Journal of Mathematical Inequalities
Volume 18, Number 3 (2024), 1159–1170 doi:10.7153/jmi-2024-18-65

SOLVABILITY OF A NONLINEAR DIFFERENCE EQUATION OF THE FIFTEENTH ORDER

STEVO STEVIĆ*, BRATISLAV IRIČANIN AND WITOLD KOSMALA

(*Communicated by T. Burić*)

Abstract. We find some closed-form formulas for the general solution to the difference equation

$$x_{n+1}=\frac{x_{n-2}x_{n-8}x_{n-14}}{x_{n-5}x_{n-11}(a+bx_{n-2}x_{n-8}x_{n-14})},\quad n\in\mathbb{N}_0,$$

where $a,b\in\mathbb{C}$, $a^2+b^2\neq 0$, and $x_{-j}\in\mathbb{C}\setminus\{0\}$, $j=\overline{0,14}$, explaining the formulas for the four special cases of the difference equation which have appeared quite recently in the literature. We also give several comments on the claims on the behaviour of solutions to the four difference equations given therein, as well as some counterexamples.

1. Introduction

Let $\mathbb{N}$ be the set of all positive natural numbers, $\mathbb{Z}$ be the set of integers, $\mathbb{N}_l=\{n\in\mathbb{Z}: n\geqslant l\}$, where $l\in\mathbb{Z}$, $\mathbb{R}$ be the set of reals, and $\mathbb{C}$ be the set of complex numbers. If $s,t\in\mathbb{Z}$ satisfy the condition $s\leqslant t$, then $j=\overline{s,t}$, is a notation which we use for the following frequently used expression: $s\leqslant j\leqslant t$, for $j\in\mathbb{Z}$. If $k\in\mathbb{Z}$, we regard

$$\prod_{j=k}^{k-1} d_j=1, \tag{1}$$

where d_j is a sequence of numbers (real or complex) defined on a subset J of the set $\mathbb{Z}$.

The first nontrivial results on solvability of difference equations were given in [4] and [7], and the investigations, mostly on the linear difference equations and systems of difference equations, was continued, among others, in [8, 17, 18, 19]. For some results on nonlinear difference equations and systems see the old literature [5, 10, 16, 20, 21], as well as the later ones [9, 12, 22, 24, 33, 51] (see also the references cited therein). Solvable difference equations and systems of difference equations occur in many areas of mathematics and science. Some of the solvable nonlinear difference equations appeared in numerical mathematics [6], some of them in combinatorics [14,

Mathematics subject classification (2020): Primary 39A20, 39A45.
Keywords and phrases: Nonlinear difference equation, solvable difference equation, closed-form formula.

* Corresponding author.

Paper JMI-18-65

33], and some in other areas of science and mathematics [1, 14, 23]. Finding invariants and applying them, is a closely related topic see, for instance, [26, 27, 28, 29, 34, 35] (some of the invariants have been essentially studied by Euler in [8], yet).

Use of computers and some packages for symbolic computations, among other things, renewed an interest in the research domain. However, there have been a lot of problems with many recent papers on solvability of difference equations and systems. In some of our recent papers we addressed to them, see, for instance, [43] and [48], where number of comments and theoretical explanations are given by us. Generally speaking, the solvability of linear difference equations and systems is what decides the solvability of nonlinear ones (see, for example, [1, 2, 11, 13, 32, 39, 40, 41, 43, 44, 46, 47, 48]). Solvable difference equations also occur in various applications (see, for example, [7, 12, 33, 51]) and in dealing with some difference inequalities [3, 37, 38].

Similar situation is with nonlinear systems of difference equations. The investigation of the long term behaviour of their solutions has attracted some attention since the mid of the '90ies (see, for instance, [25, 26, 27, 30, 31, 34, 35] and the related references therein). Among other things, we were motivated by these investigations and started studying the long term behavior of their solutions, as well as their solvability (see, for example, [40, 41, 42, 43, 45, 48, 49, 50] and some of the references therein).

