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MULTIPLE–TERM IMPROVEMENTS OF JENSEN'S INEQUALITY FOR (p,h)–CONVEX AND (p,h)–LOG CONVEX FUNCTIONS

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Abstract. In this paper, we present several new multiple-term improvements of Jensen's inequality for (p,h)-convex and (p,h)-log convex functions. As applications of our results, we present new bounds by employing means and Hölder type inequalities for the symmetric norms for τ-measurable operators. We make links between our findings and a number of well-known discoveries in the literature.

1. Introduction

The theory of convex functions has played an important role due to their significance in various fields of mathematics, consisting of analysis, optimization, mathematical physics, functional analysis, and operator theory. Let us recall that a real-valued function f defined on an interval $I \subset \mathbb{R}$ is a convex function if it satisfies

$$f(\alpha x + \beta y) \leqslant \alpha f(x) + \beta f(y), \tag{1.1}$$

for every $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. If inequality (1.1) is reversed, the function f is said to be concave. Also, the function f is said to be log-convex (log-concave) if f is positve and $\log f$ is convex (log-concave, respectively).

The inequality (1.1) has been refined in the literature, and many applications were presented for scalars, matrices and operators. We refer the reader to [1, 10, 16, 22] for further discussion. The well-known Jensen inequality extends (1.1) to n parameters in the following way

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leqslant \sum_{i=1}^{n} w_i f(x_i), \tag{1.2}$$

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where $f : I \to \mathbb{R}$ is a convex function, $\{x_1, \dots, x_n\} \subset I$ and $\{w_1, \dots, w_n\} \subset [0,1]$ with $\sum_{i=1}^n w_i = 1$. By applying Jensen's inequality (1.2) to the function $\log f$ we get the following inequality

$$f\left(\sum_{i=1}^n w_i x_i\right) \leqslant \prod_{i=1}^n f^{w_i}(x_i), \tag{1.3}$$

for the same parameters above, where the function f is $\log$-convex. The literature has given a great deal of attention to improve or reverse (1.2), and hence (1.3). In 2006, S. S. Dragomir [4] shown a celebrated refinement and reverse of the following form

$$\begin{aligned} nw_{\min}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\sum_{i=1}^n \frac{x_i}{n}\right)\right) &\leqslant \sum_{i=1}^n w_i f(x_i) - f\left(\sum_{i=1}^n w_i x_i\right) \\ &\leqslant nw_{\max}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\sum_{i=1}^n \frac{x_i}{n}\right)\right), \end{aligned} \tag{1.4}$$

where $w_{\min} = \min\{w_1, \dots, w_n\}$ and $w_{\max} = \max\{w_1, \dots, w_n\}$. Very recently, M. A. Ighachane and M. Bouchangour [9] based on the so called weak submajorization theory to generelize (1.4) to

$$\begin{aligned} &\phi\left(h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j}{\mu_j}\Big)\sum_{i=1}^n h(\mu_i)f(x_i)\right) - \phi\left(h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j}{\mu_j}\Big)f\Big(\Big[\sum_{i=1}^n \mu_i x_i^p\Big]^{\frac{1}{p}}\Big)\right) \\ &\leqslant \phi\left(\sum_{i=1}^n h(\nu_i)f(x_i)\right) - \phi\circ f\Big(\Big[\sum_{i=1}^n \nu_i x_i^p\Big]^{\frac{1}{p}}\Big) \\ &\leqslant \phi\left(h\Big(\max_{1\leqslant j\leqslant n}\frac{\nu_j}{\mu_j}\Big)\sum_{i=1}^n h(\mu_i)f(x_i)\right) - \phi\left(h\Big(\max_{1\leqslant j\leqslant n}\frac{\nu_j}{\mu_j}\Big)f\Big(\Big[\sum_{i=1}^n \mu_i x_i^p\Big]^{\frac{1}{p}}\Big)\right), \end{aligned} \tag{1.5}$$

where ϕ is an increasing convex function, h is a multiplicative and super-additive function, and f is a positve (p,h)-convex function defined on I with sequences $\{x_1, \dots, x_n\} \subset I$ and $\{\nu_1, \dots, \nu_n, \mu_1, \dots, \mu_n\} \subset (0,1)$ satisfying $\sum_{i=1}^n \nu_i = \sum_{i=1}^n \mu_i = 1$, see [9, Theorems 3.3 and 3.5] for the details.

In [16], Sababheh has presented a new refinement of Jensen's inequality by adding as many refining terms as we wish. Namely, for a convex function $f : I \to \mathbb{R}$, $\{x_1^{(1)}, \dots, x_n^{(1)}\} \subset I$ and $\{w_1^{(1)}, \dots, w_n^{(1)}\} \subset (0,1)$ with $\sum_{i=1}^n w_i^{(1)} = 1$, then for every $N \in \mathbb{N}$, the author proved the following inequality

$$f\left(\sum_{i=1}^n w_i^{(1)} x_i^{(1)}\right) + \sum_{k=1}^N nw_{min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f\left(x_i^{(k)}\right) - f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)\right) \leqslant \sum_{i=1}^n w_i^{(1)} f\left(x_i^{(1)}\right), \tag{1.6}$$

where the construction of $x_i^{(k)}$, $w_i^{(k)}$ and $w_{\min}^{(k)}$ is defined as in Section 2. In the same paper [9], the authors also extended the inequality (1.6) to the more general setting of (p,h)-convexity:

$$\sum_{i=1}^{n} h(w_i^{(1)}) f(x_i^{(1)}) \geqslant f\Big(\Big[\sum_{i=1}^{n} w_i^{(1)} (x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big) + \sum_{k=1}^{N} h(n w_{\min}^{(k)}) \Big\{ h\Big(\frac{1}{n}\Big) \sum_{i=1}^{n} f\left(x_i^{(k)}\right) - f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n} (x_i^{(k)})^p\Big]^{\frac{1}{p}}\Big) \Big\}, \tag{1.7}$$

see [9, Theorem 4.1]. Based on recent results regarding Jensen's inequality via the weak submajorization theory, readers are encouraged to explore the following recent publication [2, 23].

Inspired by the above mentioned results, we give an improvement of the inequality (1.7) in the present paper. Using the improved inequality, we establish a multiple-term generalization for the left-hand inequality of the inequality (1.5). Further we present new real power form inequalities for Jensen's inequality for (p,h)-convex functions.

The basic purpose of this research is to establish further refinements and generalisations of Jensen's inequality for (p,h)-convex and $\log$-(p,h)-convex fnctions. To be more precise, we propose some generalisations of increasing convex function form for Jensen's inequality, whose special case is the main results of [9]. These contents will be present in Section 2. Next, we propose some new real power form inequalities for Jensen's inequality extending the results of [12] in subsection 2.2 and section 3. Finally, in Section 4 and 5, we present new inequalities that lead to several refinements of well known inequalities for means, and Hölder type inequalities for the symmetric norms for τ-measurable operators.

2. Preliminaries and multiple-term refinements of Jensen's inequality for (p,h)-convex and $\log$-(p,h)-convex functions

The aim of this section is to propose an improvement of the inequality (1.7) and to establish multiple-term refinements of Jensen's inequality for (p,h)-convex and log-(p,h)-convex functions.

2.1. Preliminaries

To that end, we recall several necessary notions. First, let $J \subset \mathbb{R}$ be an interval containing $(0,1)$. A function $h : J \to \mathbb{R}$ is said to be super-multiplicative if for all $x,y \in J$ we have $xy \in J$ and

$$h(x)h(y) \leqslant h(xy).$$

If this inequality is reversed, then h is said to be sub-multiplicative. If h is both super-multiplicative and sub-multiplicative, then it is called multiplicative. On the other side, the function h is called super-additive if for all $x,y \in J$ we have $x+y \in J$ and

$$h(x)+h(y) \leqslant h(x+y).$$

In the case this inequality is reversed, the function h is said to be sub-additive. If the equality in this inequality holds for all $x,y \in J$, then it is called additive. Some examples on these kinds of functions can be found in [9].

Secondly, for a given real number $p \in \mathbb{R}$, we say that the set $I \subset \mathbb{R}$ is p-convex if $(\alpha x^p + (1-\alpha)y^p)^{\frac{1}{p}} \in I$ for all $x, y \in I$ and all $\alpha \in [0,1]$. Hereafter, we always assume that a given set $I \subset \mathbb{R}$ is p-convex for some real number p. Now, for a given function h defined on J, a function $f : I \to \mathbb{R}$ is said to be (p,h)-convex if the following inequality

$$f\Big(\Big[\alpha x^p + (1-\alpha)y^p\Big]^{\frac{1}{p}}\Big) \leqslant h(\alpha)f(x) + h(1-\alpha)f(y) \tag{2.1}$$

holds for all $x, y \in I$. If the inequality (2.1) is reversed, the function f is called (p,h)-concave.

