $f(x)$ has a formal Maclaurin power series expansion
>
> $$f(x)=\sum_{j=0}^{\infty}f^{(j)}(0)\frac{x^j}{j!}.\tag{12}$$

If $f^{(n+1)}(0) \neq 0$ for some $n \in \mathbb{N}_0$, then we call the function

$$\begin{cases} \dfrac{1}{f^{(n+1)}(0)} \dfrac{(n+1)!}{x^{n+1}} \left[ f(x) - \displaystyle\sum_{j=0}^{n} f^{(j)}(0) \frac{x^j}{j!} \right], & x \neq 0 \\ 1, & x = 0 \end{cases}$$

the normalized tail of the Maclaurin power series expansion (12).

Basing on the Maclaurin power series expansion (4) and utilizing the idea and thought mentioned above, recently Qi defined the normalized tail

$$\begin{cases} \dfrac{(2n)!}{2^{2n}(2^{2n}-1)} \dfrac{1}{|B_{2n}|x^{2n-1}} \left[ \tan x - \displaystyle\sum_{j=1}^{n-1} \frac{2^{2j}(2^{2j}-1)}{(2j)!} |B_{2j}| x^{2j-1} \right], & 0 < |x| < \dfrac{\pi}{2}; \\ 1, & x = 0. \end{cases}$$

This normalized tail is a generalization of the functions in (3), which appeared in (1) and (2). Qi and his coauthors have investigated this normalized tail in a forthcoming paper.

In the paper [5, p. 798] and the handbook [6, pp. 42 and 55], we find the Maclaurin power series expansion

$$\begin{aligned} \tan^2 x &= \sum_{j=1}^{\infty} \frac{2^{2j+2}(2^{2j+2}-1)(2j+1)}{(2j+2)!} |B_{2j+2}| x^{2j} \\ &= x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \frac{1382x^{10}}{14{,}175} + \frac{21{,}844x^{12}}{467{,}775} + \cdots \end{aligned} \tag{13}$$

for $|x| < \frac{\pi}{2}$. Basing on the series expansion (13), imitating the above observations, and employing the above initiating idea and thought to design the normalized tails, we now build the normalized tail

$$h_n(x) = \begin{cases} \dfrac{(2n+2)! \left[ \tan^2 x - \displaystyle\sum_{\ell=1}^{n-1} \frac{2^{2\ell+2}(2^{2\ell+2}-1)(2\ell+1)}{(2\ell+2)!} |B_{2\ell+2}| x^{2\ell} \right]}{2^{2n+2}(2^{2n+2}-1)(2n+1)|B_{2n+2}| x^{2n}}, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{14}$$

and denote its logarithm by $H_n(x) = \ln h_n(x)$, where $n \in \mathbb{N}$ and $|x| < \frac{\pi}{2}$.

Considering the relation $\sec^2 x = 1 + \tan^2 x$, the normalized tail $h_n(x)$ can be reformulated as

$$h_n(x) = \begin{cases} \dfrac{(2n+2)! \left[ \sec^2 x - \displaystyle\sum_{\ell=0}^{n-1} \frac{2^{2\ell+2}(2^{2\ell+2}-1)(2\ell+1)}{(2\ell+2)!} |B_{2\ell+2}| x^{2\ell} \right]}{2^{2n+2}(2^{2n+2}-1)(2n+1)|B_{2n+2}| x^{2n}}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

for $n \in \mathbb{N}$ and $|x| < \frac{\pi}{2}$. Hence, the quantity $h_n(x)$ is also the normalized tail of the Maclaurin power series expansion of the square $\sec^2 x$ about $x = 0$.

It is obvious that $H_1(x) = 2G(x)$ and

$$h_n(x) = \sum_{\ell=0}^{\infty} \frac{1}{\binom{2n+2\ell+2}{2\ell}} \frac{2n+2\ell+1}{2n+1} \frac{2^{2n+2\ell+2}-1}{2^{2n+2}-1} \left|\frac{B_{2n+2\ell+2}}{B_{2n+2}}\right| \frac{(2x)^{2\ell}}{(2\ell)!} \tag{15}$$

for $n \in \mathbb{N}$ and $|x| < \frac{\pi}{2}$. The series expression (15) shows that the even function $h_n(x)$ for $n \in \mathbb{N}$ is positive and increasing on $\left(0, \frac{\pi}{2}\right)$. As a result, for $n \in \mathbb{N}$, the function $H_n(x)$ is defined and even on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, is decreasing on $\left(-\frac{\pi}{2}, 0\right)$, and is increasing on $\left(0, \frac{\pi}{2}\right)$.

In this paper, we investigate the following two problems.

1. Prove that the even function $H_n(x)$ for $n \in \mathbb{N}$ is convex on $\left(0, \frac{\pi}{2}\right)$.
2. Establish a Maclaurin power series expansion of the function $H_n(x)$ around the point $x = 0$ for $n \in \mathbb{N}$.

