Journal of
Mathematical
Inequalities
Volume 18, Number 3 (2024), 1067–1081 doi:10.7153/jmi-2024-18-59

IMPROVED INEQUALITIES FOR THE BEREZIN NUMBER

DAVOOD AFRAZ, RAHMATOLLAH LASHKARIPOUR AND MOJTABA BAKHERAD*

(*Communicated by T. Burić*)

Abstract. A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Theta$ are continuous on $\mathscr{H}$. The Berezin number(radius) of an operator T is defined by $\mathbf{ber}(T) = \sup_{\tau\in\Theta} \left|\langle T\hat{k}_\tau, \hat{k}_\tau\rangle\right|$, where the operator T acts on the reproducing kernel Hilbert space $\mathscr{H} = \mathscr{H}(\Theta)$ over some(non-empty) set Θ. In this paper, we give some Berezin number inequalities. Moreover, we present some inequalities involving the weighted Berezin number of operators on the reproducing kernel Hilbert space.

1. Introduction

Suppose that $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ is a complex Hilbert space and $\mathscr{L}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators defined on $\mathscr{H}$. For an operator $T \in \mathscr{L}(\mathscr{H})$, the Cartesian decomposition of an operator $T \in \mathscr{L}(\mathscr{H})$ can be written as $T = \Re(T) + i\Im(T)$, where $\Re(T) = \frac{T+T^*}{2}$ and $\Im(T) = \frac{T-T^*}{2i}$.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Theta$ are continuous on $\mathscr{H}$. Applying the Riesz representation theorem there is a unique element $k_\tau \in \mathscr{H}$ such that $f(\tau) = \langle f, k_\tau\rangle$ for all $f \in \mathscr{H}$ and every $\tau \in \Theta$. The function k on $\Theta \times \Theta$ defined by $k(z,\tau) = k_\tau(z)$ is called the reproducing kernel of $\mathscr{H}$, see [1]. It was shown that $k_\tau(z)$ can be represented by $k_\tau(z) = \sum_{n=1}^{\infty} \overline{e_n(\tau)} e_n(z)$ for any orthonormal basis $\{e_n\}_{n\geqslant 1}$ of $\mathscr{H}$. For example for the Hardy-Hilbert space $\mathscr{H}^2 = \mathscr{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\{z^n\}_{n\geqslant 1}$ is an orthonormal basis, and so the reproducing kernel of $\mathscr{H}^2$ is the function $k_\tau(z) = \sum_{n=1}^{\infty} \overline{\tau}_n z^n = (1-\overline{\tau}z)^{-1}$, $\tau \in \mathbb{D}$. Let $\hat{k}_\tau = \frac{k_\tau}{\|k_\tau\|}$ be the normalized reproducing kernel of the space $\mathscr{H}$ (RKHS). For a given bounded linear operator T on $\mathscr{H}$, the Berezin symbol(or Berezin transform) of T is the bounded function $\widetilde{T}$ on Θ defined by

$$\widetilde{T}(\tau) = \left\langle T\hat{k}_\tau(z), \hat{k}_\tau(z)\right\rangle, \ \tau \in \Theta.$$

Mathematics subject classification (2020): Primary 47A63; Secondary 15A60, 47A60.
Keywords and phrases: Berezin radius, Berezin norm, reproducing kernel, Schwarz inequality.
* Corresponding author.

An important property of the Berezin symbol is that for all $T,S \in \mathscr{L}(\mathscr{H})$ if $\widetilde{T}(\tau) = \widetilde{S}(\tau)$ for all $\tau \in \Theta$, then $T = S$, see Zhu [26] and for more details, see [4, 9, 12, 17, 18]. So, the map $T \to \widetilde{T}$ is injective [13]. The Berezin set and the Berezin number(radius) of an operator T are defined, respectively, by

$$\mathbf{Ber}(T) = \mathbf{Range}\left(\widetilde{T}\right) \quad \text{and} \quad \mathbf{ber}(T) = \sup_{\tau\in\Theta}\left|\langle T\hat{k}_\tau, \hat{k}_\tau\rangle\right|.$$

Recall that the numerical range and the numerical radius number of $T \in \mathscr{L}(\mathscr{H})$ are defined, respectively, by

$$W(T) := \{\langle Tx, x\rangle : x \in \mathscr{H} \text{ and } \|x\| = 1\} \quad \text{and} \quad w(T) := \sup_{\langle Tx,x\rangle \in W(T)} |\langle Tx, x\rangle|.$$

For more facts about the numerical radius, we refer the reader to [7, 20, 21, 22] and references therein. It is clear that the Berezin transform $\widetilde{A}$ is the bounded function on Θ whose values in the numerical range of the operator A and hence $\mathbf{Ber}(A) \subseteq W(A)$ and $\mathbf{ber}(A) \leqslant w(A)$ for all $A \in \mathscr{L}(\mathscr{H})$. Karaev [19] showed that for $A = S \otimes S \in \mathscr{L}\left(\mathscr{H}^2\right)$, where S is the shift operator defined by $Sf(z) = zf(z)$ on the Hardy-Hilbert space $\mathscr{H}^2 = \mathscr{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\widetilde{A}(\lambda) = |\lambda|^2(1-|\lambda|^2)$, and thus $\mathbf{Ber}(A) = \left[0, \frac{1}{4}\right] \subsetneq [0,1] = W(A)$ and $\mathbf{ber}(A) = \frac{1}{4} < 1 = w(A)$. For more information about the Berezin number and the Berezin norm, see [5, 9, 11, 14, 15, 18] and references therein.

The Berezin norm of an operator $T \in \mathscr{L}(\mathscr{H})$ is defined by

$$\|T\|_{\mathbf{ber}} := \sup_{\tau,\mu\in\Theta}\left|\langle T\hat{k}_\tau, \hat{k}_\mu\rangle\right|.$$

Clearly, by the Cauchy-Schwarz inequality, we have $\mathbf{ber}(T) \leqslant \|T\|_{\mathbf{ber}}$. Moreover for $T,S \in \mathscr{L}(\mathscr{H})$, it is clear from the above definitions of the Berezin radius and the Berezin norms that the following properties hold:

(1) $\mathbf{ber}(\alpha T) = |\alpha|\mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$;

(2) $\mathbf{ber}(T+S) \leqslant \mathbf{ber}(T) + \mathbf{ber}(S)$;

(3) $\|\lambda T\|_{\mathbf{ber}} = |\lambda|\,\|T\|_{\mathbf{ber}}$ for all $\lambda \in \mathbb{C}$;

(4) $\|T+S\|_{\mathbf{ber}} \leqslant \|T\|_{\mathbf{ber}} + \|S\|_{\mathbf{ber}}$.