Finally, throughout this section, we also denote by $w^{(1)} = \left\{w_1^{(1)}, \ldots, w_n^{(1)}\right\} \subset (0,1)$ a convex sequence, satisfying $\sum_{i=1}^n w_i^{(1)} = 1$. Define

$$J_1 = \left\{i : w_i^{(1)} = w_{\min}^{(1)}\right\},$$

where $w_{\min}^{(1)} = \min\left\{w_i^{(1)} : 1 \leqslant i \leqslant n\right\}$ and $|J_1|$ stands for the cardinality of J_1. For $k \geqslant 2$, let $w^{(k)}$ be a sequence defined inductively in the following way

$$w_i^{(k)} = \begin{cases} w_i^{(k-1)} - w_{\min}^{(k-1)} & \text{if } i \notin J_{k-1}, \\ \frac{1}{|J_{k-1}|} n w_{\min}^{(k-1)} & \text{if } i \in J_{k-1}, \end{cases} \tag{2.2}$$

where $J_{k-1} = \left\{i : w_i^{(k-1)} = w_{\min}^{(k-1)}\right\}$ and $w_{\min}^{(k)} = \min\left\{w_1^{(k)}, \ldots, w_n^{(k)}\right\}$ for $k \geqslant 1$. Now, let us set $x^{(1)} = \left\{x_1^{(1)}, \ldots, x_n^{(1)}\right\} \subset I$, we provide a new sequence $x^{(k)}$ defined by

$$x_i^{(k)} = \begin{cases} x_i^{(k-1)} & \text{if } i \notin J_{k-1}, \\ \left(\frac{1}{n}\sum_{i=1}^n \left(x_i^{(k-1)}\right)^p\right)^{\frac{1}{p}} & \text{if } i \in J_{k-1}, \end{cases} \tag{2.3}$$

for all $1 \leqslant i \leqslant n$. Here, the order of the $\left\{x_i^{(1)}\right\}$ follows the order in which they are associated with the $\left\{w_i^{(1)}\right\}$, that is, $x_1^{(1)}$ is the value multiplied with $w_1^{(1)}$, and so on.

REMARK 2.1. Before stating the first main result of the section, we advance a significant observation as follows.

(i) If a non-negative function h defined on J is both super-multiplicative and super-additive, then h is an increasing function on J.

(ii) Let h be as in (i) and $f : I \to [0,\infty)$ be a (p,h)-convex function. Employing the same techniques as in the proof of [11, Remark 2.3], we can prove the following inequality

$$\begin{aligned} nh(w)&\left\{\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\left[\frac{1}{n}\sum_{i=1}^n x_i^p\right]^{\frac{1}{p}}\right)\right\} \\ &\geqslant h(nw)\left\{h\left(\frac{1}{n}\right)\sum_{i=1}^n f(x_i) - f\left(\left[\frac{1}{n}\sum_{i=1}^n x_i^p\right]^{\frac{1}{p}}\right)\right\}, \end{aligned}$$

where $\{x_1,\ldots,x_n\}\subset I$ and $w,nw\in J$.

2.2. Multiple-term refinements for (p,h)-convex and (p,h)-log-convex functions

The following theorem provides an improvement of [9, Theorem 4.1].

THEOREM 2.2. *Let h be a non-negative super-multiplicative and super-additive function defined on $[0,\infty)$ and suppose that $f:I\to\mathbb{R}^+$ is a (p,h)-convex function. If $N\in\mathbb{N}$, $\{x_1^{(1)},\ldots,x_n^{(1)}\}\subset I$ and $\{w_1^{(1)},\ldots,w_n^{(1)}\}\subset(0,1)$ with $\sum_{i=1}^n w_i^{(1)}=1$, we then have*

$$\begin{aligned}\sum_{i=1}^n h(w_i^{(1)})f(x_i^{(1)}) &\geqslant f\Big(\Big[\sum_{i=1}^n w_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\ &\quad+n\sum_{k=1}^N h(w_{\min}^{(k)})\Big\{\frac{1}{n}\sum_{i=1}^n f\Big(x_i^{(k)}\Big)-f\Big(\Big[\frac{1}{n}\sum_{i=1}^n (x_i^{(k)})^p\Big]^{\frac{1}{p}}\Big)\Big\}.\end{aligned}$$

Proof. We show it by induction on N. For $N=1$, we consider the difference

$$\begin{aligned}&\sum_{i=1}^n h(w_i^{(1)})f(x_i^{(1)})-nh(w_{\min}^{(1)})\Big\{\frac{1}{n}\sum_{i=1}^n f(x_i^{(1)})-f\Big(\Big[\frac{1}{n}\sum_{i=1}^n (x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\Big\}\\ &=\sum_{i=1}^n[h(w_i^{(1)})-h(w_{\min}^{(1)})]f(x_i^{(1)})+nh(w_{\min}^{(1)})f\Big(\Big[\frac{1}{n}\sum_{i=1}^n (x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\ &\geqslant\sum_{i=1}^n[h(w_i^{(1)}-w_{\min}^{(1)})]f(x_i^{(1)})+nh(w_{\min}^{(1)})f\Big(\Big[\frac{1}{n}\sum_{i=1}^n (x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\ &\geqslant f\Big(\Big[\sum_{i=1}^n (w_i^{(1)}-w_{\min}^{(1)})(x_i^{(1)})^p+nw_{\min}^{(1)}\frac{1}{n}\sum_{i=1}^n (x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\ &=f\Big(\Big[\sum_{i=1}^n w_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big),\end{aligned}$$

where we have just used Jensen's inequality for $2n$ parameters to show the second inequality above, i.e., the claimed inequality holds for $N=1$. We now suppose that it is valid for some $N\in\mathbb{N}$, that is,

$$\begin{aligned}\sum_{i=1}^n h(\mu_i^{(1)})f(y_i^{(1)}) &\geqslant f\Big(\Big[\sum_{i=1}^n \mu_i^{(1)}(y_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\ &\quad+n\sum_{k=1}^N h(\mu_{\min}^{(k)})\Big\{\frac{1}{n}\sum_{i=1}^n f\Big(y_i^{(k)}\Big)-f\Big(\Big[\frac{1}{n}\sum_{i=1}^n (y_i^{(k)})^p\Big]^{\frac{1}{p}}\Big)\Big\},\end{aligned}$$

where $\{\mu_1^{(1)},\ldots,\mu_n^{(1)}\}$ is any convex sequence and any elements $\{y_1^{(1)},\ldots,y_n^{(1)}\}\subset I$.

On the one hand, it follows from Remark 2.1, the super-multiplicative and super-additive of h, and the (p,h)-convexity of f that

$$\begin{aligned}
S &:= \sum_{i=1}^{n} h(w_i^{(1)}) f(x_i^{(1)}) - nh(w_{\min}^{(1)}) \left\{ \frac{1}{n} \sum_{i=1}^{n} f(x_i^{(1)}) - f\left(\left[\frac{1}{n} \sum_{i=1}^{n} (x_i^{(1)})^p \right]^{\frac{1}{p}} \right) \right\} \\
&\geqslant \sum_{i=1}^{n} h(w_i^{(1)}) f(x_i^{(1)}) - h(nw_{\min}^{(1)}) \left\{ h\left(\frac{1}{n}\right) \sum_{i=1}^{n} f(x_i^{(1)}) - f\left(\left[\frac{1}{n} \sum_{i=1}^{n} (x_i^{(1)})^p \right]^{\frac{1}{p}} \right) \right\} \\
&\geqslant \sum_{i=1}^{n} [h(w_i^{(1)}) - h(w_{\min}^{(1)})] f(x_i^{(1)}) + h(nw_{\min}^{(1)}) f\left(\left[\frac{1}{n} \sum_{i=1}^{n} (x_i^{(1)})^p \right]^{\frac{1}{p}} \right) \\
&\geqslant \sum_{i=1}^{n} h(w_i^{(1)} - w_{\min}^{(1)}) f(x_i^{(1)}) + h(nw_{\min}^{(1)}) f\left(\left[\frac{1}{n} \sum_{i=1}^{n} (x_i^{(1)})^p \right]^{\frac{1}{p}} \right) \\
&\geqslant \sum_{i=1}^{n} h(w_i^{(2)}) f(x_i^{(2)}),
\end{aligned}$$

where we have just used the definitions of $(w_i^{(k)})$ and $(x_i^{(k)})$ in (2.2) and (2.3). For simplicity, we write $\mu_i^{(1)}$ and $y_i^{(1)}$ to stand for $w_i^{(2)}$ and $x_i^{(2)}$, respectively. From this, we have $\mu_i^{(k)} = w_i^{(k+1)}$ and $y_i^{(k)} = x_i^{(k+1)}$ for $k \geqslant 1$. On the other hand, it is easy to see that