We will state and prove solutions of these two problems in the third and fourth sections of this paper.

## 2. Lemmas

For proceeding smoothly, we need the following lemmas which are very effective and applicable extensively.

LEMMA 1. *The function* $\phi(x) = (2^x - 1)\zeta(x)$ *is logarithmically convex on* $(1, \infty)$, *where*

$$\zeta(x) = \sum_{q=1}^{\infty} \frac{1}{q^x}, \quad \Re(x) > 1$$

*is the Riemann zeta function* [17, Chapter 25]. *Consequently, the sequence*

$$\frac{2^{2\ell+2}-1}{(\ell+1)(2\ell+1)(2^{2\ell}-1)} \left|\frac{B_{2\ell+2}}{B_{2\ell}}\right| \tag{16}$$

*is increasing in* $\ell \in \mathbb{N}$.

*Proof.* It is straightforward that

$$\begin{aligned}\phi(x) &= 2^x \sum_{q=1}^{\infty} \frac{1}{q^x} - \sum_{q=1}^{\infty} \frac{1}{q^x} \\ &= 2^x \sum_{q=1}^{\infty} \frac{1}{(2q-1)^x} + 2^x \sum_{q=1}^{\infty} \frac{1}{(2q)^x} - \sum_{q=1}^{\infty} \frac{1}{q^x} \\ &= \sum_{q=1}^{\infty} \frac{1}{(q-1/2)^x},\end{aligned}$$

$$[\ln\phi(x)]' = -\sum_{q=1}^{\infty}\frac{\ln(q-1/2)}{(q-1/2)^x}\bigg/\sum_{q=1}^{\infty}\frac{1}{(q-1/2)^x}$$

and

$$\begin{aligned}[\ln\phi(x)]'' &= \frac{\left[\sum_{j=1}^{\infty}\frac{1}{(j-1/2)^x}\right]^2}{\left[\sum_{j=1}^{\infty}\frac{\ln(j-1/2)}{(j-1/2)^x}\right]^2-\sum_{j=1}^{\infty}\frac{[\ln(j-1/2)]^2}{(j-1/2)^x}\sum_{j=1}^{\infty}\frac{1}{(j-1/2)^x}}\\
&= \frac{1}{\left[\sum_{j=1}^{\infty}\frac{1}{(j-1/2)^x}\right]^2}\sum_{j=1}^{\infty}\sum_{q=1}^{\infty}\frac{[\ln(j-1/2)]^2-\ln(j-1/2)\ln(q-1/2)}{(j-1/2)^x(q-1/2)^x}\\
&= \frac{1}{2\left[\sum_{j=1}^{\infty}\frac{1}{(j-1/2)^x}\right]^2}\sum_{j=1}^{\infty}\sum_{q=1}^{\infty}\frac{[\ln(j-1/2)-\ln(q-1/2)]^2}{(j-1/2)^x(q-1/2)^x}\\
&> 0.\end{aligned}$$

Consequently, the function $\phi(x)$ is logarithmically convex on $(1,\infty)$.

In [21, p. 5, (1.14)], we find that

$$B_{2q} = (-1)^{q+1}\frac{2(2q)!}{(2\pi)^{2q}}\zeta(2q), \quad q\in\mathbb{N}.$$

Then

$$\frac{2^{2\ell+2}-1}{(\ell+1)(2\ell+1)(2^{2\ell}-1)}\frac{|B_{2\ell+2}|}{|B_{2\ell}|} = \frac{1}{2\pi^2}\frac{\left(2^{2\ell+2}-1\right)\zeta(2\ell+2)}{(2^{2\ell}-1)\zeta(2\ell)}.$$

Since the function $\phi(x) = (2^x-1)\zeta(x)$ is logarithmically convex on $(1,\infty)$, then the first derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln[(2^x-1)\zeta(x)] = \frac{[(2^x-1)\zeta(x)]'}{(2^x-1)\zeta(x)}$$

increases in $x\in(1,\infty)$. Accordingly, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(2^{2x+2}-1)\zeta(2x+2)}{(2^{2x}-1)\zeta(2x)}\right] &= \frac{(2^{2x+2}-1)\zeta(2x+2)}{(2^{2x}-1)\zeta(2x)}\left(\frac{[(2^{2x+2}-1)\zeta(2x+2)]'}{(2^{2x+2}-1)\zeta(2x+2)}\right.\\
&\qquad\left.-\frac{[(2^{2x}-1)\zeta(2x)]'}{(2^{2x}-1)\zeta(2x)}\right)\\
&> 0\end{aligned}$$

for $x\in\left(\frac{1}{2},\infty\right)$. Hence, the function $\frac{(2^{2x+2}-1)\zeta(2x+2)}{(2^{2x}-1)\zeta(2x)}$ increases in $x\in\left(\frac{1}{2},\infty\right)$, and then the sequence in (16) increases in $\ell\in\mathbb{N}$. $\square$