The following four difference equations

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{\pm x_{n-5}x_{n-11} \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n \in \mathbb{N}_0, \tag{2}$$

have been recently considered in [36], where some formulas for their solutions are given. It is said therein that the formulas can be proved by the method of mathematical induction and of given 72 formulas only two are proved by this classical method, whereas for the other ones it is said that they are proved similarly. They did not give any information on how the formulas were obtained, nor they presented any theory which explains any of these formulas.

In this paper we consider the following generalization of the difference equations in (2):

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{x_{n-5}x_{n-11}(a + bx_{n-2}x_{n-8}x_{n-14})}, \quad n \in \mathbb{N}_0, \tag{3}$$

where the parameter $a, b \in \mathbb{C}$ satisfy the condition $a^2 + b^2 \neq 0$, and $x_{-j} \in \mathbb{C} \setminus \{0\}$, $j = \overline{0, 14}$.

We present a method how the equation can be solved, considerably extending and theoretically explaining the formulas in [36]. We employ ideas related to some in our previous investigations. We also give several comments on the claims on the behaviour of solutions to the four difference equations given in [36].

2. Main result

First, we quote an auxiliary result.

LEMMA 1. *Consider the difference equation*

$$x_n = ax_{n-1} + b, \quad n \in \mathbb{N}_0, \tag{4}$$

where $a, b, x_{-1} \in \mathbb{C}$. *Then, the following statements are true.*

(a) If $a \neq 1$, *then the general solution to equation* (4) *is*

$$x_n = a^{n+1}x_{-1} + b\frac{a^{n+1}-1}{a-1}, \tag{5}$$

for $n \in \mathbb{N}_{-1}$.

(b) If $a = 1$, *then the general solution to equation* (4) *is*

$$x_n = x_{-1} + b(n+1), \tag{6}$$

for $n \in \mathbb{N}_{-1}$.

Formula (5) was proved first by Lagrange in [17]. It was later generalized by him for the case of the linear difference equation with nonconstant coefficients (see [18]). Another method was later presented by Laplace in [19] (see, also, [10, 12, 14, 22]; in [23] are given three standard methods for solving the generalized difference equation, which correspond to the three standard methods for solving the nonhomogeneous linear differential equation of first order).

The following theorem is the main result in this paper. The theorem shows the solvability of equation (3). In the proof of the theorem is described a procedure for finding the general solution to the equation. We use several ideas and methods, which are related to some of the ones presented, for example, in [39, 40, 41, 42, 43, 44, 45, 46, 47].

THEOREM 1. *Consider equation* (3), *where* $a, b \in \mathbb{C}$, $a^2 + b^2 \neq 0$. *Then, the equation is solvable. Moreover, if* $a \neq 1$, *then the general solution to equation* (3) *is given by*

$$x_{18k+3l+j+1} = x_{3l+j-17}\prod_{s=0}^{k}\frac{a^{6s+l-1}(a-1)+x_{-2+j}x_{-8+j}x_{-14+j}b(a^{6s+l-1}-1)}{a^{6s+l+1}(a-1)+x_{-2+j}x_{-8+j}x_{-14+j}b(a^{6s+l+1}-1)}, \tag{7}$$

for $k \geqslant -1$, $l = \overline{1,6}$ *and* $j = \overline{0,2}$, *whereas if* $a = 1$, *the general solution to the equation is given by*

$$x_{18k+3l+j+1} = x_{3l+j-17}\prod_{s=0}^{k}\frac{1+x_{-2+j}x_{-8+j}x_{-14+j}b(6s+l-1)}{1+x_{-2+j}x_{-8+j}x_{-14+j}b(6s+l+1)}, \tag{8}$$

for $k \geqslant -1$, $l = \overline{1,6}$ *and* $j = \overline{0,2}$.