$$\begin{aligned}
\sum_{i=1}^{n} \mu_i^{(1)} = \sum_{i=1}^{n} w_i^{(2)} &= \sum_{i \notin J_1} (w_i^{(1)} - w_{\min}^{(1)}) + \sum_{i \in J_1} \frac{nw_{\min}^{(1)}}{|J_1|} \\
&= \sum_{i=1}^{n} w_i^{(1)} - \sum_{i \in J_1} w_i^{(1)} - \sum_{i \notin J_1} w_i^{(1)} + nw_{\min}^{(1)} \\
&= \sum_{i=1}^{n} w_i^{(1)} = 1
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{n} \mu_i^{(1)} (y_i^{(1)})^p = \sum_{i=1}^{n} w_i^{(2)} (y_i^{(2)})^p &= \sum_{i \notin J_1} (w_i^{(1)} - w_{\min}^{(1)}) (x_i^{(1)})^p + \sum_{i \in J_1} \frac{nw_{\min}^{(1)}}{|J_1|} \sum_{j=1} \frac{(x_j^{(1)})^p}{n} \\
&= \sum_{i=1}^{n} w_i^{(1)} (x_i^{(1)})^p - \sum_{i=1}^{n} w_{\min}^{(1)} (x_i^{(1)})^p + \sum_{i \in J_1} \frac{nw_{\min}^{(1)}}{|J_1|} \sum_{j=1}^{n} \frac{(x_j^{(1)})^p}{n} \\
&= \sum_{i=1}^{n} w_i^{(1)} (x_i^{(1)})^p,
\end{aligned}$$

or,

$$\left(\sum_{i=1}^{n} \mu_i^{(1)} (y_i^{(1)})^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} w_i^{(1)} (x_i^{(1)})^p \right)^{\frac{1}{p}}.$$

These two facts, together with the induction step, give us that

$$\begin{aligned} S &\geqslant \sum_{i=1}^{n} h(\mu_i^{(1)}) f(y_i^{(1)}) \\ &\geqslant f\Big(\Big[\sum_{i=1}^{n} \mu_i^{(1)} (y_i^{(1)})^p\Big]^{\frac{1}{p}}\Big) + n\sum_{k=1}^{N} h(\mu_{\min}^{(k)}) \left\{\frac{1}{n}\sum_{i=1}^{n} f\left(y_i^{(k)}\right) - f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}\left(y_i^{(k)}\right)^p\Big]^{\frac{1}{p}}\Big)\right\} \\ &= f\Big(\Big[\sum_{i=1}^{n} w_i^{(1)} (x_i^p)^{(1)}\Big]^{\frac{1}{p}}\Big) + n\sum_{k=2}^{N+1} h(w_{\min}^{(k)}) \left\{\frac{1}{n}\sum_{i=1}^{n} f\left(x_i^{(k)}\right) - f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^{(k)}\right)^p\Big]^{\frac{1}{p}}\Big)\right\}, \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{i=1}^{n} h(w_i^{(1)}) f(x_i^{(1)}) \geqslant{}& f\Big(\Big[\sum_{i=1}^{n} w_i^{(1)} (x_i^p)^{(1)}\Big]^{\frac{1}{p}}\Big) \\ &+ n\sum_{k=1}^{N+1} h(w_{\min}^{(k)}) \left\{\frac{1}{n}\sum_{i=1}^{n} f\left(x_i^{(k)}\right) - f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^{(k)}\right)^p\Big]^{\frac{1}{p}}\Big)\right\}. \end{aligned}$$

This completes the proof. □

REMARK 2.3. The positivity condition of f in Theorem 2.2 can be relaxed if h is an identity function, specifically, the following improved Jensen-type inequality holds for any p-convex function $f: I \to \mathbb{R}$:

$$\begin{aligned} \sum_{i=1}^{n} w_i^{(1)} f(x_i^{(1)}) \geqslant{}& f\Big(\Big[\sum_{i=1}^{n} w_i^{(1)} (x_i^p)^{(1)}\Big]^{\frac{1}{p}}\Big) \\ &+ n\sum_{k=1}^{N+1} w_{\min}^{(k)} \left\{\frac{1}{n}\sum_{i=1}^{n} f\left(x_i^{(k)}\right) - f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^{(k)}\right)^p\Big]^{\frac{1}{p}}\Big)\right\}, \end{aligned}$$

where $\{x_1^{(1)},\dots,x_n^{(1)}\} \subset I$ and $\{w_1^{(1)},\dots,w_n^{(1)}\} \subset (0,1)$ with $\sum_{i=1}^n w_i^{(1)} = 1$.

The second main result of this section supplies a multiple-term refinement for [9, Theorem 2.2], which is stated as follows.

THEOREM 2.4. *Let h be a non-negative super-multiplicative and super-additive function defined on $[0,\infty)$ and $f: I \to \mathbb{R}$ be a positve (p,h)-convex function. For a sequence $\{x_1^{(1)},\dots,x_n^{(1)}\} \subset I$ and two weight sequences $\{\nu_1^{(1)},\dots,\nu_n^{(1)}\}$ and $\{\mu_1^{(1)},\dots,$ $\mu_n^{(1)}\}$ in $(0,1)$ with $\sum_{i=1}^n \nu_i^{(1)} = \sum_{i=1}^n \mu_i^{(1)} = 1$, we denote by $J = \{i : \nu_i^{(1)} - \min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}$ $= 0\}$,*

$$w_i^{(1)} = \begin{cases} \nu_i^{(1)} - \min\limits_{1\leqslant j\leqslant n} \dfrac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)} & \text{for } i \notin J, \\ \frac{1}{|J|} \min\limits_{1\leqslant j\leqslant n} \dfrac{\nu_j^{(1)}}{\mu_j^{(1)}} & \text{for } i \in J, \end{cases}$$

and

$$y_i^{(1)} = \begin{cases} x_i^{(1)} & \text{for } i \notin J, \\ \left(\sum_{i=1}^n \mu_i^{(1)} (x_i^{(1)})^p\right)^{\frac{1}{p}} & \text{for } i \in J. \end{cases}$$

Then, for every $N \in \mathbb{N}$, *we have*

$$\begin{aligned} &\sum_{i=1}^n h(\nu_i^{(1)}) f(x_i^{(1)}) - f\left(\left[\sum_{i=1}^n \nu_i^{(1)} (x_i^{(1)})^p\right]^{\frac{1}{p}}\right) \\ &\qquad \geqslant h\left(\min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right) \left\{\sum_{i=1}^n h(\mu_i^{(1)}) f(x_i^{(1)}) - f\left(\left[\sum_{i=1}^n \mu_i^{(1)} (x_i^{(1)})^p\right]^{\frac{1}{p}}\right)\right\} \\ &\qquad\qquad + n \sum_{k=1}^N h(w_{\min}^{(k)}) \left\{\frac{1}{n}\sum_{i=1}^n f(y_i^{(k)}) - f\left(\left[\frac{1}{n}\sum_{i=1}^n (y_i^{(k)})^p\right]^{\frac{1}{p}}\right)\right\}. \end{aligned}$$

Proof. It follows from the proof of [9, Theorem 4.1], the super-multiplicative and super-additive of h that

$$\begin{aligned} M &:= \sum_{i=1}^n h(\nu_1^{(1)}) f(x_i^{(1)}) - h\left(\min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right) \left\{\sum_{i=1}^n h(\mu_i^{(1)}) f(x_i^{(1)}) - f\left(\left[\sum_{i=1}^n \mu_i^{(1)} (x_i^{(1)})^p\right]^{\frac{1}{p}}\right)\right\} \\ &= \sum_{i=1}^n \left[h(\nu_1^{(1)}) - h\left(\min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}} \mu_i^{(1)}\right)\right] f(x_i^{(1)}) \\ &\qquad + h\left(\min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right) f\left(\left[\sum_{i=1}^n \mu_i^{(1)} (x_i^{(1)})^p\right]^{\frac{1}{p}}\right) \\ &\geqslant \sum_{i\notin J} h\left(\nu_i^{(1)} - \min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}} \mu_i^{(1)}\right) f(x_i^{(1)}) + h\left(\min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right) f\left(\left[\sum_{i=1}^n \mu_i^{(1)} (x_i^{(1)})^p\right]^{\frac{1}{p}}\right) \\ &\geqslant \sum_{i\notin J} h(w_i^{(1)}) f(x_i^{(1)}) + \sum_{i\in J} h(w_i^{(1)}) f\left(\left[\sum_{i=1}^n \mu_i^{(1)} (x_i^{(1)})^p\right]^{\frac{1}{p}}\right) \\ &= \sum_{i=1}^n h(w_i^{(1)}) f(y_i^{(1)}). \end{aligned}$$

Notice that

$$\sum_{i=1}^n w_i^{(1)} = \sum_{i\notin J} w_i^{(1)} + \sum_{i\in J} w_i^{(1)} = \sum_{i=1}^n \left(\nu_i^{(1)} - \min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}} \mu_i^{(1)}\right) + \sum_{i\in J} \frac{1}{|J|} \min_{1\leqslant j\leqslant n} \frac{\nu_j^{(1)}}{\mu_j^{(1)}} = 1$$

and

$$\begin{aligned}\sum_{i=1}^{n} w_i^{(1)}(y_i^{(1)})^p &= \sum_{i\notin J} w_i^{(1)}(x_i^{(1)})^p + \sum_{i\in J} w_i^{(1)} \sum_{j=1}^{n} \mu_j^{(1)}(x_j^{(1)})^p \\ &= \sum_{i=1}^{n}\left(\nu_i^{(1)} - \min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\right)(x_i^{(1)})^p + \min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\sum_{j=1}^{n}\mu_j^{(1)}(x_j^{(1)})^p \\ &= \sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p - \min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p + \min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\sum_{j=1}^{n}\mu_j^{(1)}(x_j^{(1)})^p \\ &= \sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p.\end{aligned}$$

Therefore, applying Theorem 2.2, we obtain

$$M \geqslant f\left(\left[\sum_{i=1}^{n}\nu_i^{(1)}(x_i^p)^{(1)}\right]^{\frac{1}{p}}\right) + n\sum_{k=1}^{N} h(w_{\min}^{(k)})\left\{\frac{1}{n}\sum_{i=1}^{n} f(y_i^{(k)}) - f\left(\left[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(k)})^p\right]^{\frac{1}{p}}\right)\right\},$$

which yields the desired inequality. □

REMARK 2.5. We deduce Theorem 4.1 in [9], when we substitute $\mu_i = \frac{1}{n}$ for $i \in \{1,\ldots,n\}$ in Theorem 2.4.