LEMMA 2. (Monotonicity rule for the ratio of two power series [2]) *Let $\alpha_\ell$ and $\beta_\ell$ for $\ell\in\mathbb{N}_0$ be real sequences and the Maclaurin power series*

$$P(x) = \sum_{\ell=0}^{\infty}\alpha_\ell x^\ell \quad \textit{and} \quad Q(x) = \sum_{\ell=0}^{\infty}\beta_\ell x^\ell$$

*converge on* $(-\rho,\rho)$ *for some scalar* $\rho>0$*. If* $\beta_\ell>0$ *and the sequence* $\frac{\alpha_\ell}{\beta_\ell}$ *increases in* $\ell\in\mathbb{N}_0$*, then the function* $x\mapsto\frac{P(x)}{Q(x)}$ *increases on* $(0,\rho)$*.*

REMARK 1. There have been several independent developments of the monotonicity rules for the ratios between two differentiable functions, two Maclaurin power series, two Laplace transforms, two integrals, and the like. For more details, please refer to the newly published papers [3, 12, 14, 15], [19, Lemma 9 and Remark 15], [20, Remark 7.2], [25, Lemma 4], the arXiv preprints [11, 13], and closely related references therein.

In July 2023, a Chinese mathematician Zhen-Hang Yang drafted a review and survey article about the monotonicity rules for many various ratios and reported it in Guangdong University of Education.

LEMMA 3. ([4, p. 40, Exercise 5)]) *For* $n\in\mathbb{N}_0$ *and two* $n$*th differentiable functions* $p(x)$ *and* $q(x)\ne0$*, let*

$$W_{(n+1)\times(n+1)}(x)=\big(\mathscr{P}_{(n+1)\times1}(x)\ \mathscr{Q}_{(n+1)\times n}(x)\big)_{(n+1)\times(n+1)}$$

*and let* $|W_{(n+1)\times(n+1)}(x)|$ *denote the determinant of the* $(n+1)\times(n+1)$ *matrix, where the* $(n+1)\times1$ *matrix* $\mathscr{P}_{(n+1)\times1}(x)$ *is of the elements* $p_{\ell,1}(x)=p^{(\ell-1)}(x)$ *for* $1\leqslant\ell\leqslant n+1$*, and the* $(n+1)\times n$ *matrix* $\mathscr{Q}_{(n+1)\times n}(x)$ *is of the elements* $q_{\ell,j}(x)=\binom{\ell-1}{j-1}q^{(\ell-j)}(x)$ *for* $1\leqslant\ell\leqslant n+1$ *and* $1\leqslant j\leqslant n$*. Then the* $n$*th derivative of the ratio* $\frac{p(x)}{q(x)}$ *can be computed by the determinantal formula*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{p(x)}{q(x)}\right]=(-1)^n\frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{q^{n+1}(x)},\quad n\in\mathbb{N}_0. \tag{17}$$

## 3. Convexity

In this section, with the aid of Lemmas 1 and 2, we prove the convexity of the even function $H_n(x)=\ln h_n(x)$, the logarithmic convexity of the normalized tail $h_n(z)$.

THEOREM 1. *For* $n\in\mathbb{N}$*, the function* $H_n(x)=\ln h_n(x)$ *defined via the normalized tail* $h_n(x)$ *in* (14) *is convex on* $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$*. Consequently, the inequality*

$$\frac{x\tan x\sec^2x-\displaystyle\sum_{\ell=1}^{n-1}\frac{\ell(2\ell+1)2^{2\ell+2}\left(2^{2\ell+2}-1\right)}{(2\ell+2)!}|B_{2\ell+2}|x^{2\ell}}{\tan^2x-\displaystyle\sum_{\ell=1}^{n-1}\frac{(2\ell+1)2^{2\ell+2}\left(2^{2\ell+2}-1\right)}{(2\ell+2)!}|B_{2\ell+2}|x^{2\ell}}\\ >n+\frac{x^2}{2n^2+5n+2}\frac{2^{2n+4}-1}{2^{2n+2}-1}\left|\frac{B_{2n+4}}{B_{2n+2}}\right| \tag{18}$$