Proof. First, note that if

$$x_{-j} = 0, \quad \text{for some} \quad 12 \leqslant j \leqslant 14,$$

then we have

$$x_{-j+15} = 0,$$

from which it follows that x_{-j+21} is not defined.

Further, if

$$x_{-j} = 0, \quad \text{for some} \quad 6 \leqslant j \leqslant 11,$$

then x_{-j+12} is not defined.

Finally, if

$$x_{-j} = 0, \quad \text{for some} \quad 0 \leqslant j \leqslant 5,$$

then x_{-j+6} is not defined. This analysis shows that in any of these cases are obtained solutions to equation (3) which are not defined on the equation domain, that is, on $\mathbb{N}_{-14}$.

Hence, from now on we assume that

$$x_{-j} \in \mathbb{C} \setminus \{0\}, \quad j = \overline{0,14}. \tag{9}$$

We also assume

$$a + bx_{n-2}x_{n-8}x_{n-14} \neq 0,$$

for every $n \in \mathbb{N}_0$, in order to avoid dealing with undefined solutions to the difference equation.

Further, note that equation (3) is with interlacing indices (for some detailed explanations and examples of such difference equations and systems see [43] and [48]).

Because of this, we write equation (3) in the form

$$x_{3m+j+1} = \frac{x_{3m+j-2}x_{3m+j-8}x_{3m+j-14}}{x_{3m+j-5}x_{3m+j-11}(a + bx_{3m+j-2}x_{3m+j-8}x_{3m+j-14})},$$

where $m \in \mathbb{N}_0$ and $j = \overline{0,2}$, that is,

$$x_{3m+j+1} = \frac{x_{3(m-1)+j+1}x_{3(m-3)+j+1}x_{3(m-5)+j+1}}{x_{3(m-2)+j+1}x_{3(m-4)+j+1}(a + bx_{3(m-1)+j+1}x_{3(m-3)+j+1}x_{3(m-5)+j+1})}, \tag{10}$$

where $m \in \mathbb{N}_0$ and $j = \overline{0,2}$.

Let

$$x_m^{(j)} = x_{3m+j+1}, \tag{11}$$

for $m \geqslant -5$ and $j = \overline{0,2}$.

Then, from (10) we have

$$x_m^{(j)} = \frac{x_{m-1}^{(j)}x_{m-3}^{(j)}x_{m-5}^{(j)}}{x_{m-2}^{(j)}x_{m-4}^{(j)}(a + bx_{m-1}^{(j)}x_{m-3}^{(j)}x_{m-5}^{(j)})}, \tag{12}$$

where $m \in \mathbb{N}_0$ and $j = \overline{0,2}$.

This means that the three sequences defined in (11) are solutions to the difference equation

$$z_m = \frac{z_{m-1}z_{m-3}z_{m-5}}{z_{m-2}z_{m-4}(a + bz_{m-1}z_{m-3}z_{m-5})}, \tag{13}$$

where $m \in \mathbb{N}_0$.

By using the change of variables

$$y_m^{(j)} = \frac{1}{x_m^{(j)}x_{m-2}^{(j)}x_{m-4}^{(j)}}, \tag{14}$$

where $m \geqslant -1$ and $j = \overline{0,2}$, which is admissible due to the assumption (9), the equations in (12) are transformed to the equation

$$y_m^{(j)} = ay_{m-1}^{(j)} + b, \tag{15}$$

where $m \in \mathbb{N}_0$ and $j = \overline{0,2}$.

This means that the sequences $(y_m^{(j)})_{m\geqslant -1}$, $j = \overline{0,2}$, are three solutions to the difference equation

$$u_m = au_{m-1} + b, \tag{16}$$

where $m \in \mathbb{N}_0$.

By employing Lemma 1, we see that the general solution to equation (15) in the case $a \neq 1$ is

$$y_m^{(j)} = a^{m+1}y_{-1}^{(j)} + b\frac{a^{m+1}-1}{a-1}, \tag{17}$$

for $m \geqslant -1$, $j = \overline{0,2}$, and that the general solution to equation (15) in the case $a = 1$ is

$$y_m^{(j)} = y_{-1}^{(j)} + b(m+1), \tag{18}$$

for $m \geqslant -1$, $j = \overline{0,2}$.

