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COMPARISON INEQUALITIES BETWEEN COMPLEX POLYNOMIALS FOR THE MAXIMUM MODULUS OF THEIR POLAR DERIVATIVE

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Abstract. In this paper, we prove certain comparison inequalities between two complex polynomials for the maximum modulus of their polar derivative, when the zeros of one of the polynomial are restricted. A variety of interesting results follow as special cases from our results.

1. Introduction

By $\mathbb{P}_n$, we denote the set of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree n and $P'(z)$ is the derivative of $P(z)$. For brevity, we introduce the following notations:

$$\psi := \psi_k(R,r,\beta,\gamma) = \gamma\left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\} - \beta$$

$$\phi := \phi_k(R,r,\beta,\gamma) = \gamma\left\{\left(\frac{Rk+1}{rk+1}\right)^n - |\beta|\right\} - \beta,$$

where $\beta,\gamma \in \mathbb{C}$ are such that $|\beta| \leqslant 1$ and $|\gamma| \leqslant 1$. Note that $\left(\frac{R+k}{r+k}\right)^n - |\beta| > 0$ and $\left(\frac{Rk+1}{rk+1}\right)^n - |\beta| > 0$ for $R > r$.

The study of extremal problems of functions and the results where some approaches to obtaining polynomial inequalities for various norms and with various constraints on using different methods of the geometric function theory is a classical topic in analysis. A classical result due to Bernstein [3] is that, for two polynomials $f(z)$ and $F(z)$ with degree of $f(z)$ not exceeding that of $F(z)$ and $F(z) \neq 0$ for $|z| > 1$, the inequality $|f(z)| \leqslant |F(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|f'(z)| \leqslant |F'(z)|$ on $|z| = 1$. In particular, this result allows one to establish the famous Bernstein inequality [2] for the sup-norm on the unit circle: namely, if $P(z)$ is a polynomial of degree n, it is true that

$$\max_{|z|=1}\left|P'(z)\right| \leqslant n\max_{|z|=1}\left|P(z)\right|. \tag{1.1}$$

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On the other hand, concerning the maximum modulus of $P(z)$ on the circle $|z| = R \geqslant 1$, we have another classical result known as Bernstein-Walsh lemma ([17], Corollary 12.1.3), which states that, if $f(z)$ and $F(z)$ are two polynomials with degree of $f(z)$ not exceeding that of $F(z)$ and $F(z) \neq 0$ for $|z| > 1$, the inequality $|f(z)| \leqslant |F(z)|$ on the unit circle $|z| = 1$ implies that $|f(z)| < |F(z)|$ for $|z| > 1$, unless $f(z) = e^{i\theta}F(z)$, $\theta \in \mathbb{R}$. From this, one can deduce that if $P \in \mathbb{P}_n$, then for $R \geqslant 1$,

$$\max_{|z|=R}\left|P(z)\right| \leqslant R^n \max_{|z|=1}\left|P(z)\right|. \tag{1.2}$$

The inequalities (1.1) and (1.2) are related with each other and it was observed by Bernstein [3] that (1.1) can also be deduced from (1.2) by making use of Gauss-Lucas theorem and the proof of this fact was given by Govil, Qazi and Rahman [4]. If we restrict ourselves to the class of polynomials $P \in \mathbb{P}_n$, with $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1}\left|P'(z)\right| \leqslant \frac{n}{2}\max_{|z|=1}\left|P(z)\right|, \tag{1.3}$$

and

$$\max_{|z|=R\geqslant 1}\left|P(z)\right| \leqslant \frac{R^n+1}{2}\max_{|z|=1}\left|P(z)\right|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [10], where as inequality (1.4) was proved by Ankeny and Rivlin [1]. In 2011, Govil et al. [5] proved a more general result which provide a compact generalizations of inequalities (1.1)–(1.4). In fact, they proved that, if $f(z)$ and $F(z)$ are two polynomials with degree of $f(z)$ not exceeding that of $F(z)$ and $F(z) \neq 0$ for $|z| > 1$ with $|f(z)| \leqslant |F(z)|$ on $|z| = 1$, then for any β with $|\beta| \leqslant 1$ and $R \geqslant r \geqslant 1$, we have

$$\left|f(Rz) - \beta f(rz)\right| \leqslant \left|F(Rz) - \beta F(rz)\right|, \text{ for } |z| \geqslant 1. \tag{1.5}$$

Further, as a generalization of (1.5), Liman et al. [8] in the same year 2011 and under the same hypothesis as in (1.5), proved that

$$\begin{aligned}&\left|f(Rz) - \beta f(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}f(rz)\right| \\ &\qquad \leqslant \left|F(Rz) - \beta F(rz) + \gamma\left\{\left(\frac{R+1}{r+1}\right)^n - |\beta|\right\}F(rz)\right|,\end{aligned} \tag{1.6}$$

for every $\beta, \gamma \in \mathbb{C}$ with $|\beta| \leqslant 1$, $|\gamma| \leqslant 1$ and $R > r \geqslant 1$.