*is valid for* $0<|x|<\frac{\pi}{2}$ *and* $n\in\mathbb{N}$*.*

*Proof.* Direct computation gives

$$\left[H_n\left(\frac{x}{2}\right)\right]' = \frac{\displaystyle\sum_{\ell=1}^{\infty}\frac{1}{\binom{2n+2\ell+2}{2\ell}}\frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{x^{2\ell-1}}{(2\ell-1)!}}{\displaystyle\sum_{\ell=0}^{\infty}\frac{1}{\binom{2n+2\ell+2}{2\ell}}\frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{x^{2\ell}}{(2\ell)!}}$$
$$= \frac{\displaystyle x\sum_{\ell=0}^{\infty}\frac{1}{\binom{2n+2\ell+4}{2\ell+2}}\frac{(2n+2\ell+3)\left(2^{2n+2\ell+4}-1\right)|B_{2n+2\ell+4}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{x^{2\ell}}{(2\ell+1)!}}{\displaystyle\sum_{\ell=0}^{\infty}\frac{1}{\binom{2n+2\ell+2}{2\ell}}\frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{x^{2\ell}}{(2\ell)!}}$$

and

$$\frac{\frac{1}{\binom{2n+2\ell+4}{2\ell+2}}\frac{(2n+2\ell+3)\left(2^{2n+2\ell+4}-1\right)|B_{2n+2\ell+4}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{1}{(2\ell+1)!}}{\frac{1}{\binom{2n+2\ell+2}{2\ell}}\frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{1}{(2\ell)!}}
= \frac{\ell+1}{(n+\ell+2)(2n+2\ell+1)}\frac{\left(2^{2n+2\ell+4}-1\right)|B_{2n+2\ell+4}|}{\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}. \tag{19}$$

The increasing property of the sequence (19) is equivalent to

$$\frac{m-n}{(m+1)(2m-1)}\frac{\left(2^{2m+2}-1\right)|B_{2m+2}|}{\left(2^{2m}-1\right)|B_{2m}|} \leqslant \frac{m-n+1}{(m+2)(2m+1)}\frac{\left(2^{2m+4}-1\right)|B_{2m+4}|}{\left(2^{2m+2}-1\right)|B_{2m+2}|}$$

for $m \geqslant 2$ and $n \in \mathbb{N}$ such that $m-n-1 \geqslant 0$.

In Lemma 1, we proved that the sequence (16) is increasing in $\ell \geqslant 1$. On the other hand, it is easy to verify that, for given $n \in \mathbb{N}$, the sequence $\frac{(m-n)(2m+1)}{2m-1}$ is increasing in $m \geqslant 0$. Accordingly, we acquire that the product

$$\frac{(m-n)(2m+1)}{2m-1}\frac{2^{2m+2}-1}{(m+1)(2m+1)(2^{2m}-1)}\left|\frac{B_{2m+2}}{B_{2m}}\right| = \frac{m-n}{(m+1)(2m-1)}\frac{2^{2m+2}-1}{2^{2m}-1}\left|\frac{B_{2m+2}}{B_{2m}}\right| \tag{20}$$

is increasing in $m \geqslant 2$ for fixed $n \in \mathbb{N}$ such that $m-n-1 \geqslant 0$. As a result, the ratio in (19) is increasing in $\ell \in \mathbb{N}_0$ for given $n \in \mathbb{N}$. Making use of Lemma 2, we see that the function

$$\frac{1}{x}\left[H_n\left(\frac{x}{2}\right)\right]' = \frac{\displaystyle\sum_{\ell=0}^{\infty}\frac{1}{\binom{2n+2\ell+4}{2\ell+2}}\frac{(2n+2\ell+3)\left(2^{2n+2\ell+4}-1\right)|B_{2n+2\ell+4}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{x^{2\ell}}{(2\ell+1)!}}{\displaystyle\sum_{\ell=0}^{\infty}\frac{1}{\binom{2n+2\ell+2}{2\ell}}\frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\frac{x^{2\ell}}{(2\ell)!}} \tag{21}$$

is positive and increasing on $\left(0,\frac{\pi}{2}\right)$. Therefore, for $n\in\mathbb{N}$, the derivatives $\left[H_n\left(\frac{x}{2}\right)\right]'$ and $H_n'(x)$ are positive and increasing on $\left(0,\frac{\pi}{2}\right)$. In a word, for $n\in\mathbb{N}$, the function $H_n(x)$ is convex on $\left(0,\frac{\pi}{2}\right)$.