From (14) we have

$$x_m^{(j)} = \frac{1}{y_m^{(j)}x_{m-2}^{(j)}x_{m-4}^{(j)}},$$

for $m \geqslant -1$ and $j = \overline{0,2}$, from which together with (9) it follows that

$$x_m^{(j)} = \frac{x_{m-6}^{(j)}}{y_m^{(j)}x_{m-2}^{(j)}x_{m-4}^{(j)}x_{m-6}^{(j)}}, \tag{19}$$

for $m \in \mathbb{N}$, $j = \overline{0,2}$.

Using (14) where m is replaced by $m-2$ in (19), we get

$$x_m^{(j)} = \frac{y_{m-2}^{(j)}}{y_m^{(j)}} x_{m-6}^{(j)}, \tag{20}$$

for $m \in \mathbb{N}$, $j = \overline{0,2}$.

We can write equation (20) in the form

$$x_{6k+l}^{(j)} = \frac{y_{6k+l-2}^{(j)}}{y_{6k+l}^{(j)}} x_{6(k-1)+l}^{(j)}, \tag{21}$$

for $k \in \mathbb{N}_0$, $l = \overline{1,6}$, $j = \overline{0,2}$.

Equation (21) is a simple product-type difference equation, which is easily solved and is obtained

$$x_{6k+l}^{(j)} = x_{-6+l}^{(j)} \prod_{s=0}^{k} \frac{y_{6s+l-2}^{(j)}}{y_{6s+l}^{(j)}},$$

for $k \in \mathbb{N}_0$, $l = \overline{1,6}$, $j = \overline{0,2}$, from which together with the relations in (11) it follows that

$$x_{18k+3l+j+1} = x_{3l+j-17} \prod_{s=0}^{k} \frac{y_{6s+l-2}^{(j)}}{y_{6s+l}^{(j)}}, \tag{22}$$

$k \in \mathbb{N}_0$, $l = \overline{1,6}$, $j = \overline{0,2}$.

If $a \neq 1$, then from (17) and (22) we get

$$x_{18k+3l+j+1} = x_{3l+j-17} \prod_{s=0}^{k} \frac{a^{6s+l-1} y_{-1}^{(j)} + b\frac{a^{6s+l-1}-1}{a-1}}{a^{6s+l+1} y_{-1}^{(j)} + b\frac{a^{6s+l+1}-1}{a-1}},$$

for $k \in \mathbb{N}_0$, $l = \overline{1,6}$, $j = \overline{0,2}$, from which along with (14) with $m = -1$, we have

$$x_{18k+3l+j+1} = x_{3l+j-17} \prod_{s=0}^{k} \frac{a^{6s+l-1} + bx_{-1}^{(j)} x_{-3}^{(j)} x_{-5}^{(j)} \frac{a^{6s+l-1}-1}{a-1}}{a^{6s+l+1} + bx_{-1}^{(j)} x_{-3}^{(j)} x_{-5}^{(j)} \frac{a^{6s+l+1}-1}{a-1}}, \tag{23}$$

for $k \in \mathbb{N}_0$, $l = \overline{1,6}$, $j = \overline{0,2}$, from which together with the relation (11) with $m = -1$ and the convention (1), is easily obtained formula (7) for $k \geqslant -1$, $l = \overline{1,6}$ and $j = \overline{0,2}$.