Jain [6] proved a result concerning the minimum modulus of polynomials by showing that, if $f \in \mathbb{P}_n$, and $f(z)$ has all its zeros in $|z| \leqslant 1$, then for every β with $|\beta| \leqslant 1$ and $R \geqslant 1$,

$$\min_{|z|=1}\left|f(Rz) + \beta\left(\frac{R+1}{2}\right)^n f(z)\right| \geqslant \left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| \min_{|z|=1}|f(z)|. \tag{1.7}$$

Mezerji et al. [12] besides proving some other results also obtained a generalization of (1.7) by proving that, if $f \in \mathbb{P}_n$, and $f(z)$ has all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then for every $|\beta| \leqslant 1$ and $R \geqslant 1$,

$$\min_{|z|=1}\left|f(Rz)+\beta\left(\frac{R+k}{1+k}\right)^n f(z)\right| \geqslant \frac{1}{k^n}\left|R^n+\beta\left(\frac{R+k}{1+k}\right)^n\right|\min_{|z|=k}|f(z)|. \tag{1.8}$$

Recently, Kumar [7] found that there is a room for the generalization of the condition $R \geqslant 1$ in (1.7) and (1.8) to $R \geqslant r > 0$ and proved that, if $f \in \mathbb{P}_n$, and $f(z)$ has all its zeros in $|z| \leqslant k$, $k > 0$, then for every β with $|\beta| \leqslant 1$, $|z| \geqslant 1$ and $R \geqslant r$, $Rr \geqslant k^2$,

$$\min_{|z|=1}\left|f(Rz)+\beta\left(\frac{R+k}{r+k}\right)^n f(rz)\right| \geqslant \frac{1}{k^n}\left|R^n+\beta r^n\left(\frac{R+k}{r+k}\right)^n\right|\min_{|z|=k}|f(z)|. \tag{1.9}$$

For $f \in \mathbb{P}_n$, and $\alpha \in \mathbb{C}$, the polar derivative of the polynomial $f(z)$ with respect to the point α, denoted by $D_\alpha f(z)$, is defined as

$$D_\alpha f(z) := nf(z)+(\alpha-z)f'(z).$$

Note that $D_\alpha f(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha f(z)}{\alpha}\right\} := f'(z),$$

uniformly with respect to z for $|z| \leqslant R$, $R > 0$.

Although the literature on polynomial inequalities is vast and growing and over the last four decades many different authors produced a large number of different versions and generalizations of the above inequalities. Many of these generalizations involve the comparison of polar derivative $D_\alpha P(z)$ with various choices of $P(z)$, α and other parameters. More information on the polar derivative of a polynomial can be found in the books of Rahman and Schmeisser [17] and Marden [11]. One can also see in the literature (for example, refer [7], [9], [13]–[16]), the latest research and development in this direction. Recently, Liman et al. [9] besides proving some other results also proved the following generalization of (1.5) and (1.6) to the polar derivative $D_\alpha f(z)$ of a polynomial $f(z)$ with respect to $\alpha, |\alpha| \geqslant 1$.

THEOREM A. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leqslant 1$ and $f(z)$ be a polynomial of degree $m(\leqslant n)$ such that $|f(z)| \leqslant |F(z)|$, for $|z| = 1$. If $\alpha,\beta,\gamma \in \mathbb{C}$ be such that $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$ and $|\lambda| < 1$, then for $R > r \geqslant 1$ and $|z| \geqslant 1$, we have*

$$\begin{aligned}\Bigg|z\Bigg[(n-m)\Big\{f(Rz)-\beta f(rz)\Big\}+D_\alpha f(Rz)-\beta D_\alpha f(rz)\Bigg]&\\ +\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)-\beta f(rz)\Big\}\Bigg|&\\ \leqslant \Bigg|z\Big\{D_\alpha F(Rz)-\beta D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)-\beta F(rz)\Big\}\Bigg|.&\end{aligned} \tag{1.10}$$

Equality holds in (1.10) *for* $f(z) = e^{i\eta}F(z),\ \eta \in \mathbb{R}$.