From (15) and (21) and by the increasing property of $\frac{H_n'(x)}{x}$, it follows that

$$\frac{H_n'(x)}{x} > \lim_{x\to0^+}\frac{H_n'(x)}{x} = \frac{2}{2n^2+5n+2}\frac{\left(2^{2n+4}-1\right)|B_{2n+4}|}{\left(2^{2n+2}-1\right)|B_{2n+2}|}$$

and

$$H_n'(x) = \frac{2\tan x\sec^2 x-\sum_{\ell=1}^{n-1}\frac{2^{2\ell+2}\left(2^{2\ell+2}-1\right)(2\ell+1)}{(2\ell+2)!}|B_{2\ell+2}|2\ell x^{2\ell-1}}{\tan^2 x-\sum_{\ell=1}^{n-1}\frac{2^{2\ell+2}\left(2^{2\ell+2}-1\right)(2\ell+1)}{(2\ell+2)!}|B_{2\ell+2}|x^{2\ell}} - \frac{2n}{x}$$

on $\left(0,\frac{\pi}{2}\right)$ for $n\in\mathbb{N}$. Consequently, we derive

$$\frac{2\tan x\sec^2 x-\sum_{\ell=1}^{n-1}\frac{2^{2\ell+2}\left(2^{2\ell+2}-1\right)(2\ell+1)}{(2\ell+2)!}|B_{2\ell+2}|2\ell x^{2\ell-1}}{\tan^2 x-\sum_{\ell=1}^{n-1}\frac{2^{2\ell+2}\left(2^{2\ell+2}-1\right)(2\ell+1)}{(2\ell+2)!}|B_{2\ell+2}|x^{2\ell}} > \frac{2n}{x}+\frac{2x}{2n^2+5n+2}\frac{\left(2^{2n+4}-1\right)|B_{2n+4}|}{\left(2^{2n+2}-1\right)|B_{2n+2}|}$$

on $\left(0,\frac{\pi}{2}\right)$ for $n\in\mathbb{N}$. The inequality (18) is thus proved. The proof of Theorem 1 is thus complete. □

COROLLARY 1. *For given $m\in\mathbb{N}$ and $n\in\mathbb{N}_0$, the sequence*

$$\frac{m-n}{(m+1)(2m-1)}\frac{2^{2m+2}-1}{2^{2m}-1}\left|\frac{B_{2m+2}}{B_{2m}}\right| \tag{22}$$

*is increasing in $m$ satisfying $m>n\in\mathbb{N}_0$.*

*Proof.* By calculus, when regarding $m$ as a continuous variable, we have

$$\frac{\mathrm{d}}{\mathrm{d}m}\left[\frac{(m-n)(2m+1)}{2m-1}\right] = \frac{(2m-1)^2+4n-2}{(2m-1)^2} > 0$$

for either $m,n>\frac{1}{2}$ or $m\geqslant\frac{1}{2}\left(1+\sqrt{2}\right)=1.207\cdots$ and $n=0$. This implies that,

1. when $m,n\in\mathbb{N}$, the sequence $\frac{(m-n)(2m+1)}{2m-1}$ is increasing in $m\in\mathbb{N}$ for fixed $n\in\mathbb{N}$;
2. when $m\in\mathbb{N}$ and $n=0$, the sequence $\frac{(m-n)(2m+1)}{2m-1}=\frac{m(2m+1)}{2m-1}$ is increasing in $m\geqslant 2$.

Combining these two items with the increasing property of the sequence (16), we derive that the sequence in (20) is increasing in $m > n \in \mathbb{N}$ for fixed $n \in \mathbb{N}$ or in $m \geqslant 2$ for $n = 0$. Moreover, numerical calculation shows

$$\left[\frac{m}{(m+1)(2m-1)}\frac{\left(2^{2m+2}-1\right)|B_{2m+2}|}{(2^{2m}-1)|B_{2m}|}\right]\bigg|_{m=1} = \frac{1}{2} \\ < \left[\frac{m}{(m+1)(2m-1)}\frac{\left(2^{2m+2}-1\right)|B_{2m+2}|}{(2^{2m}-1)|B_{2m}|}\right]\bigg|_{m=2} = \frac{2}{3}.$$

Consequently, the sequence (22) is increasing in $m$ satisfying $m > n \in \mathbb{N}_0$. □

## 4. Maclaurin power series expansion

In this section, in light of the derivative formula (17), we expand the even function $H_n(x) = \ln h_n(x)$ defined via the normalized tail $h_n(x)$ in (14) into a Maclaurin power series about $x = 0$.

THEOREM 2. *For $\ell \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let*

$$M_{\ell,n} = (2\ell)!!(2n+\ell+1)\frac{2^{2n+\ell+2}-1}{(2n+\ell+2)!}|B_{2n+\ell+2}|.$$

*Then we have*

$$H_n(x) = -\sum_{\ell=1}^{\infty} D_{2\ell}(n)\left[\frac{(2n+2)!}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\right]^{2\ell}\frac{x^{2\ell}}{(2\ell)!},$$