A generalization of the Cartesian decomposition was introduced in [25], called the weighted real and imaginary part of T defined by

$$\Re_\alpha(T) = (1-\alpha)T^* + \alpha T \quad \text{and} \quad \Im_\alpha(T) = \frac{(1-\alpha)T - \alpha T^*}{i} \quad \text{for all } \alpha \in [0,1].$$

Obviously for $\alpha = \frac{1}{2}$, $\Re_\alpha(T) = \Re(T)$ and $\Im_\alpha(T) = \Im(T)$. It is easy to see that for every operator $T \in \mathscr{L}(\mathscr{H})$, $\Re_\alpha(T) + i\Im_\alpha(T) = (1-2\alpha)T^* + T$.

If $\mathscr{H}=\mathscr{H}(\Theta)$ is a reproducing kernel Hilbert space for $T\in\mathscr{L}(\mathscr{H})$ and $\alpha\in[0,1]$, the weighed Berezin radius is defined by the following formulas [10]

$$\mathbf{ber}_\alpha(T):=\sup_{\theta\in\mathbb{R}}\mathbf{ber}(\Re_\alpha(e^{i\theta}T))=\sup_{\theta\in\mathbb{R}}\mathbf{ber}((1-\alpha)e^{-i\theta}T^*+\alpha e^{i\theta}T). \tag{1}$$

Similar to the Berezin radius inequality, the weighted Berezin radius also satisfies the triangle inequality

$$\mathbf{ber}_\alpha(T+S)\leqslant\mathbf{ber}_\alpha(T)+\mathbf{ber}_\alpha(S)\quad\text{for } S,T\in\mathscr{L}(\mathscr{H}).$$

It is easy to observe that for $\alpha=\frac{1}{2}$, $\mathbf{ber}_\alpha(T)=\mathbf{ber}(T)$.

For two operators $S,T\in\mathscr{L}(\mathscr{H})$ the Euclidean Berezin number and the Euclidean Berezin norm are defined, respectively, by

$$\mathbf{ber_e}(S,T):=\sup_{\tau\in\Theta}\sqrt{\left|\left\langle S\hat{k}_\tau,\hat{k}_\tau\right\rangle\right|^2+\left|\left\langle T\hat{k}_\tau,\hat{k}_\tau\right\rangle\right|^2}$$

and

$$\|(S,T)\|_{e,\mathbf{ber}}:=\sup_{\tau,\mu\in\Theta}\sqrt{\left|\left\langle S\hat{k}_\tau,\hat{k}_\mu\right\rangle\right|^2+\left|\left\langle T\hat{k}_\tau,\hat{k}_\mu\right\rangle\right|^2}.$$

One of the most basic yet useful inequalities that govern the inner product is the so-called Cauchy-Schwarz inequality, which asserts that

$$|\langle x,y\rangle|\leqslant\|x\|\|y\|\quad\text{for all } x,y\in\mathscr{H}.$$

Buzano in [8] showed an extension of the Cauchy-Schwarz inequality as follows:

$$\left|\langle x,e\rangle\langle e,x\rangle\right|\leqslant\frac{1}{2}\left(|\langle x,y\rangle|+\|x\|\|y\|\right)\quad\text{for all } x,y,e\in\mathscr{H}, \tag{2}$$

where $\|e\|=1$. For an arbitrary operator $T\in\mathscr{L}(\mathscr{H})$, the mixed Schwarz inequality has been established by Kato in [16]. This inequality asserts that

$$|\langle Tx,y\rangle|\leqslant\sqrt{\langle|T|x,x\rangle\langle|T^*|y,y\rangle}\quad\text{for all } x,y\in\mathscr{H}.$$

Recently, Sahoo et al. [23] proved that

$$\left|\left\langle AXBx,y\right\rangle\right|\leqslant\frac{\|X\|}{2}\left(|\langle ABx,y\rangle|+\|Bx\|\|A^*y\|\right) \tag{3}$$

for all $x,y\in\mathscr{H}$ and all $A,X,B\in\mathscr{L}(\mathscr{H})$, where X is positive. The above inequalities have many applications to obtain numerical radius and Berezin number inequalities.

In [6], the authors obtained

$$\mathbf{ber}^r\left(A^*XB\right)\leqslant\frac{\|X\|^r}{2^r}\left\|\left(A^*A+B^*B\right)^r\right\|_{\mathbf{ber}} \tag{4}$$

for $A,B,X \in \mathscr{L}(\mathscr{H})$ and $r \geqslant 1$. Another version of this inequality was proved by [2], as follows:

$$\mathbf{ber}^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leqslant \mathbf{ber}\left(\frac{1}{p}\left[\sum_{i=1}^n B_i^* f^2\left(|X_i|\right) B_i\right]^{\frac{rp}{2}} + \frac{1}{p}\left[\sum_{i=1}^n A_i^* g^2\left(|X_i^*|\right) A_i\right]^{\frac{rq}{2}}\right) \quad (5)$$

for $A_i,B_i,X_i \in \mathscr{L}(\mathscr{H})$ $(1 \leqslant i \leqslant n)$, nonnegative continuous functions f,g on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \geqslant 0)$, $\frac{1}{p}+\frac{1}{q}=1$, $r \geqslant 1$ and $pr \geqslant qr \geqslant 2$.

In this paper, we give some Berezin number inequalities. More precisely, in the second section, we obtain some Berezin inequalities including the product of operators. For instance, we prove

$$\mathbf{ber}\left(A^*XB\right) \leqslant \frac{\|X\|}{2}\left(\mathbf{ber}\left(A^*B\right) + \frac{1}{2}\mathbf{ber}\left(|A|^2+|B|^2\right)\right),$$

where $A,B,X \in \mathscr{L}(\mathscr{H})$ such that X is positive. In the third section, we show some the Euclidean Berezin number and the Berezin norm inequalities. Furthermore, in the last section, we present some inequalities involving the weighted Berezin radius of operators on the reproducing kernel Hilbert space.

2. Some inequalities related to the Berezin number

The following well known lemma are essential to prove our results in this section.

LEMMA 1. [2] *Let $A,B \in \mathscr{L}(\mathscr{H})$. Then*

$$\boldsymbol{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leqslant \max\{\boldsymbol{ber}(A), \boldsymbol{ber}(B)\}.$$

Our first result of this section is the following.