If $a = 1$, then from (18) and (22) we get

$$x_{18k+3l+j+1} = x_{3l+j-17} \prod_{s=0}^{k} \frac{y_{-1}^{(j)} + b(6s+l-1)}{y_{-1}^{(j)} + b(6s+l+1)}, \tag{24}$$

for $k \in \mathbb{N}_0$, $l = \overline{1,6}$, $j = \overline{0,2}$, from which together (14) with $m = -1$, we have

$$x_{18k+3l+j+1} = x_{3l+j-17} \prod_{s=0}^{k} \frac{1 + bx_{-1}^{(j)} x_{-3}^{(j)} x_{-5}^{(j)} (6s+l-1)}{1 + bx_{-1}^{(j)} x_{-3}^{(j)} x_{-5}^{(j)} (6s+l+1)}, \tag{25}$$

for $k \in \mathbb{N}_0$, $l = \overline{1,6}$, $j = \overline{0,2}$, from which together with the relation (11) with $m = -1$ and the convention (1), is easily obtained formula (8) for $k \geqslant -1$, $l = \overline{1,6}$ and $j = \overline{0,2}$. □

REMARK 1. Using the formulas (7) and (8) for the general solutions to equation (3), one can describe the well-defined solutions. Namely, a solution to equation (3) is well defined if none of the denominators in the formula (7) (or, respectively, in the formula (8)) vanishes.

REMARK 2. By employing the formulas in (7) and (8) we can obtain all the 72 formulas for the general solutions to the four equations in (2). The simple task is left to the reader as an exercise.

3. On the claims in [36] on the behaviour of solutions to (3)

In this section we give some comments on the claims given in [36] on the behaviour of solutions to equation (3).

3.1. The first equation in (2)

In Theorem 2.2 in [36] is formulated the following claim.

CLAIM 1. *The equation*

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{x_{n-5}x_{n-11} + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n \in \mathbb{N}_0, \tag{26}$$

has a unique equilibrium point $\bar{x}$ which is the number zero and this equilibrium is not locally asymptotically stable. Also $\bar{x}$ is non hyperbolic.

To prove the claim the authors of [36] try to find the equilibria of the equation (26) by solving the algebraic equation

$$\bar{x} = \frac{\bar{x}^3}{\bar{x}^2 + \bar{x}^5}. \tag{27}$$

From (27) they concluded that

$$\bar{x}^3 + \bar{x}^6 = \bar{x}^3, \tag{28}$$

and consequently that $\bar{x} = 0$.

However, the equation in (27) is not defined for $\bar{x} = 0$. This, in fact, shows that $\bar{x} = 0$ is not an equilibrium. Hence, their claim is not true. Then, they conducted some calculations which make the same mistake. The problem is that the algebraic equations (27) and (28) are not equivalent. Bearing in mind this fact, the second part of the above claim, that is, that $\bar{x}$ is non hyperbolic, even makes no sense.

3.2. The second equation in (2)

In Theorem 3.2 in [36] is formulated the following claim.

CLAIM 2. *The equation*

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{x_{n-5}x_{n-11} - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n \in \mathbb{N}_0, \tag{29}$$

has a unique equilibrium point $\bar{x} = 0$*, which is not locally asymptotically stable.*

However, the claim is also not true because $\bar{x} = 0$ is not an equilibrium of equation (29). Namely, $\bar{x}$ will be an equilibrium of equation (29) if it satisfies the algebraic equation

$$\bar{x} = \frac{\bar{x}^3}{\bar{x}^2 - \bar{x}^5},$$

which is not satisfied by $\bar{x} = 0$.

3.3. The third equation in (2)

In Theorem 4.2 in [36] is formulated the following claim.

CLAIM 3. *The equation*

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{-x_{n-5}x_{n-11} + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n \in \mathbb{N}_0, \tag{30}$$

has three equilibrium points which are $0, \ \pm\sqrt[3]{2}$*, and these equilibrium points are not locally asymptotically stable.*

It further says that the proof is the same as the proof of Theorem 2.2 in [36], because of which it is omitted. However, the claim is also not true because $\bar{x} = 0$ is not an equilibrium of equation (30), that is, it is not a solution to the algebraic equation

$$\bar{x} = \frac{\bar{x}^3}{-\bar{x}^2 + \bar{x}^5}. \tag{31}$$

Moreover, if $\bar{x} \neq 0$, then from (31) we have

$$\bar{x}^3(\bar{x}^3 - 2) = 0,$$

from which it follows that $\bar{x} = \sqrt[3]{2}$. This means that $-\sqrt[3]{2}$ is not an equilibrium of equation (30), which is another mistake in Claim 3.