*where*

$$D_{2\ell}(n) = \begin{vmatrix}
0 & \binom{0}{0}M_{0,n} & 0 & 0 & \cdots & 0 \\
M_{2,n} & 0 & \binom{1}{1}M_{0,n} & 0 & \cdots & 0 \\
0 & \binom{2}{0}M_{2,n} & 0 & \binom{2}{2}M_{0,n} & \cdots & 0 \\
M_{4,n} & 0 & \binom{3}{1}M_{2,n} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
0 & \binom{\ell-3}{0}M_{2\ell-4,n} & 0 & \binom{\ell-3}{2}M_{2\ell-6,n} & \cdots & 0 \\
M_{2\ell-2,n} & 0 & \binom{\ell-2}{1}M_{2\ell-4,n} & 0 & \cdots & 0 \\
0 & \binom{\ell-1}{0}M_{2\ell-2,n} & 0 & \binom{\ell-1}{2}M_{2\ell-4,n} & \cdots & \binom{\ell-1}{\ell-1}M_{0,n} \\
M_{2\ell,n} & 0 & \binom{\ell}{1}M_{2\ell-2,n} & 0 & \cdots & 0
\end{vmatrix}.$$

*Proof.* For $\ell \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let

$$E_{\ell,n} = \frac{2^\ell}{\binom{2n+\ell+2}{\ell}} \frac{(2n+\ell+1)\left(2^{2n+\ell+2}-1\right)|B_{2n+\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}.$$

Then

$$E_{\ell,n} = \frac{(2n+2)!}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|} M_{\ell,n}.$$

We note that $B_{2\ell+1} = 0$ for $\ell \geqslant 1$.

Differentiating on both sides of (15) gives

$$H_n'(x) = \frac{\displaystyle\sum_{\ell=0}^{\infty} \frac{2^{2\ell+2}}{\binom{2n+2\ell+4}{2\ell+2}} \frac{(2n+2\ell+3)\left(2^{2n+2\ell+4}-1\right)|B_{2n+2\ell+4}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|} \frac{x^{2\ell+1}}{(2\ell+1)!}}{\displaystyle\sum_{\ell=0}^{\infty} \frac{2^{2\ell}}{\binom{2n+2\ell+2}{2\ell}} \frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|} \frac{x^{2\ell}}{(2\ell)!}}.$$

Let

$$q_n(x) = \sum_{\ell=0}^{\infty} \frac{2^{2\ell}}{\binom{2n+2\ell+2}{2\ell}} \frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|} \frac{x^{2\ell}}{(2\ell)!}$$

and

$$\begin{aligned} p_n(x) &= q_n'(x) \\ &= \sum_{\ell=0}^{\infty} \frac{2^{2\ell+2}}{\binom{2n+2\ell+4}{2\ell+2}} \frac{(2n+2\ell+3)\left(2^{2n+2\ell+4}-1\right)|B_{2n+2\ell+4}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|} \frac{x^{2\ell+1}}{(2\ell+1)!} \end{aligned}$$

for $n \in \mathbb{N}$. Then

$$p_n^{(2\ell)}(0) = q_n^{(2\ell+1)}(0) = 0 = E_{2\ell+1,n}$$

for $\ell \in \mathbb{N}_0$ and

$$\begin{aligned} p_n^{(2\ell-1)}(0) &= q_n^{(2\ell)}(0) \\ &= \frac{2^{2\ell}}{\binom{2n+2\ell+2}{2\ell}} \frac{(2n+2\ell+1)\left(2^{2n+2\ell+2}-1\right)|B_{2n+2\ell+2}|}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|} \\ &= E_{2\ell,n} \end{aligned}$$