THEOREM 1. *Suppose that $A,B,X \in \mathscr{L}(\mathscr{H})$ such that X is positive. Then*

$$\boldsymbol{ber}\left(A^*XB\right) \leqslant \frac{\|X\|}{2}\left(\boldsymbol{ber}\left(A^*B\right) + \frac{1}{2}\boldsymbol{ber}\left(|A|^2+|B|^2\right)\right).$$

Proof. Let $\tau \in \Theta$ be arbitrary. From the inequality (3), we have

$$\begin{aligned}
\left|\left\langle A^*XB\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| &\leqslant \frac{\|X\|}{2}\left(\left|\left\langle A^*B\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| + \|A\hat{k}_\tau\|\|B\hat{k}_\tau\|\right) \\
&= \frac{\|X\|}{2}\left(\left|\left\langle A^*B\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| + \sqrt{\left\langle |A|^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\left\langle |B|^2\hat{k}_\tau, \hat{k}_\tau\right\rangle}\right) \\
&\leqslant \frac{\|X\|}{2}\left(\left|\left\langle A^*B\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| + \frac{1}{2}\left(\left\langle |A|^2\hat{k}_\tau, \hat{k}_\tau\right\rangle + \left\langle |B|^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\right)\right) \\
&\leqslant \frac{\|X\|}{2}\left(\left|\left\langle A^*B\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| + \frac{1}{2}\left\langle\left(|A|^2+|B|^2\right)\hat{k}_\tau, \hat{k}_\tau\right\rangle\right).
\end{aligned}$$

By taking the supremum over $\tau \in \Theta$ in the above inequality, we obtain

$$\mathbf{ber}\left(A^*XB\right) \leqslant \frac{\|X\|}{2}\left(\mathbf{ber}\left(A^*B\right)+\frac{1}{2}\mathbf{ber}\left(|A|^2+|B|^2\right)\right). \quad \square$$

REMARK 1. Suppose that $A,B \in \mathscr{L}(\mathscr{H})$. If we put $X=I$, $r=1$, $p=q=2$ and $f(t)g(t)=\sqrt{t}$ $(t \in [0,\infty))$ in the inequality (5), we have

$$\mathbf{ber}\left(A^*B\right) \leqslant \frac{1}{2}\mathbf{ber}\left(|A|^2+|B|^2\right). \tag{6}$$

Now, Theorem 1 and the inequality (6) imply that

$$\begin{aligned}\mathbf{ber}\left(A^*XB\right) &\leqslant \frac{\|X\|}{2}\left(\mathbf{ber}\left(A^*B\right)+\frac{1}{2}\mathbf{ber}\left(|A|^2+|B|^2\right)\right)\\ &\leqslant \frac{\|X\|}{2}\mathbf{ber}\left(|A|^2+|B|^2\right).\end{aligned}$$

The above inequalities show that Theorem 1 is a refinement of the inequality (4) for $r=1$.

LEMMA 2. *Assume that $A \in \mathscr{L}(\mathscr{H})$. Then*

$$\frac{1}{2}\mathbf{ber}(A) \leqslant \mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) \leqslant \mathbf{ber}(A).$$

Proof. For every $(\tau,\mu) \in \Theta \times \Theta$, let $\hat{\mathbf{k}}_{(\tau,\mu)} = \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}$ be the reproducing kernel of $\mathscr{H} \oplus \mathscr{H}$ i.e., $\|k_\tau\|^2+\|k_\mu\|^2=1$. Then, we have

$$\begin{aligned}\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) &= \sup\left\{\left|\left\langle \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\mu \end{bmatrix}\right\rangle\right| : (\tau,\mu) \in \Theta \times \Theta, \|k_\tau\|^2+\|k_\mu\|^2=1\right\}\\ &\geqslant \sup\left\{\left|\left\langle \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} k_\tau \\ k_\tau \end{bmatrix}, \begin{bmatrix} k_\tau \\ k_\tau \end{bmatrix}\right\rangle\right| : (\tau,\tau) \in \Theta \times \Theta, \|k_\tau\|^2+\|k_\tau\|^2=1\right\}\\ &= \sup\left\{\left|\left\langle \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\hat{k}_\tau}{\sqrt{2}} \\ \frac{\hat{k}_\tau}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{\hat{k}_\tau}{\sqrt{2}} \\ \frac{\hat{k}_\tau}{\sqrt{2}} \end{bmatrix}\right\rangle\right| : (\tau,\tau) \in \Theta \times \Theta, \|\hat{k}_\tau\|=1\right\}\\ &= \sup\left\{\left|\left\langle A\frac{\hat{k}_\tau}{\sqrt{2}}, \frac{\hat{k}_\tau}{\sqrt{2}}\right\rangle\right| : \tau \in \Theta, \|\hat{k}_\tau\|=1\right\}\\ &= \frac{1}{2}\sup\left\{\left|\left\langle A\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| : \tau \in \Theta, \|\hat{k}_\tau\|=1\right\}\\ &= \frac{1}{2}\mathbf{ber}(A).\end{aligned}$$

Further, Lemma 1 implies that

$$\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) \leqslant \mathbf{ber}(A).$$

Then, by the two above inequalities, we get the desired result. □

THEOREM 2. *Let $A_i, B_i, X_i \in \mathscr{L}(\mathscr{H})$ for all $i = 1, 2, \cdots, n$ such that X_i's $(i = 1, 2, \cdots, n)$ are positive. Then*

$$\boldsymbol{ber}\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leqslant \left(\max_{1 \leqslant i \leqslant n} \|X_i\|\right)\left(\boldsymbol{ber}\left(\sum_{i=1}^{n} A_i^* B_i\right) + \frac{1}{2}\boldsymbol{ber}\left(\sum_{i=1}^{n}\left(|A_i|^2 + |B_i|^2\right)\right)\right).$$

Proof. Consider the operator matrices

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_n & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ B_n & 0 & \cdots & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{pmatrix}.$$

Therefore

$$A^* X B = \begin{pmatrix} \sum_{i=1}^{n} A_i^* X_i B_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Utilizing Lemma 2, we have

$$\begin{aligned} \mathbf{ber}\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) &= 2\mathbf{ber}(A^* X B) \\ &\leqslant \|X\|\left(\mathbf{ber}(A^* B) + \frac{1}{2}\mathbf{ber}\left(|A|^2 + |B|^2\right)\right) \qquad \text{(by Theorem 1)} \\ &= \left(\max_{1 \leqslant i \leqslant n} \|X_i\|\right)\left(\mathbf{ber}\left(\sum_{i=1}^{n} A_i^* B_i\right) + \frac{1}{2}\mathbf{ber}\left(\sum_{i=1}^{n}\left(|A_i|^2 + |B_i|^2\right)\right)\right). \end{aligned}$$

□

COROLLARY 1. *Assume that $S, T \in \mathscr{L}(\mathscr{H})$. Then*

$$\boldsymbol{ber}(ST) \leqslant \frac{\|S\|^{\frac{1}{2}}}{2}\boldsymbol{ber}\left(|S^*| + |T|^2\right).$$