Besides, the case of the equilibrium $\sqrt[3]{2}$ has not been considered in [36] at all.

3.4. The fourth equation in (2)

In Theorem 5.2 in [36] is formulated the following claim.

CLAIM 4. *The equation*

$$x_{n+1}=\frac{x_{n-2}x_{n-8}x_{n-14}}{-x_{n-5}x_{n-11}-x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}},\quad n\in\mathbb{N}_0, \tag{32}$$

has three equilibrium points which are $0,\ \pm\sqrt[3]{2}$*, and these equilibrium points are not locally asymptotically stable.*

It also says that the proof is the same as the proof of Theorem 2.2 in [36], because of which it is omitted. However, the claim is also not true because $\bar{x}=0$ is not an equilibrium of equation (32), since it does not satisfy the algebraic equation

$$\bar{x}=\frac{\bar{x}^3}{-\bar{x}^2-\bar{x}^5}. \tag{33}$$

Moreover, if $\bar{x}\neq 0$, then from (31) we have

$$\bar{x}^3(\bar{x}^3+2)=0,$$

from which it follows that $\bar{x}=-\sqrt[3]{2}$. This means that $\sqrt[3]{2}$ is not an equilibrium of equation (32), which is another mistake in Claim 4.

Besides, the case of equilibrium $-\sqrt[3]{2}$ has not been considered in [36] at all.

4. Conclusions

We present a method for solving equation (3), explaining some formulas for the four special cases of the difference equation which have appeared recently in [36]. To do this we employ several methods which can be useful in the investigations of similar difference equations. Our present study can motivate some further investigations of the theoretical and practical solvability of the related classes of difference equations.

Author Contributions. Stevo Stević proposed the investigation. All the authors contributed to the study. All authors have read and agreed to this version of the manuscript.

Funding. This work was partially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract number: 451-03-65/2024-03/200103. This research is also partially supported by the Science Fund of the Republic of Serbia, #GRANT No 7632, Project "Mathematical Methods in Image Processing under Uncertainty" – MaMIPU.

Use of AI tools declaration. The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of Interest. The authors declare no conflicts of interest.

REFERENCES

[1] D. ADAMOVIĆ, *Solution to problem 194*, Mat. Vesnik, **23** (1971), 236–242.

[2] I. BAJO AND E. LIZ, *Global behaviour of a second-order nonlinear difference equation*, J. Differ. Equat. Appl. **17** (10) (2011), 1471–1486.

[3] L. BERG AND S. STEVIĆ, *On the asymptotics of the difference equation* $y_n(1+y_{n-1}\cdots y_{n-k+1}) = y_{n-k}$, J. Difference Equ. Appl. **17** (4) (2011), 577–586.

[4] D. BERNOULLI, *Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur*, Commentarii Acad. Petropol. III, 1728 (1732), 85–100, (in Latin).

[5] G. BOOLE, *A Treatise on the Calculus of Finite Differences*, Third Edition, Macmillan and Co., London, 1880.

[6] B. P. DEMIDOVICH AND I. A. MARON, *Computational Mathematics*, Mir Publishers, Moscow, 1973.

[7] A. DE MOIVRE, *Miscellanea analytica de seriebus et quadraturis*, Londini, 1730, (in Latin).

[8] L. EULER, *Introductio in Analysin Infinitorum, Tomus Primus*, Lausannae, 1748, (in Latin).

[9] T. FORT, *Finite Differences and Difference Equations in the Real Domain*, Oxford Univ. Press, London, 1948.

[10] J. F. W. HERSCHEL, *Collection of Examples of the Applications of the Calculus of Finite Differences*, J. Smith, Cambridge, 1820.