REMARK 2.6. We deduce the main result in [17], when we substitute $\mu_i = \frac{1}{n}$ for $i \in \{1,\ldots,n\}$, $h(x) = x$ and $p = 1$ in Theorem 2.7.

Replacing f by $\log f$ in Theorem 2.4, we get the (p,h)-log-convex version of the previous result as follows.

THEOREM 2.7. *Let h be a non-negative super-multiplicative and super-additive function defined on $[0,\infty)$ and $f : I \to \mathbb{R}^+$ be a (p,h)-log-convex function. For a sequence $\{x_1^{(1)},\ldots,x_n^{(1)}\} \subset I$ and two weight sequences $\{\nu_1^{(1)},\ldots,\nu_n^{(1)}\}$ and $\{\mu_1^{(1)},\ldots,\mu_n^{(1)}\}$ in $(0,1)$ with $\sum_{i=1}^{n}\nu_i^{(1)} = \sum_{i=1}^{n}\mu_i^{(1)} = 1$, we use the notations $w_i^{(1)}$ and $y_i^{(1)}$ as in Theorem 2.4. Then, for every $N \in \mathbb{N}$, we have*

$$\frac{\prod_{i=1}^{n} f^{h(\nu_i^{(1)})}(x_i^{(1)})}{f([\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p]^{\frac{1}{p}})} \geqslant \left(\frac{\prod_{i=1}^{n} f^{h(\mu_i^{(1)})}(x_i^{(1)})}{f([\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p]^{\frac{1}{p}})}\right)^{h\left(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right)} \times \prod_{k=1}^{N}\left(\frac{\prod_{i=1}^{n} f^{\frac{1}{n}}(y_i^{(k)})}{f([\frac{1}{n}\sum_{i=1}^{n}(y_i^{(k)})^p]^{\frac{1}{p}})}\right)^{nh(w_{\min}^{(k)})}.$$

3. Further inequalities for (p,h)-convex and (p,h)-log-convex functions via the theory of weak submajorization

The main purpose of this section is to extend Theorems 2.4 and 2.7 with respective to $N=2$ to a more general framework via the theory of weak submajorization. One of the generalizations of Theorem 2.4 is stated as follows.

THEOREM 3.1. *Let $\phi:[0,\infty)\to\mathbb{R}$ be an increasing and convex function. Under the notations as in Theorem 2.4, we have*

$$\phi\Big(\sum_{i=1}^{n}h(\nu_i^{(1)})f(x_i^{(1)})\Big)-\phi\circ f\Big(\Big[\sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)$$
$$\geqslant\phi\Big(h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)\sum_{i=1}^{n}h(\mu_i^{(1)})f(x_i^{(1)})\Big)-\phi\Big(h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)f\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\Big)$$
$$+\sum_{k=1}^{2}\Big\{\phi\Big(h(w_{\min}^{(k)})\sum_{i=1}^{n}f(y_i^{(k)})\Big)-\phi\Big(nh(w_{\min}^{(k)})f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(k)})^p\Big]^{\frac{1}{p}}\Big)\Big)\Big\}.$$

In order to prove this theorem, we additionally need the following two lemmas. To this end, we recall the theory of weak submajorization. Throughout this section, we denote by $X^*=(X_1^*,\dots,X_n^*)$ the vector obtained from the vector $X=(X_1,\dots,X_n)\in\mathbb{R}^n$ by rearranging the components of it in decreasing order. Then, for two vectors $X=(X_1,\dots,X_n)$ and $Y=(Y_1,\dots,Y_n)$ in $\mathbb{R}^n, Y$ is said to be weakly sub-majorized by X, written $X\succ_w Y$, if

$$\sum_{i=1}^{k}X_i^*\geqslant\sum_{i=1}^{k}Y_i^*$$

for all $k=1,\dots,n$.

The following result characterizes the theory of weak sub-majorization via increasing convex functions and can be found in [14, pp. 13].

LEMMA 3.2. *Let $X=(X_i)_{i=1}^n$, $Y=(Y_i)_{i=1}^n\in\mathbb{R}^n$ and $J\subset\mathbb{R}$ be an interval containing the components of X and Y. If $X\succ_w Y$ and $\psi:J\to\mathbb{R}$ is a continuous increasing convex function, then*

$$\sum_{i=1}^{n}\psi(X_i)\geqslant\sum_{i=1}^{n}\psi(Y_i).$$

The next lemma presents the concrete vectors used in the proof of the theorem.

LEMMA 3.3. *Under the notations as in Theorem 3.1, we consider two vectors $X=(X_1,X_2,X_3,X_4)$ and $Y=(Y_1,Y_2,Y_3,Y_4)$ with components*

$$X_1=f\Big(\Big[\sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big),\quad X_2=h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)\sum_{i=1}^{n}h(\mu_i^{(1)})f(x_i^{(1)}),$$

$$X_3=h(w_{\min}^{(1)})\sum_{i=1}^{n}f(y_i^{(1)}),\quad X_4=h(w_{\min}^{(2)})\sum_{i=1}^{n}f(y_i^{(2)});$$

and

$$Y_1=\sum_{i=1}^{n}h(\nu_i^{(1)})f(x_i^{(1)}),\quad Y_2=h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)f\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big),$$

$$Y_3=nh(w_{\min}^{(1)})f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(1)})^p\Big]^{\frac{1}{p}}\Big),\quad Y_4=nh(w_{\min}^{(2)})f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(2)})^p\Big]^{\frac{1}{p}}\Big).$$

Then, we have $X\prec_w Y$, namely, the vectors X^* and Y^* have components satisfying that

$$X_1^*\leqslant Y_1^*, \tag{3.1}$$

$$X_1^*+X_2^*\leqslant Y_1^*+Y_2^*, \tag{3.2}$$

$$X_1^*+X_2^*+X_3^*\leqslant Y_1^*+Y_2^*+Y_3^*, \tag{3.3}$$

$$X_1^*+X_2^*+X_3^*+X_4^*\leqslant Y_1^*+Y_2^*+Y_3^*+Y_4^*. \tag{3.4}$$

Proof. First of all, inequality (3.4) is obvious by Theorem 2.4 with $N=2$. In oder to prove inequality (3.1), we have to show $Y_1\geqslant X_i$ for all $i=1,2,3,4$. Indeed, we have $Y_1\geqslant X_1$ by Jensen's inequality. Now, utilizing the super-multiplicative, super-additive and the positve of h, we have

$$\begin{aligned}
Y_1-X_2&=\sum_{i=1}^{n}h(\nu_i^{(1)})f(x_i^{(1)})-h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)\sum_{i=1}^{n}h(\mu_i^{(1)})f(x_i^{(1)})\\
&=\sum_{i=1}^{n}\Big[h(\nu_i^{(1)})-h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)h(\mu_i^{(1)})\Big]f(x_i^{(1)})\\
&\geqslant\sum_{i=1}^{n}\Big[h(\nu_i^{(1)})-h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\Big)\Big]f(x_i^{(1)})\\
&\geqslant\sum_{i\notin J}h\Big(\nu_i^{(1)}-\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\Big)f(x_i^{(1)})\\
&\geqslant 0,
\end{aligned}$$

which implies that $Y_1\geqslant X_2$. Next, we consider the difference

$$\begin{aligned}
Y_1-X_3&=\sum_{i=1}^{n}h(\nu_i^{(1)})f(x_i^{(1)})-h(w_{\min}^{(1)})\sum_{i=1}^{n}f(y_i^{(1)})\\
&=\sum_{i\notin J}\big[h(\nu_i^{(1)})-h(w_{\min}^{(1)})\big]f(x_i^{(1)})+\sum_{i\in J}h(\nu_i^{(1)})f(x_i^{(1)})\\
&\qquad-|J|h(w_{\min}^{(1)})f\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)
\end{aligned}$$

$$\geqslant \sum_{i\notin J}\left[h(v_i^{(1)})-h(w_{\min}^{(1)})\right]f(x_i^{(1)})+\sum_{i\in J}h(v_i^{(1)})f(x_i^{(1)})$$
$$-|J|h(w_{\min}^{(1)})\sum_{i=1}^{n}h(\mu_i^{(1)})f\left(x_i^{(1)}\right)$$
$$=\sum_{i\notin J}\left[h(v_i^{(1)})-h(w_{\min}^{(1)})-|J|h(w_{\min}^{(1)})h(\mu_i^{(1)})\right]f(x_i^{(1)})$$
$$+\sum_{i\in J}\left[h(v_i^{(1)})-|J|h(w_{\min}^{(1)})h(\mu_i^{(1)})\right]f(x_i^{(1)})$$
$$\geqslant \sum_{i\notin J}\left[h(v_i^{(1)})-h(w_{\min}^{(1)}+|J|w_{\min}^{(1)}\mu_i^{(1)})\right]f(x_i^{(1)})$$
$$+\sum_{i\notin J}\left[h(v_i^{(1)})-h(|J|w_{\min}^{(1)}\mu_i^{(1)})\right]f(x_i^{(1)}).$$