While making an attempt towards the generalizations of the above inequalities, the author found that there is a room for the extension of the condition $R > r \geqslant 1$ in (1.10) to $R > r,\ rR \geqslant k^2$ with $k > 0$, which in turn induces inequalities towards more generalized form. The essence in the papers of Liman et al. [9] and Kumar [7] is the origin of thought for the new inequalities presented in this paper.

2. Main results

Here, we shall establish certain comparison inequalities between complex polynomials for the maximum modulus of their polar derivative, when the zeros of one of the polynomial are restricted. The obtained results include certain interesting generalizations of (1.5)–(1.10) and related results. We begin by proving the following generalization and extension of Theorem A.

THEOREM 2.1. *Let* $F \in \mathbb{P}_n$*, having all its zeros in* $|z| \leqslant k,\ k > 0$ *and* $f(z)$ *be a polynomial of degree* $m (\leqslant n)$ *such that*

$$\left|f(z)\right| \leqslant |F(z)|, \text{ for } |z| = k.$$

If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ *be such that* $|\alpha| \geqslant 1,\ |\beta| \leqslant 1,\ |\gamma| \leqslant 1$ *and* $|\lambda| < 1$*, then for* $R > r$*,* $rR \geqslant k^2$ *and* $|z| \geqslant 1$*, we have*

$$\begin{aligned}\Bigg|z\Bigg[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Bigg] \\ +\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\Bigg| \\ \leqslant \left|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\right|. \qquad (2.1)\end{aligned}$$

The result is sharp and equality in (2.1) *holds for* $f(z) = e^{i\eta}F(z)$*,* η *is real and* $F(z)$ *has all its zeros in* $|z| \leqslant k$.

We now present and discuss some consequences of Theorem 2.1. Suppose $f \in \mathbb{P}_n$, and $f(z) \neq 0$ in $|z| < k$, the polynomial $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})} \in \mathbb{P}_n$, and $Q(z)$ has all its zeros in $|z| \leqslant \frac{1}{k}$. Note that

$$|Q(z)| = \frac{1}{k^n}|f(k^2z)|, \text{ for } |z| = \frac{1}{k}.$$

Applying Theorem 2.1 with $F(z)$ replaced by $k^nQ(z)$, we get the following result.

COROLLARY 2.1. *If* $f \in \mathbb{P}_n$*, and* $f(z) \neq 0$ *in* $|z| < k,\ k > 0$*, then for every* $|\alpha| \geqslant 1,\ |\beta| \leqslant 1,\ |\gamma| \leqslant 1$ *and* $|\lambda| < 1$*, we have for* $R > r,\ rR \geqslant \frac{1}{k^2}$ *and* $|z| \geqslant 1$*,*

$$\begin{aligned}\left|z\Big\{D_\alpha f(Rk^2z)+\phi D_\alpha f(rk^2z)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rk^2z)+\phi f(rk^2z)\Big\}\right| \\ \leqslant k^n\left|z\Big\{D_\alpha Q(Rz)+\phi D_\alpha Q(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{Q(Rz)+\phi Q(rz)\Big\}\right|, \qquad (2.2)\end{aligned}$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$.

Equality holds in (2.2) *for* $f(z) = e^{i\eta}Q(z),\ \eta \in \mathbb{R}$.

REMARK 2.1. For $k = 1$ and $\gamma = 0$, Corollary 2.1 in particular yields a result of Liman et al. ([9], Corollary 1.4).

Taking $\beta = \lambda = 0$ in Corollary 2.1, we get the following result.

COROLLARY 2.2. *If* $f \in \mathbb{P}_n$, *and* $f(z) \neq 0$ *in* $|z| < k,\ k > 0$, *then for every* $|\alpha| \geqslant 1,\ |\gamma| \leqslant 1$, *we have for* $R > r,\ rR \geqslant \frac{1}{k^2}$ *and* $|z| \geqslant 1$,

$$\left|D_\alpha f(Rk^2z) + \gamma\left(\frac{Rk+1}{rk+1}\right)^n D_\alpha f(rk^2z)\right|$$
$$\leqslant k^n\left|D_\alpha Q(Rz) + \gamma\left(\frac{Rk+1}{rk+1}\right) D_\alpha Q(rz)\right|, \tag{2.3}$$

where $Q(z) = z^n\overline{f(\frac{1}{\bar{z}})}$.