Proof. Let $S = U|S|$ be the polar decomposition of S. By setting $A = U|S|^{\frac{1}{2}}$, $X = |S|^{\frac{1}{2}}$ and $B = T$ in Theorem 1, we deduce that

$$\mathbf{ber}(ST) \leqslant \frac{\|S\|^{\frac{1}{2}}}{2}\left(\mathbf{ber}\left(U|S|^{\frac{1}{2}}T\right) + \frac{1}{2}\mathbf{ber}\left(|S^*| + |T|^2\right)\right). \tag{7}$$

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|\left\langle U|S|^{\frac{1}{2}}T\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| &= \left|\left\langle T\hat{k}_\tau, |S|^{\frac{1}{2}}U^*\hat{k}_\tau\right\rangle\right| \\
&\leqslant \|T\hat{k}_\tau\|\||S|^{\frac{1}{2}}U^*\hat{k}_\tau\| \\
&= \sqrt{\left\langle |T|^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\left\langle U|S|U^*\hat{k}_\tau, \hat{k}_\tau\right\rangle} \\
&= \sqrt{\left\langle |T|^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\left\langle |S^*|\hat{k}_\tau, \hat{k}_\tau\right\rangle} \\
&\leqslant \frac{1}{2}\left\langle\left(|T|^2 + |S^*|\right)\hat{k}_\tau, \hat{k}_\tau\right\rangle.
\end{aligned}$$

By taking the supremum over $\tau \in \Theta$ in the above inequality, we get

$$\mathbf{ber}\left(U|S|^{\frac{1}{2}}T\right) \leqslant \frac{1}{2}\mathbf{ber}\left(|T|^2 + |S^*|\right). \tag{8}$$

Therefore, the inequalities (7) and (8) imply that

$$\mathbf{ber}(ST) \leqslant \frac{\|S\|^{\frac{1}{2}}}{2}\mathbf{ber}\left(|S^*| + |T|^2\right). \quad \square$$

THEOREM 3. *Suppose that $T \in \mathscr{L}(\mathscr{H})$ with the polar decomposition $T = U|T|$. Then*

$$\boldsymbol{ber}(T) \leqslant \frac{\|T\|^{\frac{1}{2}}}{2}\left(\boldsymbol{ber}\left(U|T|^{\frac{1}{2}}\right) + \sqrt{\boldsymbol{ber}\left(|T|^t\right)\boldsymbol{ber}\left(|T^*|^{1-t}\right)}\right)$$

and

$$\boldsymbol{ber}(T) \leqslant \frac{\|T\|^{1-t}}{2}\left(\boldsymbol{ber}\left(U|T|^t\right) + \sqrt{\boldsymbol{ber}\left(|T|^t\right)\boldsymbol{ber}\left(|T^*|^t\right)}\right)$$

for $0 \leqslant t \leqslant 1$.

Proof. Let $\tau \in \Theta$ be arbitrary. Letting $A = U|T|^{\frac{1-t}{2}}$, $B = |T|^{\frac{t}{2}}$ and $X = |T|^{\frac{1}{2}}$, in the inequality (3), we deduce that

$$\begin{aligned}
\left|\left\langle T\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| &\leqslant \frac{\|T\|^{\frac{1}{2}}}{2}\left(\left|\left\langle U|T|^{\frac{1}{2}}\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| + \||T|^{\frac{t}{2}}\hat{k}_\tau\|\||T|^{\frac{1-t}{2}}U^*\hat{k}_\tau\|\right) \\
&= \frac{\|T\|^{\frac{1}{2}}}{2}\left(\left|\left\langle U|T|^{\frac{1}{2}}\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| + \sqrt{\left\langle |T|^t\hat{k}_\tau, \hat{k}_\tau\right\rangle\left\langle U|T|^{1-t}U^*\hat{k}_\tau, \hat{k}_\tau\right\rangle}\right) \\
&= \frac{\|T\|^{\frac{1}{2}}}{2}\left(\left|\left\langle U|T|^{\frac{1}{2}}\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| + \sqrt{\left\langle |T|^t\hat{k}_\tau, \hat{k}_\tau\right\rangle\left\langle |T^*|^{1-t}\hat{k}_\tau, \hat{k}_\tau\right\rangle}\right).
\end{aligned}$$

By taking the supremum over $\tau \in \Theta$ in the above inequality, we have

$$\mathbf{ber}(T) \leqslant \frac{\|T\|^{\frac{1}{2}}}{2}\left(\mathbf{ber}\left(U|T|^{\frac{1}{2}}\right) + \sqrt{\mathbf{ber}\left(|T|^{t}\right)\mathbf{ber}\left(|T^*|^{1-t}\right)}\right).$$

The second inequality follows similarly by setting $A = U|T|^{\frac{t}{2}}$, $B = |T|^{\frac{t}{2}}$ and $X = |T|^{1-t}$. □

3. Euclidean Berezin number inequalities

The next lemma is the generalized mixed Schwarz inequality was introduced in [24].

LEMMA 3. [24, Lemma 2.1] *Let $x,y,z \in \mathscr{H}$. Then*

$$|\langle x,y\rangle|^2 + |\langle x,z\rangle|^2 \leqslant \|x\|\sqrt{|\langle x,y\rangle|^2\|y\|^2 + |\langle x,z\rangle|^2\|z\|^2 + 2|\langle x,y\rangle||\langle x,z\rangle||\langle y,z\rangle|}.$$

LEMMA 4. [24] *Let $x,y,e \in \mathscr{H}$ with $\|e\| = 1$. Then*

$$|\langle x,e\rangle|^2 + |\langle y,e\rangle|^2 \leqslant \|\langle e,x\rangle x + \langle e,y\rangle y\|.$$

THEOREM 4. *Suppose that $A,B \in \mathscr{L}(\mathscr{H})$. Then*

$$\boldsymbol{ber}_e^2(A,B) \leqslant \sqrt{\left\|\boldsymbol{ber}^2(A)\,|A^*|^2 + \boldsymbol{ber}^2(B)\,|B^*|^2\right\|_{\boldsymbol{ber}} + 2\boldsymbol{ber}(A^*)\boldsymbol{ber}(B)\boldsymbol{ber}(A^*B)}.$$