[11] B. IRIČANIN, S. STEVIĆ, *On some rational difference equations*, Ars Combin. **92** (2009), 67–72.

[12] C. JORDAN, *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1965.

[13] G. KARAKOSTAS, *The forbidden set, solvability and stability of a circular system of complex Riccati type difference equations*, AIMS Mathematics **8** (11) (2023), 28033–28050.

[14] V. A. KRECHMAR, *A Problem Book in Algebra*, Mir Publishers, Moscow, 1974.

[15] S. F. LACROIX, *Traité des Differénces et des Séries*, J. B. M. Duprat, Paris, 1800, (in French).

[16] S. F. LACROIX, *An Elementary Treatise on the Differential and Integral Calculus, with an Appendix and Notes by J. Herschel*, J. Smith, Cambridge, 1816.

[17] J.-L. LAGRANGE, *Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes*, Miscellanea Taurinensia, t. I, (1759), 33–42 (Lagrange OEuvres, I, 23–36, 1867) (in French).

[18] J.-L. LAGRANGE, *OEuvres*, t. II, Gauthier-Villars, Paris, 1868, (in French).

[19] P. S. LAPLACE, *Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la théorie des hasards*, Mémoires de l' Académie Royale des Sciences de Paris 1773, t. VII, (1776) (Laplace OEuvres, VIII, 69–197, 1891), (in French).

[20] P. S. LAPLACE, *Sur le, calcul des fonctions génératrices*, Journal de l'École Polytechnique, Tome VIII, Cah. 15, (1810), 229–265, (in French).

[21] A. A. MARKOFF, *Differenzenrechnung*, Teubner, Leipzig, 1896, (in German).

[22] L. M. MILNE-THOMSON, *The Calculus of Finite Differences*, MacMillan and Co., London, 1933.

[23] D. S. MITRINOVIĆ AND J. D. KEČKIĆ, *Metodi Izračunavanja Konačnih Zbirova*, Naučna Knjiga, Beograd, 1984, (in Serbian).

[24] N. E. NÖRLUND, *Vorlesungen über Differenzenrechnung*, Berlin, Springer, 1924, (in German).

[25] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *On a system of two nonlinear difference equations*, J. Math. Anal. Appl. **219** (2) (1998), 415–426.

[26] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *On the behavior of the solutions of a system of two nonlinear difference equations*, Comm. Appl. Nonlinear Anal. **5** (2) (1998), 47–59.

[27] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *Invariants for systems of two nonlinear difference equations*, Differ. Equ. Dyn. Syst. **7** (1999), 181–196.

[28] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *Invariants and oscillation for systems of two nonlinear difference equations*, Nonlinear Anal. Theory Methods Appl. **46** (2001), 967–978.

[29] G. PAPASCHINOPOULOS, C. J. SCHINAS, AND G. STEFANIDOU, *On a k-order system of Lyness-type difference equations*, Adv. Difference Equ. Vol. 2007, Article ID 31272, (2007), 13 pages.

[30] G. PAPASCHINOPOULOS, C. J. SCHINAS AND G. STEFANIDOU, *Two modifications of the Beverton-Holt equation*, Int. J. Difference Equ. **4** (1) (2009), 115–136.

[31] G. PAPASCHINOPOULOS AND G. STEFANIDOU, *Trichotomy of a system of two difference equations*, J. Math. Anal. Appl. **289** (2004), 216–230.

[32] G. PAPASCHINOPOULOS AND G. STEFANIDOU, *Asymptotic behavior of the solutions of a class of rational difference equations*, Inter. J. Difference Equations **5** (2) (2010), 233–249.

[33] J. RIORDAN, *Combinatorial Identities*, John Wiley & Sons Inc., New York-London-Sydney, 1968.

[34] C. SCHINAS, *Invariants for difference equations and systems of difference equations of rational form*, J. Math. Anal. Appl. **216** (1997), 164–179.