Inequality (2.3) should be compared with a result recently proved by Kumar ([7], Lemma 2.2), where $f(z)$ is replaced by $D_\alpha f(z)$, $|\alpha| \geqslant 1$.

REMARK 2.2. For $r = 1$, Corollary 2.2 gives the polar derivative analogue of a result due to Mezerji et al. ([12], Lemma 4).

If we take $\beta = 0$ in Theorem 2.1, we get the following:

COROLLARY 2.3. *Let* $F \in \mathbb{P}_n$, *having all zeros in* $|z| \leqslant k,\ k > 0$ *and* $f(z)$ *be a polynomial of degree* $m(\leqslant n)$ *such that*

$$|f(z)| \leqslant |F(z)|,\ \text{for}\ |z| = k.$$

If $\alpha, \gamma, \lambda \in \mathbb{C}$ *be such that* $|\alpha| \geqslant 1,\ |\gamma| \leqslant 1$ *and* $|\lambda| < 1$, *then for* $R > r,\ rR \geqslant k^2$ *and* $|z| \geqslant 1$, *we have*

$$\Bigg|z\left[(n-m)\left\{f(Rz) + \gamma\left(\frac{R+k}{r+k}\right)^n f(rz)\right\} + D_\alpha f(Rz) + \gamma\left(\frac{R+k}{r+k}\right)^n D_\alpha f(rz)\right]$$
$$+ \frac{n\lambda}{2}(|\alpha|-1)\left\{f(Rz) + \gamma\left(\frac{R+k}{r+k}\right)^n f(rz)\right\}\Bigg|$$
$$\leqslant \Bigg|z\left\{D_\alpha F(Rz) + \gamma\left(\frac{R+k}{r+k}\right)^n D_\alpha F(rz)\right\}$$
$$+ \frac{n\lambda}{2}(|\alpha|-1)\left\{F(Rz) + \left(\frac{R+k}{r+k}\right)^n F(rz)\right\}\Bigg|. \tag{2.4}$$

Equality holds in (2.4) *for* $f(z) = e^{i\eta}F(z),\ \eta \in \mathbb{R}$ *and* $F(z)$ *has all its zeros in* $|z| \leqslant k$.

If we apply Theorem 2.1 to polynomials $f(z)$ and $\frac{z^n}{k^n}\min\limits_{|z|=k}|f(z)|$, we get the following result.

COROLLARY 2.4. *If $f \in \mathbb{P}_n$, and $f(z)$ has all its zeros in $|z| \leqslant k$, $k > 0$, then for every $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ such that $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, $|\gamma| \leqslant 1$ and $|\lambda| < 1$, we have for $R > r$, $rR \geqslant k^2$ and $|z| \geqslant 1$,*

$$\left|z\Big\{D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\right| \geqslant \frac{n|z|^n}{k^n}\left|\alpha(R^{n-1}+\psi r^{n-1}) + \frac{\lambda}{2}(|\alpha|-1)(R^n+\psi r^n)\right|\min_{|z|=k}|f(z)|. \tag{2.5}$$

Equality holds in (2.5) holds for $f(z) = az^n$, $a \neq 0$.

Taking $\lambda = 0$ in Corollary 2.4, we get the following result.

COROLLARY 2.5. *If $f \in \mathbb{P}_n$, and $f(z)$ has all its zeros in $|z| \leqslant k$, $k > 0$ then for every $\alpha, \beta, \gamma, \in \mathbb{C}$ such that $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, $|\gamma| \leqslant 1$ and for $R > r$, $rR \geqslant k^2$, we have*

$$\min_{|z|=1}\left|D_\alpha f(Rz) + \psi D_\alpha f(rz)\right| \geqslant \frac{n|\alpha|}{k^n}\left|R^{n-1}+\psi r^{n-1}\right|\min_{|z|=k}|f(z)|. \tag{2.6}$$

Equality holds in (2.6) holds for $f(z) = az^n$, $a \neq 0$.

REMARK 2.3. For $\beta = 0$, the above inequality (2.6) gives the polar derivative analogue of (1.9).

REMARK 2.4. For $k = 1$, Theorem 2.1 in particular gives a recently proved result of Mir and Sheikh [15], and for $k = 1$ and $\gamma = 0$, it in particular yields Theorem A.