Proof. Let $\tau \in \Theta$ be arbitrary. By substituting $x = A\hat{k}_\tau$, $y = B\hat{k}_\tau$, and $e = \hat{k}_\tau$, in Lemma 4, we deduce that

$$\begin{aligned}
&\left|\langle A\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2 + \left|\langle B\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2 \\
&\leqslant \left\|\langle \hat{k}_\tau, A\hat{k}_\tau\rangle A\hat{k}_\tau + \langle \hat{k}_\tau, B\hat{k}_\tau\rangle B\hat{k}_\tau\right\| \\
&= \left\|\left(\langle A^*\hat{k}_\tau,\hat{k}_\tau\rangle A + \langle B^*\hat{k}_\tau,\hat{k}_\tau\rangle B\right)\hat{k}_\tau\right\| \\
&= \sqrt{\left\langle\left(\langle A\hat{k}_\tau,\hat{k}_\tau\rangle A^* + \langle B\hat{k}_\tau,\hat{k}_\tau\rangle B^*\right)\left(\langle A^*\hat{k}_\tau,\hat{k}_\tau\rangle A + \langle B^*\hat{k}_\tau,\hat{k}_\tau\rangle B\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle} \\
&= \sqrt{\left\langle\left(\left|\langle A\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|A^*|^2 + \left|\langle B\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|B^*|^2 + 2\Re\left(\langle A^*\hat{k}_\tau,\hat{k}_\tau\rangle\langle B\hat{k}_\tau,\hat{k}_\tau\rangle A^*B\right)\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle} \\
&= \sqrt{\left\langle\left(\left|\langle A\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|A^*|^2 + \left|\langle B\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|B^*|^2\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle + \left\langle 2\Re\left(\langle A^*\hat{k}_\tau,\hat{k}_\tau\rangle\langle B\hat{k}_\tau,\hat{k}_\tau\rangle A^*B\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle} \\
&\leqslant \sqrt{\left\langle\left(\left|\langle A\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|A^*|^2 + \left|\langle B\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|B^*|^2\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle + 2\left|\left\langle\Re\left(\langle A^*\hat{k}_\tau,\hat{k}_\tau\rangle\langle B\hat{k}_\tau,\hat{k}_\tau\rangle A^*B\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle\right|} \\
&\leqslant \sqrt{\left\langle\left(\left|\langle A\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|A^*|^2 + \left|\langle B\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2|B^*|^2\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle + 2\left|\left\langle\langle A^*\hat{k}_\tau,\hat{k}_\tau\rangle\langle B\hat{k}_\tau,\hat{k}_\tau\rangle A^*B\hat{k}_\tau,\hat{k}_\tau\right\rangle\right|} \\
&\leqslant \sqrt{\left\langle\left(\mathbf{ber}^2(A)\,|A^*|^2 + \mathbf{ber}^2(B)\,|B^*|^2\right)\hat{k}_\tau,\hat{k}_\tau\right\rangle + 2\mathbf{ber}(A^*)\mathbf{ber}(B)\left|\langle A^*B\hat{k}_\tau,\hat{k}_\tau\rangle\right|} \\
&\leqslant \sqrt{\left\|\mathbf{ber}^2(A)\,|A^*|^2 + \mathbf{ber}^2(B)\,|B^*|^2\right\|_{\mathbf{ber}} + 2\mathbf{ber}(A^*)\mathbf{ber}(B)\mathbf{ber}(A^*B)}.
\end{aligned}$$

By taking the supremum over $\tau \in \Theta$, we have

$$\mathbf{ber}_e^2(A,B) \leqslant \sqrt{\left\| \mathbf{ber}^2(A)\,|A^*|^2 + \mathbf{ber}^2(B)\,|B^*|^2 \right\|_{\mathbf{ber}} + 2\mathbf{ber}(A^*)\mathbf{ber}(B)\mathbf{ber}(A^*B)}.$$

This completes the proof. □

Using the generalized mixed Schwarz inequality, we establish the next result.

THEOREM 5. *Assume that $A,B \in \mathscr{L}(\mathscr{H})$. Then*

$$\|(A,B)\|_{e,\boldsymbol{ber}}^2 \leqslant \Big(\boldsymbol{ber}\left(|A|^2 + i\,|B|^2\right) \boldsymbol{ber}\left(|A^*|^2 + i\,|B^*|^2\right) + 2\boldsymbol{ber}\left(|A| + i\,|B|\right) \boldsymbol{ber}\left(|A^*| + i\,|B^*|\right) \boldsymbol{ber}\left(BA^*\right) \Big)^{\frac{1}{2}}.$$

Proof. Let $\tau \in \Theta$ be arbitrary. Letting $y = A^*\hat{k}_\mu$, $z = B^*\hat{k}_\mu$, and $x = \hat{k}_\tau$, in Lemma 3, we get