For $i\notin J$, we have

$$v_i^{(1)}-w_{\min}^{(1)}\geqslant v_i^{(1)}-w_i^{(1)}=\min_{1\leqslant j\leqslant n}\frac{v_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}=|J|\frac{1}{|J|}\min_{1\leqslant j\leqslant n}\frac{v_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\geqslant |J|w_{\min}^{(1)}\mu_i^{(1)}.$$

For $i\in J$, we have

$$v_i^{(1)}=\min_{1\leqslant j\leqslant n}\frac{v_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}=|J|\frac{1}{|J|}\min_{1\leqslant j\leqslant n}\frac{v_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\geqslant |J|w_{\min}^{(1)}\mu_i^{(1)}.$$

On the other hand, it follows from the super-multiplicative and the super-additive and the positve of h that h is increasing. This, combined with the previous two facts, implies that

$$Y_1-X_3\geqslant \sum_{i\in J}\left[h(v_i^{(1)})-h(w_{\min}^{(1)}+|J|w_{\min}^{(1)}\mu_i^{(1)})\right]f(x_i^{(1)})$$
$$+\sum_{i\notin J}\left[h(v_i^{(1)})-h(|J|w_{\min}^{(1)}\mu_i^{(1)})\right]f(x_i^{(1)})\geqslant 0,$$

that is, $Y_1\geqslant X_3$. Similarly, from the estimate of S in the proof of Theorem 2.2, we find that

$$\sum_{i=1}^{n}h(w_i^{(1)})f(y_i^{(1)})\geqslant \sum_{i=1}^{n}h(w_i^{(2)})f(y_i^{(2)}).$$

Thus, combining with the increasing of h, we deduce that

$$Y_1-X_4=\sum_{i=1}^{n}h(v_i^{(1)})f(x_i^{(1)})-h(w_{\min}^{(2)})\sum_{i=1}^{n}f(y_i^{(2)})$$
$$\geqslant \sum_{i=1}^{n}h(v_i^{(1)})f(x_i^{(1)})-\sum_{i=1}^{n}h(w_i^{(2)})f(y_i^{(2)})$$
$$\geqslant \sum_{i=1}^{n}h(v_i^{(1)})f(x_i^{(1)})-\sum_{i=1}^{n}h(w_i^{(1)})f(y_i^{(1)})$$

$$
\begin{aligned}
&\geqslant \sum_{i\notin J}\Big[h(\nu_i^{(1)})-h\Big(\nu_i^{(1)}-\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\Big)\Big]f(x_i^{(1)})\\
&\qquad +\sum_{i\in J}h(\nu_i^{(1)})f(x_i^{(1)})-\sum_{i\in J}h\Big(\frac{1}{|J|}\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)f\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\
&\geqslant \sum_{i\notin J}h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\Big)f(x_i^{(1)})+\sum_{i\in J}h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\mu_i^{(1)}\Big)f(x_i^{(1)})\\
&\qquad -h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)f\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\
&\geqslant h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)\Big\{\sum_{i=1}^{n}h(\mu_i^{(1)})f(x_i^{(1)})-f\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\Big\}\\
&\geqslant 0,
\end{aligned}
$$

which is equivalent to $Y_1 \geqslant X_4$.

To show inequality (3.3), we have check the following inequalities

$$X_1+X_2+X_3 \leqslant Y_1+Y_2+Y_3, \tag{3.5}$$

$$X_1+X_2+X_4 \leqslant Y_1+Y_2+Y_4, \tag{3.6}$$

$$X_1+X_3+X_4 \leqslant Y_1+Y_3+Y_4, \tag{3.7}$$

$$X_2+X_3+X_4 \leqslant Y_1+Y_2+Y_3. \tag{3.8}$$

Indeed, inequality (3.5) is evident because Theorem 2.4 with $N=1$. Also, by the (p,h)-convixty of f and the non-negative of h, we have $X_2 \geqslant Y_2$ and $X_3 \geqslant Y_3$. From this and Theorem 2.4, it follows that

$$
\begin{aligned}
X_1+X_2+X_4 &= (X_1+X_2+X_3+X_4)-X_3\\
&\leqslant (Y_1+Y_2+Y_3+Y_4)-X_3\\
&\leqslant (Y_1+Y_2+Y_3+Y_4)-Y_3\\
&= Y_1+Y_2+Y_4,
\end{aligned}
$$

and

$$
\begin{aligned}
X_1+X_3+X_4 &= (X_1+X_2+X_3+X_4)-X_2\\
&\leqslant (Y_1+Y_2+Y_3+Y_4)-X_2\\
&\leqslant (Y_1+Y_2+Y_3+Y_4)-Y_2\\
&= Y_1+Y_3+Y_4,
\end{aligned}
$$

namely, inequalities (3.6) and (3.7) are valid. By invoking the estimate M in the proof of Theorem 2.4, we have

$$Y_1+Y_2-X_2=\sum_{i=1}^{n}h(w_i^{(1)})f(y_i^{(1)}).$$

Combining this with the estimate for S in the proof of Theorem 2.2, we infer that

$$\begin{aligned}
&Y_1+Y_2+Y_3-X_2-X_3-X_4\\
&=\sum_{i=1}^n h(w_i^{(1)})f(y_i^{(1)})-nh(w_{\min}^{(1)})\left\{\frac{1}{n}\sum_{i=1}^n f(y_i^{(1)})-f\left(\left[\frac{1}{n}\sum_{i=1}^n (y_i^{(1)})^p\right]^{\frac{1}{p}}\right)\right\}\\
&\qquad -h(w_{\min}^{(2)})\sum_{i=1}^n f(y_i^{(2)})\\
&\geqslant \sum_{i=1}^n h(w_i^{(2)})f(y_i^{(2)})-h(w_{\min}^{(2)})\sum_{i=1}^n f(y_i^{(2)})\\
&\geqslant \sum_{i=1}^n [h(w_i^{(2)})-h(w_{\min}^{(2)})]f(y_i^{(2)})\\
&\geqslant 0,
\end{aligned}$$

where the third estimate above is based on the increasing of h. So,

$$X_2+X_3+X_4\leqslant Y_1+Y_2+Y_3. \tag{3.9}$$

Finally, to test inequality (3.6), we have to prove the following inequalities

$$X_1+X_2\leqslant Y_1+Y_2, \tag{3.10}$$
$$X_1+X_3\leqslant Y_1+Y_3, \tag{3.11}$$
$$X_1+X_4\leqslant Y_1+Y_4, \tag{3.12}$$
$$X_2+X_3\leqslant Y_1+Y_2, \tag{3.13}$$
$$X_2+X_4\leqslant Y_1+Y_2, \tag{3.14}$$
$$X_3+X_4\leqslant Y_1+Y_3. \tag{3.15}$$

Indeed, inequality (3.10) is obvious by Theorem 2.4. Next, since $X_2\geqslant Y_2$ and Theorem 2.4, we have

$$X_1+X_3=(X_1+X_2+X_3)-X_2\leqslant (Y_1+Y_2+Y_3)-X_2\leqslant (Y_1+Y_2+Y_3)-Y_2=Y_1+Y_3$$

i.e., inequality (3.11) holds true. Also, by Theorem 2.4, it is easy to see that

$$X_1+X_4=\sum_{i=1}^4 X_i-(X_2+X_3)\leqslant \sum_{i=1}^4 Y_i-(X_2+X_3)\leqslant \sum_{i=1}^4 Y_i-(Y_2+Y_3)=Y_1+Y_4$$

because $Y_2\leqslant X_2$ and $Y_3\leqslant X_3$, namely, inequality (3.12) is valid. Similarly, it follows from (3.9) and inequalities $Y_3\leqslant X_3$ and $Y_2\leqslant X_2$ that inequalities (3.14) and (3.15) hold. On the other hand, by the increasing of h, we also have

$$Y_1+Y_2-X_2=\sum_{i=1}^n h(w_i^{(1)})f(y_i^{(1)})\geqslant \sum_{i=1}^n h(w_{\min}^{(1)})f(y_i^{(1)})=X_3,$$

which implies that $X_2+X_3\leqslant Y_1+Y_2$, namely, inequality (3.13) is satisfied. □

Now, we are prepared to present a proof of the theorem.