THEOREM 2.2. *Let $F \in \mathbb{P}_n$, having all its zeros in $|z| \leqslant k$, $k > 0$ and $f(z)$ be a polynomial of degree $m (\leqslant n)$ such that*

$$\left|f(z)\right| \leqslant |F(z)|, \text{ for } |z| = k.$$

If $\alpha, \beta, \gamma \in \mathbb{C}$ be such that $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$ and $|\gamma| \leqslant 1$, then for $R > r$, $rR \geqslant k^2$ and $|z| \geqslant 1$, we have

$$\left|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\Big]\right| + \frac{n}{2}(|\alpha|-1)\left|F(Rz)+\psi F(rz)\right| \leqslant \left|z\Big\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\Big\}\right| + \frac{n}{2}(|\alpha|-1)\left|f(Rz)+\psi f(rz)\right|. \tag{2.7}$$

Equality holds in (2.7) for $f(z) = e^{i\eta}F(z)$, $\eta \in \mathbb{R}$ and $F(z)$ has all its zeros in $|z| \leqslant k$.

REMARK 2.5. For $k = 1$, Theorem 2.2 in particular gives a recently proved result of Mir and Sheikh [15] and for $\gamma = 0$ and $k = 1$, it gives in particular a result of Liman et al. ([9], Theorem 2).

From Theorem 2.2, we also have the following result.

COROLLARY 2.5. *If $f \in \mathbb{P}_n$, and $f(z)$ does not vanish in $|z| < k$, $k > 0$ then for every $\alpha, \beta, \gamma \in \mathbb{C}$ with $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, $|\gamma| \leqslant 1$, we have for $R > r$, $rR \geqslant \frac{1}{k^2}$ and $|z| \geqslant 1$,*

$$\left|z\left\{D_\alpha f(Rk^2z)+\phi D_\alpha f(rk^2z)\right\}\right|+\frac{n}{2}(|\alpha|-1)k^n\left|Q(Rz)+\phi Q(rz)\right|$$
$$\leqslant k^n\left|z\left\{D_\alpha Q(Rz)+\phi D_\alpha Q(rz)\right\}\right|+\frac{n}{2}(|\alpha|-1)\left|f(Rk^2z)+\phi f(rk^2z)\right|, \tag{2.8}$$

where $Q(z)=z^n\overline{f(\frac{1}{\bar{z}})}$.

REMARK 2.6. We recover a result of Liman et al. ([9], Corollary 2.3) from Corollary 2.5, when we take $\gamma = 0$ and $k = 1$.

3. Auxiliary results

We need the following lemmas to prove our theorems. The first lemma is due to Govil et al. [5].

LEMMA 3.1. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leqslant k$, $k \geqslant 0$, then for every $R > r$, $rR \geqslant k^2$,*

$$\left|f(Rz)\right| > \left(\frac{R+k}{r+k}\right)^n\left|f(rz)\right|, \ \textit{for} \ |z|=1.$$

LEMMA 3.2. *Let $f \in \mathbb{P}_n$, having all its zeros in $|z| \leqslant 1$, then for every α with $|\alpha| \geqslant 1$,*

$$2\left|zD_\alpha f(z)\right| \geqslant n(|\alpha|-1)\left|f(z)\right|, \ \textit{for} \ |z|=1.$$

The above lemma is due to Shah [18].

LEMMA 3.3. *Let $f \in \mathbb{P}_n$, having all its zeros in a circular domain Ω and $\alpha \in \mathbb{C}\setminus\Omega$. Then all the zeros of*

$$D_\alpha f(z) := nf(z)+(\alpha-z)f'(z),$$

lie in Ω.

The above lemma is due to Laguerre ([11], p. 49).

4. Proofs of Theorems

Proof of Theorem 2.1. Recall that $F(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ and $f(z)$ is a polynomial of degree at most n such that

$$\left|f(z)\right| \leqslant |F(z)|, \ \text{for} \ |z|=k, \tag{4.1}$$

therefore, if $F(z)$ has a zero of multiplicity ν at $z=ke^{i\theta_0}$, then $f(z)$ must also have a zero of multiplicity at least ν at $z=ke^{i\theta_0}$. We assume that $\frac{f(z)}{F(z)}$ is not a constant, otherwise, the inequality (2.1) is obvious. It follows by the Maximum Modulus Principle that

$$\left|f(z)\right| < |F(z)|, \text{ for } |z| > k.$$

Suppose $F(z)$ has m zeros on $|z|=k$, where $0 \leqslant m < n$, so that we can write