$$\begin{aligned}
&\left|\left\langle A\hat{k}_\tau, \hat{k}_\mu \right\rangle\right|^2 + \left|\left\langle B\hat{k}_\tau, \hat{k}_\mu \right\rangle\right|^2 \\
&\leqslant \Big(\left|\left\langle A\hat{k}_\tau, \hat{k}_\mu \right\rangle\right|^2 \|A^*\hat{k}_\mu\|^2 + \left|\left\langle B\hat{k}_\tau, \hat{k}_\mu \right\rangle\right|^2 \|B^*\hat{k}_\mu\|^2 \\
&\qquad + 2\left|\left\langle A\hat{k}_\tau, \hat{k}_\mu \right\rangle\right| \left|\left\langle B\hat{k}_\tau, \hat{k}_\mu \right\rangle\right| \left|\left\langle BA^*\hat{k}_\mu, \hat{k}_\mu \right\rangle\right| \Big)^{\frac{1}{2}} \\
&\leqslant \left(\|A\hat{k}_\tau\|^2 \|A^*\hat{k}_\mu\|^2 + \|B\hat{k}_\tau\|^2 \|B^*\hat{k}_\mu\|^2 + 2\left|\left\langle A\hat{k}_\tau, \hat{k}_\mu \right\rangle\right| \left|\left\langle B\hat{k}_\tau, \hat{k}_\mu \right\rangle\right| \left|\left\langle BA^*\hat{k}_\mu, \hat{k}_\mu \right\rangle\right| \right)^{\frac{1}{2}} \\
&\qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= \Big(\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |A^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle + \left\langle |B|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |B^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle \\
&\qquad + 2\left|\left\langle A\hat{k}_\tau, \hat{k}_\mu \right\rangle\right| \left|\left\langle B\hat{k}_\tau, \hat{k}_\mu \right\rangle\right| \left|\left\langle BA^*\hat{k}_\mu, \hat{k}_\mu \right\rangle\right| \Big)^{\frac{1}{2}} \\
&\leqslant \Big(\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |A^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle + \left\langle |B|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |B^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle \\
&\qquad + 2\sqrt{\left\langle |A|\,\hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |A^*|\,\hat{k}_\mu, \hat{k}_\mu \right\rangle \left\langle |B|\,\hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |B^*|\,\hat{k}_\mu, \hat{k}_\mu \right\rangle} \left|\left\langle BA^*\hat{k}_\mu, \hat{k}_\mu \right\rangle\right| \Big)^{\frac{1}{2}} \\
&\qquad \text{(by the mixed Cauchy-Schwarz inequality)} \\
&\leqslant \Big(\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |A^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle + \left\langle |B|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |B^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle \\
&\qquad + 2\left(\left\langle |A|\,\hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |A^*|\,\hat{k}_\mu, \hat{k}_\mu \right\rangle \left\langle |B|\,\hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle |B^*|\,\hat{k}_\mu, \hat{k}_\mu \right\rangle \right) \left|\left\langle BA^*\hat{k}_\mu, \hat{k}_\mu \right\rangle\right| \Big)^{\frac{1}{2}} \\
&\qquad \text{(by the arithmetic-geometric mean inequality)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \Bigg(\left(\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle^2 + \left\langle |B|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle^2 \right)^{\frac{1}{2}} \left(\left\langle |A^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle^2 + \left\langle |B^*|^2 \hat{k}_\mu, \hat{k}_\mu \right\rangle^2 \right)^{\frac{1}{2}} \\
&\qquad + 2 \left(\left\langle |A| \hat{k}_\tau, \hat{k}_\tau \right\rangle^2 + \left\langle |B| \hat{k}_\tau, \hat{k}_\tau \right\rangle^2 \right)^{\frac{1}{2}} \\
&\qquad \times \left(\left\langle |A^*| \hat{k}_\mu, \hat{k}_\mu \right\rangle^2 + \left\langle |B^*| \hat{k}_\mu, \hat{k}_\mu \right\rangle^2 \right)^{\frac{1}{2}} \left| \left\langle BA^* \hat{k}_\mu, \hat{k}_\mu \right\rangle \right| \Bigg)^{\frac{1}{2}} \\
&\leqslant \Bigg(\left| \left\langle \left(|A|^2 + i |B|^2 \right) \hat{k}_\tau, \hat{k}_\tau \right\rangle \right| \left| \left\langle \left(|A^*|^2 + i |B^*|^2 \right) \hat{k}_\mu, \hat{k}_\mu \right\rangle \right| \\
&\qquad + 2 \left| \left\langle \left(|A| + i |B| \right) \hat{k}_\tau, \hat{k}_\tau \right\rangle \right| \left| \left\langle \left(|A^*| + i |B^*| \right) \hat{k}_\mu, \hat{k}_\mu \right\rangle \right| \left| \left\langle BA^* \hat{k}_\mu, \hat{k}_\mu \right\rangle \right| \Bigg)^{\frac{1}{2}} \\
&\qquad \text{(since } |x+iy| = \sqrt{x^2+y^2} \text{ for any } x, y \in \mathbb{R} \text{)} \\
&\leqslant \Bigg(\mathbf{ber}\left(|A|^2 + i |B|^2 \right) \mathbf{ber}\left(|A^*|^2 + i |B^*|^2 \right) \\
&\qquad + 2 \mathbf{ber}\left(|A| + i |B| \right) \mathbf{ber}\left(|A^*| + i |B^*| \right) \mathbf{ber}\left(BA^* \right) \Bigg)^{\frac{1}{2}}.
\end{aligned}
$$

Now, by taking the supremum over $\tau \in \Theta$, we deduce that

$$
\begin{aligned}
\|(A,B)\|_{e,\mathbf{ber}}^2 \leqslant \Bigg(&\mathbf{ber}\left(|A|^2 + i |B|^2 \right) \mathbf{ber}\left(|A^*|^2 + i |B^*|^2 \right) \\
&+ 2 \mathbf{ber}\left(|A| + i |B| \right) \mathbf{ber}\left(|A^*| + i |B^*| \right) \mathbf{ber}\left(BA^* \right) \Bigg)^{\frac{1}{2}},
\end{aligned}
$$

which completes the proof of the statement. □

EXAMPLE 1. Let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$. Consider $\mathbb{C}^2$ as a RKHS on the set $\{1,2\}$. By putting 2 by 2 matrices $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in Theorems 4 and 5, we have

$$
\begin{aligned}
\mathbf{ber}_e^2(A,B) = 1 \lneqq &\sqrt{\left\| \mathbf{ber}^2(A)\, |A^*|^2 + \mathbf{ber}^2(B)\, |B^*|^2 \right\|_{\mathbf{ber}} + 2\mathbf{ber}(A^*)\mathbf{ber}(B)\mathbf{ber}(A^*B)} \\
&\approx 1.41
\end{aligned}
$$

and

$$
\begin{aligned}
\|(A,B)\|_{e,\mathbf{ber}}^2 = 2 \lneqq \Bigg(&\mathbf{ber}\left(|A|^2 + i |B|^2 \right) \mathbf{ber}\left(|A^*|^2 + i |B^*|^2 \right) \\
&+ 2 \mathbf{ber}\left(|A| + i |B| \right) \mathbf{ber}\left(|A^*| + i |B^*| \right) \mathbf{ber}\left(BA^* \right) \Bigg)^{\frac{1}{2}} \approx 3.25.
\end{aligned}
$$

4. Some inequalities for the weighted Berezin number

Through the present section, we give some new inequalities for the weighted Berezin radius of operators on the reproducing kernel Hilbert space $\mathscr{H} = \mathscr{H}(\Theta)$. First, note that by the subadditivity of the Berezin radius, we deduce that

$$\mathbf{ber}(\Re_\alpha(T)) = \mathbf{ber}\left((1-\alpha)T^* + \alpha T\right) \leqslant \mathbf{ber}_\alpha(T)$$

and

$$\mathbf{ber}(\Im_\alpha(T)) = \mathbf{ber}\left((1-\alpha)T - \alpha T^*\right) \leqslant \mathbf{ber}_\alpha(T),$$

where $T \in \mathscr{L}(\mathscr{H})$, $\Re_\alpha(T) = (1-\alpha)T^* + \alpha T$, $\Im_\alpha(T) = \frac{(1-\alpha)T - \alpha T^*}{i}$ and $\alpha \in [0,1]$.

Now, we give our first lemma with concerning the Cartesian decomposition.

LEMMA 5. *Suppose that $T \in \mathscr{L}(\mathscr{H})$ and $\alpha \in [0,1]$. Then*

$$\mathbf{ber}_\alpha(T) = \sup_{s^2+t^2=1} \mathbf{ber}\left(s\Re_\alpha(T) \pm t\Im_\alpha(T)\right), \tag{9}$$

where $\Re_\alpha(T) = (1-\alpha)T^* + \alpha T$ and $\Im_\alpha(T) = \frac{(1-\alpha)T - \alpha T^*}{i}$.