Proof Theorem 3.1. Let us consider two vectors

$$X=(X_1,X_2,X_3,X_4) \quad \text{and} \quad Y=(Y_1,Y_2,Y_3,Y_4)$$

defined as in Lemma 3.3, we have $X \prec_w Y$. Hence, applying Lemma 3.2 to the function ϕ, we obtain

$$\phi(Y_1)+\phi(Y_2)+\phi(Y_3)+\phi(Y_4) \geqslant \phi(X_1)+\phi(X_2)+\phi(X_3)+\phi(X_4),$$

or equivalently,

$$\phi(Y_1)-\phi(X_1) \geqslant [\phi(X_2)-\phi(Y_2)]+[\phi(X_3)-\phi(Y_3)]+[\phi(X_4)-\phi(Y_4)].$$

This completes the proof. $\square$

According to Remark 2.1, we derive a consequence of Theorem 3.1, as follows.

COROLLARY 3.4. *Under the notations as in Theorem 3.1, we have*

$$\begin{aligned}
&\phi\Big(\sum_{i=1}^{n} h(\nu_i^{(1)})f(x_i^{(1)})\Big)-\phi\circ f\Big(\Big[\sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\
&\geqslant \phi\Big(h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)\sum_{i=1}^{n}h(\mu_i^{(1)})f(x_i^{(1)})\Big)\\
&\quad-\phi\Big(h\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)f\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\Big)\\
&\quad+\sum_{k=1}^{2}\Big\{\phi\Big(h(nw_{\min}^{(k)})h\Big(\frac{1}{n}\Big)\sum_{i=1}^{n}f(y_i^{(k)})\Big)-\phi\Big(h(nw_{\min}^{(k)})f\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(k)})^p\Big]^{\frac{1}{p}}\Big)\Big)\Big\}.
\end{aligned}$$

By choosing $\phi(x)=x^{\lambda}$ with $\lambda\geqslant 1$ in Theorem 3.1, we get the following result.

COROLLARY 3.5. *Under the notations as in Theorem 2.4 and $\lambda \geqslant 1$, we have*

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} h(\nu_i^{(1)})f(x_i^{(1)})\Big)^{\lambda}-f^{\lambda}\Big(\Big[\sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\
&\geqslant h^{\lambda}\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)\Big(\sum_{i=1}^{n}h(\mu_i^{(1)})f(x_i^{(1)})\Big)^{\lambda}-h^{\lambda}\Big(\min_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\Big)f^{\lambda}\Big(\Big[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\Big]^{\frac{1}{p}}\Big)\\
&\quad+\sum_{k=1}^{2}\Big\{h^{\lambda}(w_{\min}^{(k)})\Big(\sum_{i=1}^{n}f(y_i^{(k)})\Big)^{\lambda}-(nh(w_{\min}^{(k)}))^{\lambda}f^{\lambda}\Big(\Big[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(k)})^p\Big]^{\frac{1}{p}}\Big)\Big\}.
\end{aligned}$$

Replacing f by $\log f$ in Theorem 3.1, we obtain the following.

THEOREM 3.6. *Let $\phi : [0,\infty) \to \mathbb{R}$ be an increasing and convex function. Assume that h is a non-negative super-multiplicative and super-additive defined on $[0,\infty)$ and $f : I \to (0,\infty)$ is a (p,h)-$\log$-convex function. For a sequence $\{x_1^{(1)},\dots,x_n^{(1)}\} \subset I$, and two weight sequences $\{\nu_1^{(1)},\dots,\nu_n^{(1)}\}$, $\{\mu_1^{(1)},\dots,\mu_n^{(1)}\}$ in $(0,1)$, we construct sequences $\{w_1^{(1)},\dots,w_1^{n}\}$ and $\{y_1^{(1)},\dots,y_n^{(1)}\}$ as in Theorem 2.4. Then, for every $N \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\phi\circ\log\left(\prod_{i=1}^{n} f^{h(\nu_i^{(1)})}(x_i^{(1)})\right) - \phi\circ\log\circ f\left(\left[\sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p\right]^{\frac{1}{p}}\right)\\
&\geqslant \phi\circ\log\left(\left(\prod_{i=1}^{n} f^{h(\mu_i^{(1)})}(x_i^{(1)})\right)^{h\left(\min\limits_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right)}\right)\\
&\quad -\phi\circ\log\left(f^{h\left(\min\limits_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right)}\left(\left[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\right]^{\frac{1}{p}}\right)\right)\\
&\quad +\sum_{k=1}^{2}\left\{\phi\circ\log\left(\prod_{i=1}^{n} f^{h(w_{\min}^{(k)})}(y_i^{(k)})\right) - \phi\circ\log\left(f^{nh(w_{\min}^{(k)})}\left(\left[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(k)})^p\right]^{\frac{1}{p}}\right)\right)\right\}.
\end{aligned}
$$

By choosing $\phi(x) = \exp(\lambda x)$ with $\lambda > 0$ in Theorem 3.6, leads to the following consequence.

COROLLARY 3.7. *Let h be a non-negative super-multiplicative and super-additive defined on $[0,\infty)$ and $f : I \to (0,\infty)$ be a (p,h)-$\log$-convex function. For a sequence $\{x_1^{(1)},\dots,x_n^{(1)}\} \subset I$, and two weight sequences $\{\nu_1^{(1)},\dots,\nu_n^{(1)}\}$, $\{\mu_1^{(1)},\dots,\mu_n^{(1)}\}$ in $(0,1)$, we construct sequences $\{w_1^{(1)},\dots,w_1^{n}\}$ and $\{y_1^{(1)},\dots,y_n^{(1)}\}$ as in Theorem 2.4. Then, for every $N \in \mathbb{N}$ and $\lambda > 0$, we have*

$$
\begin{aligned}
&\left(\prod_{i=1}^{n} f^{h(\nu_i^{(1)})}(x_i^{(1)})\right)^{\lambda} - f^{\lambda}\left(\left[\sum_{i=1}^{n}\nu_i^{(1)}(x_i^{(1)})^p\right]^{\frac{1}{p}}\right)\\
&\quad\geqslant \left(\prod_{i=1}^{n} f^{h(\mu_i^{(1)})}(x_i^{(1)})\right)^{\lambda h\left(\min\limits_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right)} - f^{\lambda h\left(\min\limits_{1\leqslant j\leqslant n}\frac{\nu_j^{(1)}}{\mu_j^{(1)}}\right)}\left(\left[\sum_{i=1}^{n}\mu_i^{(1)}(x_i^{(1)})^p\right]^{\frac{1}{p}}\right)\\
&\qquad +\sum_{k=1}^{2}\left\{\prod_{i=1}^{n} f^{\lambda h(w_{\min}^{(k)})}(y_i^{(k)}) - f^{\lambda nh(w_{\min}^{(k)})}\left(\left[\frac{1}{n}\sum_{i=1}^{n}(y_i^{(k)})^p\right]^{\frac{1}{p}}\right)\right).
\end{aligned}
$$

REMARK 3.8. We deduce the main result of [12], when we substitute $n = 2$, $h(x) = x$ and $p = 1$ in Corollary 3.4.

4. Some applications of the main results

4.1. Scalar inequalities

In this subsection, we provide specific applications of the inequalities derived earlier. When $x > 0$ and $p \in (-\infty, 1)$ the function $f(x) = x^{\frac{1}{p}}$ is convex. Applying Corollary 3.5, for $h(x) = x$ and $p = 1$ we obtain the following new bounds for the difference between the arithmetic and power means. Here, we recall that given positive numbers $x_1, \cdots, x_n$ and $\alpha_1, \cdots, \alpha_n$ such that $\sum_{i=1}^n \alpha_i = 1$, the quantity $A := \sum_{i=1}^n \alpha_i x_i$ is called the arithmetic mean of the $\{x_i\}$. On the other hand, if $p \in \mathbb{R}$, the power mean of $\{x_i\}$ is defined by $M_p := \left(\sum_{i=1}^n \alpha_i x_i^p\right)^{\frac{1}{p}}$. When $p = 0$, the power mean is calculated via a limit to obtain the geometric mean, namely $\prod_{i=1}^n x_i^{\alpha_i}$. It is well known that, as a function of p, $\left(\sum_{i=1}^n \alpha_i x_i^p\right)^{\frac{1}{p}}$ is an increasing function. Thus, when $p \leqslant 1$, we have $\left(\sum_{i=1}^n \alpha_i x_i^p\right)^{\frac{1}{p}} \leqslant \sum_{i=1}^n \alpha_i x_i$. The following is a refinement for this celebrated result.