[35] C. SCHINAS, *Invariants for some difference equations*, J. Math. Anal. Appl. **212** (1997), 281–291.

[36] D. SIMSEK, B. OGUL AND F. ABDULLAYEV, *Dynamical behavior of solution of fifteenth-order rational difference equation*, Filomat **38** (3) (2024), 997–1008.

[37] S. STEVIĆ, *On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ II*, Dyn. Contin. Discrete Impuls. Syst. **10a** (6) (2003), 911–916.

[38] S. STEVIĆ, *Boundedness character of a class of difference equations*, Nonlinear Anal. TMA **70** (2009), 839–848.

[39] S. STEVIĆ, *On the difference equation $x_n = x_{n-k}/(b + cx_{n-1}\cdots x_{n-k})$*, Appl. Math. Comput. **218** (2012), 6291–6296.

[40] S. STEVIĆ, *On the system $x_{n+1} = y_n x_{n-k}/(y_{n-k+1}(a_n + b_n y_n x_{n-k}))$, $y_{n+1} = x_n y_{n-k}/(x_{n-k+1}(c_n + d_n x_n y_{n-k}))$*, Appl. Math. Comput. **219** (2013), 4526–4534.

[41] S. STEVIĆ, *On the system of difference equations $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$*, Appl. Math. Comput. **219** (2013), 4755–4764.

[42] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDA, *On a third-order system of difference equations with variable coefficients*, Abstr. Appl. Anal. Vol. 2012, Article ID 508523, (2012), 22 pages.

[43] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDA, *On some solvable difference equations and systems of difference equations*, Abstr. Appl. Anal. Vol. 2012, Article ID 541761, (2012), 11 pages.

[44] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDA, *On the difference equation $x_n = a_n x_{n-k}/(b_n + c_n x_{n-1}\cdots x_{n-k})$*, Abstr. Appl. Anal. Vol. 2012, Article ID 409237, (2012), 19 pages.

[45] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDA, *On a solvable system of rational difference equations*, J. Difference Equ. Appl. **20** (5–6) (2014), 811–825.

[46] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDA, *Solvability of nonlinear difference equations of fourth order*, Electron. J. Differential Equations Vol. 2014, Article No. 264, (2014), 14 pages.

[47] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDA, *On a fifth-order difference equation*, J. Comput. Anal. Appl. **20** (7) (2016), 1214–1227.

[48] S. STEVIĆ, B. IRIČANIN, W. KOSMALA AND Z. ŠMARDA, *Note on the bilinear difference equation with a delay*, Math. Methods Appl. Sci. **41** (2018), 9349–9360.

[49] S. STEVIĆ, B. IRIČANIN AND Z. ŠMARDA, *On a close to symmetric system of difference equations of second order*, Adv. Difference Equ. Vol. 2015, Article No. 264, (2015), 17 pages.

[50] S. STEVIĆ, B. IRIČANIN AND Z. ŠMARDA, *On a product-type system of difference equations of second order solvable in closed form*, J. Inequal. Appl. Vol. 2015, Article No. 327, (2015), 15 pages.

[51] N. N. VOROBIEV, *Fibonacci Numbers*, Birkhäuser, Basel, 2002.

(Received June 22, 2024)

Stevo Stević
Mathematical Institute of the Serbian Academy of Sciences
Knez Mihailova 36/III, 11000 Beograd, Serbia
and
Department of Medical Research, China Medical University Hospital
China Medical University
Taichung 40402, Taiwan, Republic of China
e-mail: sscite1@gmail.com

Bratislav Iričanin
Faculty of Electrical Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73, 11000 Beograd, Serbia
and
Faculty of Mechanical and Civil Engineering in Kraljevo
University of Kragujevac
36000 Kraljevo, Serbia
e-mail: iricanin@etf.rs

Witold Kosmala
Deptartment of Mathematical Sciences
Appalachian State University
Boone, NC 28608, USA
e-mail: kosmalawa@appstate.edu