Proof. Let $\theta \in \mathbb{R}$ and $\alpha \in [0,1]$. Then

$$\begin{aligned}\Re_\alpha(e^{i\theta}T) &= \frac{1}{2}\left((1-\alpha)e^{-i\theta}T^* + \alpha e^{i\theta}T\right)\\ &= \cos\theta\left((1-\alpha)T^* + \alpha T\right) + \sin\theta\left(\frac{(1-\alpha)T - \alpha T^*}{i}\right)\\ &= \cos\theta\,\Re_\alpha(T) + \sin\theta\,\Im_\alpha(T) \qquad (10)\end{aligned}$$

If we take $s = \cos\theta$ and $t = \sin\theta$ in (10), then by (1), we have

$$\mathbf{ber}_\alpha(T) = \sup_{\theta\in\mathbb{R}} \mathbf{ber}\left(\Re_\alpha(e^{i\theta}T)\right) = \sup_{s^2+t^2=1} \mathbf{ber}\left(s\Re_\alpha(T) \pm t\Im_\alpha(T)\right). \tag{11}$$

□

REMARK 2. If $T \in \mathscr{L}(\mathscr{H})$, then by using the properties of the Berezin radius, we get

$$\begin{aligned}(1-\alpha)\alpha\,\mathbf{ber}\left(T^*T + TT^*\right) &= \mathbf{ber}\left(\frac{(\Re_\alpha(T))^2 + (\Im_\alpha(T))^2}{2}\right)\\ &\leqslant \frac{1}{2}\left(\mathbf{ber}\left((\Re_\alpha(T))^2\right) + \mathbf{ber}\left((\Im_\alpha(T))^2\right)\right)\\ &\qquad \text{(by the subadditivity of } \mathbf{ber}\text{)}\\ &\leqslant \frac{1}{2}\left(\mathbf{ber}^2\left(\Re_\alpha(T)\right) + \mathbf{ber}^2\left(\Im_\alpha(T)\right)\right)\\ &\qquad \text{(by the convexity of } f(t) = t^2\text{)}\\ &\leqslant \mathbf{ber}_\alpha(T). \qquad (12)\end{aligned}$$

In the next theorem, we give an improvement of the inequality (12).

THEOREM 6. *Assume that* $T \in \mathscr{L}(\mathscr{H})$ *and* $\alpha \in [0,1]$. *Then*

$$\sup_{s^2+t^2=1} \mathbf{ber}\left(s^2(\Re_\alpha(T))^2 + t^2(\Im_\alpha(T))^2\right) \leqslant \mathbf{ber}_\alpha^2(T), \tag{13}$$

where $\Re_\alpha(T) = (1-\alpha)T^* + \alpha T$ *and* $\Im_\alpha(T) = \frac{(1-\alpha)T - \alpha T^*}{i}$.

Proof. If $\alpha \in [0,1]$, then

$$\begin{aligned}
&\sup_{s^2+t^2=1} \mathbf{ber}\left(s^2(\Re_\alpha(T))^2 + t^2(\Im_\alpha(T))^2\right) \\
&= \frac{1}{2}\sup_{s^2+t^2=1} \mathbf{ber}\left((s\Re_\alpha(T) + t\Im_\alpha(T))^2 + (s\Re_\alpha(T) - t\Im_\alpha(T))^2\right) \\
&\leqslant \frac{1}{2}\left(\sup_{s^2+t^2=1} \mathbf{ber}\left((s\Re_\alpha(T) + t\Im_\alpha(T))^2\right) + \sup_{s^2+t^2=1} \mathbf{ber}\left((s\Re_\alpha(T) - t\Im_\alpha(T))^2\right)\right) \\
&\qquad \text{(by the subadditivity of } \mathbf{ber}) \\
&\leqslant \frac{1}{2}\left(\sup_{s^2+t^2=1} \mathbf{ber}^2\left(s\Re_\alpha(T) + t\Im_\alpha(T)\right) + \sup_{s^2+t^2=1} \mathbf{ber}^2\left(s\Re_\alpha(T) - t\Im_\alpha(T)\right)\right) \\
&\qquad \text{(by the convexity of } f(t) = t^2) \\
&\leqslant \mathbf{ber}_\alpha{}^2(T),
\end{aligned}$$

as required. □

REMARK 3. If we put $t = s = \frac{\sqrt{2}}{2}$ in Theorem 6, we get an improvement of (12) as follows:

$$(1-\alpha)\alpha\mathbf{ber}(T^*T + TT^*) \leqslant \sup_{s^2+t^2=1} \mathbf{ber}_\alpha\left(s^2(\Re_\alpha(T))^2 + t^2(\Im_\alpha(T)^2\right) \leqslant \mathbf{ber}_\alpha{}^2(T),$$

where $\alpha \in [0,1]$. In particular for $\alpha = \frac{1}{2}$, we deduce that

$$\mathbf{ber}\left(\frac{T^*T + TT^*}{4}\right) \leqslant \sup_{s^2+t^2=1} \mathbf{ber}\left(s^2(\Re(T))^2 + t^2(\Im(T)^2\right) \leqslant \mathbf{ber}^2(T).$$

In the next theorem, we present a lower bound for the Berezin radius.

THEOREM 7. *Suppose that* $T \in \mathscr{L}(\mathscr{H})$. *Then*

$$\frac{1}{2}\sup_{s^2+t^2=1} \mathbf{ber}\left(s^2T^2 + t^2(T^*)^2\right) \leqslant \mathbf{ber}^2(T). \tag{14}$$

Proof. Assume that $\hat{k}_\tau \in \mathscr{H}$. Then

$$\begin{aligned} 2\left|\langle T\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2 &= \left|\langle T\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2 + \left|\langle T^*\hat{k}_\tau,\hat{k}_\tau\rangle\right|^2 \\ &= \sup_{s^2+t^2=1}\left(s\left|\langle T\hat{k}_\tau,\hat{k}_\tau\rangle\right| + t\left|\langle T^*\hat{k}_\tau,\hat{k}_\tau\rangle\right|\right)^2 \\ &\geqslant \sup_{s^2+t^2=1}\left|\left\langle (sT \pm tT^*)\hat{k}_\tau,\hat{k}_\tau\right\rangle\right|^2. \end{aligned}$$

By taking the supremum on the both sides of the last inequality over $\hat{k}_\tau \in \mathscr{H}$, we get

$$2\mathbf{ber}^2(T) \geqslant \sup_{s^2+t^2=1} \mathbf{ber}^2(sT \pm tT^*). \tag{15}$$