COROLLARY 4.1. *Let n be a positive integer and $p \in (-\infty, 1)$. For $i = 1, 2, \ldots, n$, let $x_i^{(1)} > 0$, $\{\alpha_1^{(1)}, \ldots, \alpha_n^{(1)}\} \subset [0,1]$ be such that $\sum_{i=1}^n \alpha_i^{(1)} = 1$. Then for all real number $\lambda \geqslant 1$. We have*

$$\left(\sum_{i=1}^n \alpha_i x_i^p\right)^{\frac{\lambda}{p}} + \sum_{k=1}^3 \left(n\alpha_{min}^{(k)}\right)^{\lambda} \left(\left(\sum_{i=1}^n \frac{1}{n} x_i^{(k)}\right)^{\lambda} - \left(\sum_{i=1}^n \frac{1}{n} x_i^{(k)p}\right)^{\frac{\lambda}{p}}\right) \leqslant \left(\sum_{i=1}^n \alpha_i x_i\right)^{\lambda}. \tag{4.1}$$

Alternatively, letting $p = -1$ in Corollary 4.1, we derive the following bounds for the difference between the arithmetic and harmonic means.

COROLLARY 4.2. *Let n be a positive integer. For $i = 1, 2, \ldots, n$, let $x_i^{(1)} > 0$, $\{\alpha_1^{(1)}, \ldots, \alpha_n^{(1)}\} \subset [0,1]$ be such that $\sum_{i=1}^n \alpha_i^{(1)} = 1$. Then for all real number $\lambda \geqslant 1$. We have*

$$\left(\sum_{i=1}^n \alpha_i x_i^{-1}\right)^{-\lambda} + \sum_{k=1}^3 \left(n\alpha_{min}^{(k)}\right)^{\lambda} \left(\left(\sum_{i=1}^n \frac{1}{n} x_i^{(k)}\right)^{\lambda} - \left(\sum_{i=1}^n \frac{1}{n} x_i^{(k)-1}\right)^{-\lambda}\right) \leqslant \left(\sum_{i=1}^n \alpha_i x_i\right)^{\lambda}. \tag{4.2}$$

If we let $p \longrightarrow 0$ in Corollary 4.1, we obtain the following bounds for the difference between the arithmetic and geometric means.

COROLLARY 4.3. *Let n be a positive integer. For $i = 1, 2, \ldots, n$, let $x_i^{(1)} > 0$, $\{\alpha_1^{(1)}, \ldots, \alpha_n^{(1)}\} \subset [0,1]$ be such that $\sum_{i=1}^n \alpha_i^{(1)} = 1$. Then for all real number $\lambda \geqslant 1$. We have*

$$\left(\prod_{i=1}^n x_i^{\alpha_i}\right)^{\lambda} + \sum_{k=1}^3 \left(n\alpha_{min}^{(k)}\right)^{\lambda} \left(\left(\sum_{i=1}^n \frac{1}{n} x_i^{(k)}\right)^{\lambda} - \left(\prod_{i=1}^n x_i^{(k)\frac{1}{n}}\right)^{\lambda}\right) \leqslant \left(\sum_{i=1}^n \alpha_i x_i\right)^{\lambda}. \tag{4.3}$$

REMARK 4.4. The Corollary 4.3 present three refining terms of the main result of [3].

4.2. The special case $n=2$

For the rest of the paper, the following notations will be adopted. For $0 \leqslant \alpha \leqslant 1$ and $j \in \mathbb{N}$, let

$$\begin{cases} m_j(\alpha) = \left[2^{j-1}\alpha\right], \ d_j(\alpha) = \left[2^j\alpha\right] \text{ and} \\ A_j(\alpha) = (-1)^{d_j(\alpha)}2^{j-1}\alpha + (-1)^{d_j(\alpha)+1}\left[\frac{d_j(\alpha)+1}{2}\right], \end{cases}$$

where [ ] is the greatest integer function. Moreover, if $f:[x,y]\to\mathbb{R}$ is any function, define

$$\begin{aligned}\Delta_{(j,p,h)}f(\alpha;x,y) = h\left(\frac{1}{2}\right)&\left[f\left(\left(1-\frac{m_j(\alpha)}{2^{j-1}}\right)x+\frac{m_j(\alpha)}{2^{j-1}}y\right)\right.\\ &\left.+f\left(\left(1-\frac{m_j(\alpha)+1}{2^{j-1}}\right)x+\frac{m_j(\alpha)+1}{2^{j-1}}y\right)\right]\\ &-f\left[\left(\left(1-\frac{2m_j(\alpha)+1}{2^j}\right)x^p+\frac{2m_j(\alpha)+1}{2^j}y^p\right)^{\frac{1}{p}}\right],\end{aligned}$$

and

$$\begin{aligned}\widehat{\Delta}_{(j,p,h,t)}f(\alpha;x,y,\lambda) = &\left[f^{\lambda h\left(\frac{1}{2}\right)}\left(\left(1-\frac{m_j(\alpha)}{2^{j-1}}\right)x+\frac{m_j(\alpha)}{2^{j-1}}y\right)\right.\\ &\left.\times f^{\lambda h\left(\frac{1}{2}\right)}\left(\left(1-\frac{m_j(\alpha)+1}{2^{j-1}}\right)x+\frac{m_j(\alpha)+1}{2^{j-1}}y\right)\right]^t\\ &-f^{\lambda t}\left[\left(\left(1-\frac{2m_j(\alpha)+1}{2^j}\right)x^p+\frac{2m_j(\alpha)+1}{2^j}y^p\right)^{\frac{1}{p}}\right],\end{aligned}$$

where $0 \leqslant \alpha \leqslant 1$ and $p \in \mathbb{R}^*$.

Applying our refinement, pesented in Theorem 3.1 and Corollary 3.7 with $n=2$, implies the following two results, similar to our recent refinements obtained in [12, 13].

THEOREM 4.5. *Let h be a non-negative super-multiplicative and super-additive defined on $[0,\infty)$ and $f:[x,y]\to\mathbb{R}$ be (p,h)-convex. Then, for each $N\in\mathbb{N}$ and $0\leqslant\alpha\leqslant\beta\leqslant 1$, we have*

$$\begin{aligned}&f\left(\left[\alpha x^p+(1-\alpha)y^p\right]^{\frac{1}{p}}\right)\\ &+h\left(\frac{\alpha}{\beta}\right)\left(h(\beta)f(x)+h(1-\beta)f(y)-f\left(\left[\beta x^p+(1-\beta)y^p\right]^{\frac{1}{p}}\right)\right)\\ &+\sum_{j=1}^{N}2h\left(A_j\left(\frac{\alpha}{\beta}\right)\right)\Delta_{(j,p,h)}f\left(\alpha;x,\left[\beta x^p+(1-\beta)y^p\right]^{\frac{1}{p}}\right)\\ \leqslant\ &h(\alpha)f(x)+h(1-\alpha)f(y).\end{aligned}$$

THEOREM 4.6. *Let h be a non-negative super-multiplicative and super-additive defined on $[0,\infty)$ and $f:[x,y]\to\mathbb{R}$ be (p,h)-log-convex. Then, for $0\leqslant\alpha\leqslant\beta\leqslant 1$, we have*

$$\begin{aligned}
&f^{\lambda}\left(\left[\alpha x^p+(1-\alpha)y^p\right]^{\frac{1}{p}}\right)\\
&+\left(\left(f^{\lambda h(\beta)}(x)f^{\lambda h(1-\beta)}(y)\right)^{h\left(\frac{\alpha}{\beta}\right)}-f^{\lambda h\left(\frac{\alpha}{\beta}\right)}\left(\left[\beta x^p+(1-\beta)y^p\right]^{\frac{1}{p}}\right)\right)\\
&+\sum_{j=1}^{3}\widehat{\Delta}_{\left(j,p,h,\lambda,2h\left(A_j\left(\frac{\alpha}{\beta}\right)\right)\right)}f\left(\alpha;x,\left[\beta x^p+(1-\beta)y^p\right]^{\frac{1}{p}}\right)\\
\leqslant\ & f^{\lambda h(\alpha)}(x)f^{\lambda h(1-\alpha)}(y).
\end{aligned}$$

4.3. New inequalities for the p-norms of τ-measurable operators

Assume that $\mathfrak{A}\subset\mathfrak{L}(\mathfrak{H})$ is a weakly closed *-algebra containing the identity operator I, namely, it is a finite von Neumann algebra. A trace τ on $\mathfrak{A}$ is a map τ from $\mathfrak{A}^+=\{T\in\mathfrak{A}:T\geqslant 0\}$ to $[0,+\infty)$ that has additive, positively homogeneous and unitarily invariant properties, that is, $\tau(T)=\tau(U^*TU)$ for all $T\in\mathfrak{A}^+$ and unitary $U\in\mathfrak{A}$.