for $\ell \geqslant 1$. Accordingly, making use of the formula (17), we obtain

$$
\begin{aligned}
H_n^{(\ell+1)}(0) &= \frac{(-1)^\ell}{q_n^{\ell+1}(0)} \left|\mathscr{P}_{(\ell+1)\times 1}(0)\ \mathscr{Q}_{(\ell+1)\times\ell}(0)\right|_{(\ell+1)\times(\ell+1)} \\
&= (-1)^\ell \begin{vmatrix}
E_{1,n} & \binom{0}{0}E_{0,n} & 0 & 0 & \cdots & 0 \\
E_{2,n} & \binom{1}{0}E_{1,n} & \binom{1}{1}E_{0,n} & 0 & \cdots & 0 \\
E_{3,n} & \binom{2}{0}E_{2,n} & \binom{2}{1}E_{1,n} & \binom{2}{2}E_{0,n} & \cdots & 0 \\
E_{4,n} & \binom{3}{0}E_{3,n} & \binom{3}{1}E_{2,n} & \binom{3}{2}E_{1,n} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
E_{\ell-2,n} & \binom{\ell-3}{0}E_{\ell-3,n} & \binom{\ell-3}{1}E_{\ell-4,n} & \binom{\ell-3}{2}E_{\ell-5,n} & \cdots & 0 \\
E_{\ell-1,n} & \binom{\ell-2}{0}E_{\ell-2,n} & \binom{\ell-2}{1}E_{\ell-3,n} & \binom{\ell-2}{2}E_{\ell-4,n} & \cdots & 0 \\
E_{\ell,n} & \binom{\ell-1}{0}E_{\ell-1,n} & \binom{\ell-1}{1}E_{\ell-2,n} & \binom{\ell-1}{2}E_{\ell-3,n} & \cdots & \binom{\ell-1}{\ell-1}E_{0,n} \\
E_{\ell+1,n} & \binom{\ell}{0}E_{\ell,n} & \binom{\ell}{1}E_{\ell-1,n} & \binom{\ell}{2}E_{\ell-2,n} & \cdots & \binom{\ell}{\ell-1}E_{1,n}
\end{vmatrix} \\
&= (-1)^\ell \left[\frac{(2n+2)!}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\right]^{\ell+1} \\
&\quad\times \begin{vmatrix}
M_{1,n} & \binom{0}{0}M_{0,n} & 0 & 0 & \cdots & 0 \\
M_{2,n} & \binom{1}{0}M_{1,n} & \binom{1}{1}M_{0,n} & 0 & \cdots & 0 \\
M_{3,n} & \binom{2}{0}M_{2,n} & \binom{2}{1}M_{1,n} & \binom{2}{2}M_{0,n} & \cdots & 0 \\
M_{4,n} & \binom{3}{0}M_{3,n} & \binom{3}{1}M_{2,n} & \binom{3}{2}M_{1,n} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
M_{\ell-2,n} & \binom{\ell-3}{0}M_{\ell-3,n} & \binom{\ell-3}{1}M_{\ell-4,n} & \binom{\ell-3}{2}M_{\ell-5,n} & \cdots & 0 \\
M_{\ell-1,n} & \binom{\ell-2}{0}M_{\ell-2,n} & \binom{\ell-2}{1}M_{\ell-3,n} & \binom{\ell-2}{2}M_{\ell-4,n} & \cdots & 0 \\
M_{\ell,n} & \binom{\ell-1}{0}M_{\ell-1,n} & \binom{\ell-1}{1}M_{\ell-2,n} & \binom{\ell-1}{2}M_{\ell-3,n} & \cdots & \binom{\ell-1}{\ell-1}M_{0,n} \\
M_{\ell+1,n} & \binom{\ell}{0}M_{\ell,n} & \binom{\ell}{1}M_{\ell-1,n} & \binom{\ell}{2}M_{\ell-2,n} & \cdots & \binom{\ell}{\ell-1}M_{1,n}
\end{vmatrix}.
\end{aligned}
$$

Since the function $H_n(x)$ is even, we deduce $H_n^{(2\ell+1)}(0)=0$ for $\ell\in\mathbb{N}_0$, that is,

$$
\begin{vmatrix}
0 & \binom{0}{0}M_{0,n} & 0 & 0 & \cdots & 0 \\
M_{2,n} & 0 & \binom{1}{1}M_{0,n} & 0 & \cdots & 0 \\
0 & \binom{2}{0}M_{2,n} & 0 & \binom{2}{2}M_{0,n} & \cdots & 0 \\
M_{4,n} & 0 & \binom{3}{1}M_{2,n} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
M_{2\ell-2,n} & 0 & \binom{2\ell-3}{1}M_{2\ell-4,n} & \binom{2\ell-3}{2}M_{2\ell-5,n} & \cdots & 0 \\
0 & \binom{2\ell-2}{0}M_{2\ell-2,n} & 0 & \binom{2\ell-2}{2}M_{2\ell-4,n} & \cdots & 0 \\
M_{2\ell,n} & 0 & \binom{2\ell-1}{1}M_{2\ell-2,n} & 0 & \cdots & \binom{2\ell-1}{2\ell-1}M_{0,n} \\
0 & \binom{2\ell}{0}M_{2\ell,n} & 0 & \binom{2\ell}{2}M_{2\ell-2,n} & \cdots & 0
\end{vmatrix} = 0.
$$

Meanwhile, for $\ell \geqslant 1$, we have

$$H_n^{(2\ell)}(0) = -\left[\frac{(2n+2)!}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\right]^{2\ell} \times \begin{vmatrix} 0 & \binom{0}{0}M_{0,n} & 0 & 0 & \cdots & 0 \\ M_{2,n} & 0 & \binom{1}{1}M_{0,n} & 0 & \cdots & 0 \\ 0 & \binom{2}{0}M_{2,n} & 0 & \binom{2}{2}M_{0,n} & \cdots & 0 \\ M_{4,n} & 0 & \binom{3}{1}M_{2,n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \binom{\ell-3}{0}M_{2\ell-4,n} & 0 & \binom{\ell-3}{2}M_{2\ell-6,n} & \cdots & 0 \\ M_{2\ell-2,n} & 0 & \binom{\ell-2}{1}M_{2\ell-4,n} & 0 & \cdots & 0 \\ 0 & \binom{\ell-1}{0}M_{2\ell-2,n} & 0 & \binom{\ell-1}{2}M_{2\ell-4,n} & \cdots & \binom{\ell-1}{\ell-1}M_{0,n} \\ M_{2\ell,n} & 0 & \binom{\ell}{1}M_{2\ell-2,n} & 0 & \cdots & 0 \end{vmatrix}.$$