Therefore

$$\begin{aligned} 4\mathbf{ber}^2(T) &\geqslant \sup_{s^2+t^2=1} \mathbf{ber}^2(sT+tT^*) + \sup_{s^2+t^2=1} \mathbf{ber}^2(sT-tT^*) \\ &\geqslant \sup_{s^2+t^2=1}\left(\mathbf{ber}\left((sT+tT^*)^2\right) + \mathbf{ber}\left((sT-tT^*)^2\right)\right) \\ &\geqslant \sup_{s^2+t^2=1} \mathbf{ber}\left((sT+tT^*)^2 + (sT-tT^*)^2\right) \\ &\qquad\qquad \text{(by the subadditivity of } \mathbf{ber}\text{)} \\ &= 2\sup_{s^2+t^2=1} \mathbf{ber}\left(s^2T^2 + t^2(T^*)^2\right), \end{aligned}$$

whence we get the desired result. □

COROLLARY 2. *Let $T \in \mathscr{L}(\mathscr{H})$. Then*

$$\sqrt{2}\mathbf{ber}(T) \geqslant \max\left\{\sup_{s^2+t^2=1} \mathbf{ber}(sT+tT^*), \sup_{s^2+t^2=1} \mathbf{ber}(sT-tT^*)\right\}.$$

In particular,

$$\mathbf{ber}(T) \geqslant \max\left\{\mathbf{ber}(\Re(T)), \mathbf{ber}(\Im(T))\right\}.$$

Proof. Applying the inequality (15), we get the first result. For the second inequality, take $s = t = \frac{\sqrt{2}}{2}$ in the first inequality. □

Funding. The authors declare that there is no source of funding for this research.

Competing interests. The authors declare that they have no competing interests.

Authors' contributions. The authors contributed equally to the manuscript and read and approved the final manuscript.

Availability of data and materials. Not applicable.

REFERENCES

[1] N. ARONZAJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[2] M. BAKHERAD, *Some Berezin number inequalities for operator matrices*, Czechoslovak Math. J. **68** (4) (2018), 997–1009.

[3] M. BAKHERAD AND M. T. KARAEV, *Berezin number inequalities for Hilbert space operators*, Concr. Oper. **6** (2019), no. 1, 33–43.

[4] M. BAKHERAD, R. LASHKARIPOUR, M. HAJMOHAMADI, AND U. YAMANCI, *Complete refinements of the Berezin number inequalities*, J. Math. Inequal. **13** (2019), no. 4, 1117–1128.

[5] M. BAKHERAD AND U. YAMANCI, *New estimations for the Berezin number inequality*, J. Inequal. Appl. 2020, Paper No. **40**, 9 pp.

[6] P. BHUNIA, A. SEN, S. BARIK, AND K. PAUL, *Berezin number and Berezin norm inequalities for operator matrices*, Linear Multilinear Algebra, https://doi.org/10.1080/03081087.2023.2299388.

[7] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN, AND K. PAUL, *Lectures on Numerical Radius Inequalities*, Infosys Science Foundation Series in Mathematical Sciences, Springer, Cham 2022.

[8] M. L. BUZANO, *Generalizzazione della diseguaglianza di Cauchy-Schwarz* (Italian), Rend. Sem. Mat. Univ. e Politech. Torino **31** (1974), 405–409.

[9] F. CHIEN, M. BAKHERAD, AND M. W. ALOMARI, *Refined Berezin number inequalities via superquadratic and convex functions*, Filomat **37** (2023), no. 1, 265–277.

[10] M. T. GARAYEV, M. BAKHERAD, AND R. TAPDIGOGLU, *The weighted and the Davis-Wielandt Berezin number*, Oper. Matrices **17** (2023), no. 2, 469–484.

[11] M. T. GARAYEV, M. GÜRDAL, AND S. SALTAN, *Hardy type inequality for reproducing kernel Hilbert space operators and related problems*, Positivity **21** (4) (2017), 1615–1623.

[12] V. GÜRDAL AND M. B. GÜRDAL, *A-Davis-Wielandt-Berezin radius inequalities*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **72** (1) (2023), 182–198.

[13] V. GUILLEMIN, *Toeplitz operators in n-dimensions*, Integral Equations Operator Theory **7** (2) (1984), 145–205.

[14] M. HAJMOHAMADI, R. LASHKARIPOUR, AND M. BAKHERAD, *Some generalizations of numerical radius on off–diagonal part of 2×2 operator matrices*, J. Math. Inequal. **12** (2) (2018), 447–457.

[15] M. HAJMOHAMADI, R. LASHKARIPOUR, AND M. BAKHERAD, *Improvements of Berezin number inequalities*, Linear Multilinear Algebra **68** (2020), no. 6, 1218–1229.

[16] T. KATO, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.

[17] M. T. KARAEV, *On the Berezin symbol*, Zap. Nauch. Semin. POMI, **270** (2000), 80–89 (Russian); Translated from Zapiski Nauchnykh Seminarov POMI, **270** (2003), 2135–2140.

[18] M. T. KARAEV, *Berezin symbol and invertibility of operators on the functional Hilbert spaces*, J. Funct. Anal. **238** (2006), 181–192.

[19] M. T. KARAEV, *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory **7** (2013), no. 4, 983–1018.

[20] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), 283–293.

[21] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), 11–17.

[22] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), no. 1, 73–80.

[23] M. SABABHEH, H. R. MORADI, AND S. SAHOO, *Inner product inequalities with applications*, Linear and Multilinear Algebra, doi:10.1080/03081087.2024.2312444.

[24] M. SABABHEH AND H. R. MORADI, *New orders among Hilbert space operators*, Math. Inequal. Appl. **26** (2) (2023), 415–432.

[25] A. SHEIKHHOSSEINI, M. KHOSRAVI, AND M. SABABHEH, *The weighted numerical radius*, Ann. Funct. Anal. **13** (3) (2022), no. 1, 15 pp.

[26] K. ZHU, *Operator Theory in Function Spaces*, Marcel Dekker, 2nd. edition, 2007.

(Received January 16, 2024)

Davood Afraz
Department of Mathematics, Faculty of Mathematics
University of Sistan and Baluchestan
Zahedan, I.R. Iran
e-mail: davood.afraz@pgs.usb.ac.ir

Rahmatollah Lashkaripour
Department of Mathematics, Faculty of Mathematics
University of Sistan and Baluchestan
Zahedan, I.R. Iran
e-mail: lashkari@hamoon.usb.ac.ir

Mojtaba Bakherad
Department of Mathematics, Faculty of Mathematics
University of Sistan and Baluchestan
Zahedan, I.R. Iran
e-mail: bakherad@member.ams.org