The symbol $L_p(\mathfrak{A},\tau)$, $0<p<+\infty$, denotes the set of all linear operators T associated with $\mathfrak{A}$ measurable in τ such that

$$\mathscr{N}_p(T)=\tau(|T|^p)^{\frac{1}{p}}<+\infty.$$

Clearly, $L_p(\mathfrak{A},\tau)$ with $1\leqslant p<+\infty$ is a Banach space under the p-norm $\|\cdot\|_p$, see [24] for more details.

Hereafter, we always assume that τ is a trace on $\mathfrak{A}$ with normal, faithful, and finite properties. Following [7], the determinant of $T\in\mathfrak{A}$ is defined as

$$\mathscr{D}_\tau(T)=\begin{cases}\exp\tau(\log|T|) & \text{if } |T| \text{ is invertible},\\ \inf\limits_{\varepsilon>0}\mathscr{D}_\tau(|T|+\varepsilon I) & \text{otherwise.}\end{cases}$$

The following are several properties of the determinant for τ-measurable operators (see [5, 6]).

1. $\mathscr{D}_\tau(I)=1$ and $\mathscr{D}_\tau(TS)=\mathscr{D}_\tau(T)\mathscr{D}_\tau(S)$;
2. $\mathscr{D}_\tau(T)=\mathscr{D}_\tau(T^*)=\mathscr{D}_\tau(|T|)$;
3. $\mathscr{D}_\tau(|T|^\alpha)=\mathscr{D}_\tau(|T|)^\alpha$ for all $\alpha\in\mathbb{R}^+$;
4. $\mathscr{D}_\tau(T^{-1})=(\mathscr{D}_\tau(T))^{-1}$ when T is invertible in $\mathfrak{A}$;
5. $\mathscr{D}_\tau(T)\leqslant\mathscr{D}_\tau(S)$ when $0\leqslant T\leqslant S$;
6. $\lim\limits_{\varepsilon\to 0^+}\mathscr{D}_\tau(T+\varepsilon I)=\mathscr{D}_\tau(T)$ when $T\geqslant 0$.

The well-known Hölder's inequality for τ-measurable operators is expressed as follows.

THEOREM 4.7. ([26]) *Suppose that $T,S \in L_p(\mathfrak{A},\tau)$, $1 \leqslant p < +\infty$, are positive operators, where $Z \in \mathfrak{A}$, and $0 \leqslant \alpha \leqslant 1$. Then, we have*

$$\mathcal{N}_p(T^{1-\alpha}ZS^{\alpha}) \leqslant \mathcal{N}_p(TZ)^{1-\alpha}\mathcal{N}_p(ZS)^{\alpha}. \tag{4.4}$$

In particular,

$$\tau(T^{1-\alpha}S^{\alpha}) \leqslant \tau(T)^{1-\alpha}\tau(S)^{\alpha}.$$

It has been proven in [15] that for $T,S \in L_p(\mathfrak{A},\tau)$, $1 \leqslant p < +\infty$, are positive operators, where $Z \in \mathfrak{A}$, the function

$$f_1(t) = \mathcal{N}_p(T^{1-t}ZS^{t})$$

is log-convex on $[0,1]$, for any symmetric norm $\mathcal{N}_p$. In particular

$$\widehat{f_1}(t) = \tau(T^{1-t}ZS^{t})$$

is log-convex. By applying Corollary 3.7, with $h(x) = x$, $\lambda = 1$ and $p = 1$, to the function f_1 we get the following theorem which refines the corresponding Hölder-type inequality (4.4) for τ-measurable operators.

THEOREM 4.8. *Let $T,S \in \mathcal{M}^+$ and $Z \in \mathfrak{A}$. Then, for $0 \leqslant \alpha \leqslant \beta \leqslant 1$, we have*

$$\begin{aligned}
&\left(\mathcal{N}_p(TZ)^{1-\beta}\mathcal{N}_p(ZS)^{\beta}\right)^{\frac{\alpha}{\beta}} - \left(\mathcal{N}_p(T^{1-\beta}ZS^{\beta})\right)^{\frac{\alpha}{\beta}}\\
&+\sum_{j=1}^{3}\widehat{\Delta}_{\left(j,\lambda,2h\left(A_j\left(\frac{\alpha}{\beta}\right)\right)\right)} f_1\,(\alpha;1,\beta)\\
&\leqslant \mathcal{N}_p(TZ)^{1-\alpha}\mathcal{N}_p(ZS)^{\alpha} - \mathcal{N}_p(T^{1-\alpha}ZS^{\alpha}).
\end{aligned}$$

In particular, if $\mathcal{M}$ is a finite von Neumann algebra, then

$$\begin{aligned}
&\left(\tau(T)^{1-\beta}\tau(S)^{\beta}\right)^{\frac{\alpha}{\beta}} - \tau^{\frac{\alpha}{\beta}}\left(T^{1-\beta}S^{\beta}\right)\\
&+\sum_{j=1}^{3}\widehat{\Delta}_{\left(j,\lambda,2h\left(A_j\left(\frac{\alpha}{\beta}\right)\right)\right)} \widehat{f_1}\,(\alpha;1,\beta)\\
&\leqslant \tau(T)^{1-\alpha}\tau(S)^{\alpha} - \tau\left(T^{1-\alpha}S^{\alpha}\right).
\end{aligned}$$

Furthermore, it has been established in [15] that for $T,S \in L_p(\mathfrak{A},\tau)$, $1 \leqslant p < +\infty$, are positive operators, where $Z \in \mathfrak{A}$, the function

$$f_2(t) = \mathcal{N}_p(T^{t}ZS^{t})$$

is log-convex on $[0,1]$ for any symmetric norm $\mathcal{N}_p$. Applying again Corollary 3.7, with $h(x) = x$ and $p = 1$, to the function f_2, we obtain the following theorem.

THEOREM 4.9. *Let $T,S \in \mathscr{M}^+$ and $Z \in \mathfrak{A}$. Then, for $0 \leqslant \alpha \leqslant \beta \leqslant 1$, we have*

$$\begin{aligned}&\left(\mathscr{N}_p(X)^{\beta}\mathscr{N}_p(TZS)\right)^{\frac{\alpha}{\beta}} - \left(\mathscr{N}_p(T^{\beta}ZS^{\beta})\right)^{\frac{\alpha}{\beta}}\\&+\sum_{j=1}^{3}\widehat{\Delta}_{\left(j,\lambda,2h\left(A_j\left(\frac{\alpha}{\beta}\right)\right)\right)}f_2\left(\alpha;1,\beta\right)\\&\leqslant \mathscr{N}_p(T^tZS^t)^{1-\alpha}\mathscr{N}_p(TZS)^{\alpha} - \mathscr{N}_p(T^{\alpha}ZS^{\alpha}).\end{aligned}$$

In particular, if $X = I$, we get

$$\begin{aligned}&\left(\mathscr{N}_p(TS)\right)^{\frac{\alpha}{\beta}} - \left(\mathscr{N}_p(T^{\beta}ZS^{\beta})\right)^{\frac{\alpha}{\beta}}\\&+\sum_{j=1}^{3}\widehat{\Delta}_{\left(j,\lambda,2h\left(A_j\left(\frac{\alpha}{\beta}\right)\right)\right)}f_2\left(\alpha;1,\beta\right)\\&\leqslant \mathscr{N}_p(T^tS^t)^{1-\alpha}\mathscr{N}_p(TS)^{\alpha} - \mathscr{N}_p(T^{\alpha}S^{\alpha}).\end{aligned}$$

It has been shown in [15] that for $T,S \in L_p(\mathfrak{A},\tau)$, $1 \leqslant p < +\infty$, are positive operators, where $Z \in \mathfrak{A}$, the function

$$f_3(t) = \mathscr{N}_p(T^{1-t}ZS^t)\mathscr{N}_p(T^tZS^{1-t})$$

is log-convex on $[0,1]$ for any symmetric norm $\mathscr{N}_p$. Therefore, applying Corollary 3.7 with $h(x) = x$ and $p = 1$, we obtain the following theorem.

THEOREM 4.10. *Let $T,S \in \mathscr{M}^+$ and $Z \in \mathfrak{A}$. Then, for $0 \leqslant \alpha \leqslant \beta \leqslant 1$, we have*

$$\begin{aligned}&\left(\mathscr{N}_p(TZ)\mathscr{N}_p(ZS)\right)^{\frac{\alpha}{\beta}} - \left(\mathscr{N}_p(T^{1-\beta}ZS^{\beta})\mathscr{N}_p(T^{\beta}ZS^{1-\beta})\right)^{\frac{\alpha}{\beta}}\\&+\sum_{j=1}^{3}\widehat{\Delta}_{\left(j,\lambda,2h\left(A_j\left(\frac{\alpha}{\beta}\right)\right)\right)}f_3\left(\alpha;1,\beta\right)\\&\leqslant \mathscr{N}_p(TZ)\mathscr{N}_p(ZS) - \mathscr{N}_p(T^{1-\alpha}ZS^{\alpha})\mathscr{N}_p(T^{\alpha}ZS^{1-\alpha}).\end{aligned}$$

REMARK 4.11. It is important to note that the results obtained in this section provide new refinements of the findings presented in the last section of [11].

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