Consequently, we acquire

$$\begin{aligned} H_n(x) &= \sum_{\ell=0}^{\infty} H_n^{(\ell)}(0)\frac{x^\ell}{\ell!} \\ &= \sum_{\ell=1}^{\infty} H_n^{(2\ell)}(0)\frac{x^{2\ell}}{(2\ell)!} \\ &= -\sum_{\ell=1}^{\infty} D_{2\ell}(n)\left[\frac{(2n+2)!}{(2n+1)(2^{2n+2}-1)|B_{2n+2}|}\right]^{2\ell}\frac{x^{2\ell}}{(2\ell)!}. \end{aligned}$$

The required proof of Theorem 2 is complete. □

REMARK 2. Numerical computation yields

$$D_2(1) = \begin{vmatrix} 0 & \binom{0}{0}M_{0,1} \\ M_{2,1} & 0 \end{vmatrix} = -M_{0,1}M_{2,1} = -\frac{1}{192}$$

and

$$D_4(1) = \begin{vmatrix} 0 & \binom{0}{0}M_{0,1} & 0 & 0 \\ M_{2,1} & 0 & \binom{1}{1}M_{0,1} & 0 \\ 0 & \binom{2}{0}M_{2,1} & 0 & \binom{2}{2}M_{0,1} \\ M_{4,1} & 0 & \binom{3}{1}M_{2,1} & 0 \end{vmatrix} = -\frac{7}{122\,880}.$$

Hence, it follows that

$$\begin{aligned} H_1(x) &= -\sum_{\ell=1}^{\infty} D_{2\ell}(1)\left[\frac{4!}{3(2^4-1)|B_4|}\right]^{2\ell}\frac{x^{2\ell}}{(2\ell)!} \\ &= -\sum_{\ell=1}^{\infty} D_{2\ell}(1)16^{2\ell}\frac{x^{2\ell}}{(2\ell)!} \\ &= -D_2(1)16^2\frac{x^2}{2!} - D_4(1)16^4\frac{x^4}{4!} - \cdots \\ &= \frac{2}{3}x^2 + \frac{7}{45}x^4 + \cdots. \end{aligned}$$

This coincides with the first two terms in the series expansion

$$\begin{aligned} \ln\frac{\tan^2 x}{x^2} &= \sum_{\ell=1}^{\infty}\frac{2^{2\ell+1}\left(2^{2\ell-1}-1\right)}{\ell(2\ell)!}|B_{2\ell}|x^{2\ell} \\ &= \frac{2x^2}{3} + \frac{7x^4}{45} + \frac{124x^6}{2835} + \frac{127x^8}{8{,}450} + \frac{292x^{10}}{66{,}825} + \cdots, \quad 0<|x|<\frac{\pi}{2}, \end{aligned}$$

which can be deduced from

$$\begin{aligned} \ln\tan x &= \ln x + \sum_{\ell=1}^{\infty}\frac{2^{2\ell}\left(2^{2\ell-1}-1\right)}{\ell(2\ell)!}|B_{2\ell}|x^{2\ell} \\ &= \ln x + \frac{x^2}{3} + \frac{7x^4}{90} + \frac{62x^6}{2835} + \frac{127x^8}{18{,}900} + \frac{146x^{10}}{66{,}825} + \cdots, \quad 0<x<\frac{\pi}{2} \end{aligned}$$

found in the handbook [6, p. 55].

## 5. Conclusions

There are two main conclusions in this paper.

The first main conclusion is the convexity of the function $H_n(x)=\ln h_n(x)$ defined via the normalized tail $h_n(x)$ in (14) for $n\in\mathbb{N}$ and $x\in\left(0,\frac{\pi}{2}\right)$; see Theorem 1.

The second main conclusion is the Maclaurin power series expansion about $x=0$ of the function $H_n(x)=\ln h_n(x)$ defined via the normalized tail $h_n(x)$ in (14) for $n\in\mathbb{N}$; see Theorem 2.

Lemma 1 and Corollary 1 are interesting too.

The infantile idea and thought of the notion of normalized tails of the Maclaurin power series expansions of analytic functions originated from Qi in the papers [7, 9, 10]. Hereafter, the novel notion was enlightenedly and formally invented and designed in the papers [1, 16, 18, 22, 23, 26, 27]. From main results in [1, 16, 18, 22, 23, 26, 27] and [8, Remark 7], we can understand the significance of the notion of normalized tails of the Maclaurin power series expansions of analytic functions.

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