$$F(z) = F_1(z)F_2(z),$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z|=k$ and $F_2(z)$ is a polynomial of degree $n-m$ whose all zeros lie in $|z|<k$. This gives with the help of (4.1) that

$$f(z) = P_1(z)F_1(z),$$

where $P_1(z)$ is a polynomial of degree at most $n-m$. Now, from inequality (4.1), we get

$$|P_1(z)| \leqslant |F_2(z)|, \text{ for } |z| = k,$$

and $F_2(z) \neq 0$ for $|z|=k$. Therefore, for a given complex number δ with $|\delta|>1$, it follows from Rouché's theorem that the polynomial $P_1(z)-\delta F_2(z)$ of degree $n-m \geqslant 1$ has all its zeros in $|z|<k$. Hence the polynomial

$$P(z) = F_1(z)(P_1(z)-\delta F_2(z)) = f(z)-\delta F(z)$$

has all its zeros in $|z| \leqslant k$, with at least one zero in $|z|<k$, so that we can write

$$P(z) = (z-\eta e^{i\gamma})H(z),$$

where $\eta < k$, and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leqslant k$. Applying Lemma 3.1 to $H(z)$, we obtain for $R>r$, $rR \geqslant k^2$ and $0 \leqslant \theta < 2\pi$,

$$\begin{aligned} \left|P(Re^{i\theta})\right| &= \left|Re^{i\theta}-\eta e^{i\gamma}\right|\left|H(Re^{i\theta})\right| \\ &> \left|Re^{i\theta}-\eta e^{i\gamma}\right|\left(\frac{R+k}{r+k}\right)^{n-1}\left|H(re^{i\theta})\right| \\ &= \left(\frac{R+k}{r+k}\right)^{n-1}\frac{\left|Re^{i\theta}-\eta e^{i\gamma}\right|}{\left|re^{i\theta}-\eta e^{i\gamma}\right|}\left|re^{i\theta}-\eta e^{i\gamma}\right|\left|H(re^{i\theta})\right|. \end{aligned} \tag{4.2}$$

Now, for $0 \leqslant \theta < 2\pi$, we have

$$\begin{aligned} \left|\frac{Re^{i\theta}-\eta e^{i\gamma}}{re^{i\theta}-\eta e^{i\gamma}}\right|^2 &= \frac{R^2+\eta^2-2R\eta\cos(\theta-\gamma)}{r^2+\eta^2-2r\eta\cos(\theta-\gamma)} \\ &\geqslant \left(\frac{R+\eta}{r+\eta}\right)^2, \text{ for } R>r \text{ and } rR \geqslant k^2 \\ &> \left(\frac{R+k}{r+k}\right)^2, \text{ since } \eta<k. \end{aligned}$$

This implies

$$\left|\frac{Re^{i\theta}-\eta e^{i\gamma}}{re^{i\theta}-\eta e^{i\gamma}}\right| > \left(\frac{R+k}{r+k}\right),$$

which on using in (4.2) gives for $R>r$, $rR\geqslant k^2$ and $0\leqslant\theta<2\pi$,

$$\left|P(Re^{i\theta})\right| > \left(\frac{R+k}{r+k}\right)^n \left|P(re^{i\theta})\right|.$$

Equivalently,

$$\left|P(Rz)\right| > \left(\frac{R+k}{r+k}\right)^n \left|P(rz)\right|, \tag{4.3}$$

for $R>r$, $rR\geqslant k^2$ and $|z|=1$. This implies for every $|\beta|\leqslant 1$, $R>r$, $rR\geqslant k^2$ and $|z|=1$,

$$\begin{aligned}\left|P(Rz)-\beta P(rz)\right| &\geqslant \left|P(Rz)\right| - |\beta|\left|P(rz)\right| \\ &> \left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\}\left|P(rz)\right|. \end{aligned}\tag{4.4}$$

Again, since $r<R$, it follows that $\left(\frac{r+k}{R+k}\right)^n<1$, inequality (4.3) implies that

$$|P(rz)|<|P(Rz)|, \text{ for } |z|=1.$$

Also, all the zeros of $P(Rz)$ lie in $|z|\leqslant\frac{k}{R}$, and $R^2>rR\geqslant k^2$, we have $\frac{k}{R}<1$. A direct application of Rouché's theorem shows that the polynomial $P(Rz)-\beta f(rz)$ has all its zeros in $|z|<1$, for every $|\beta|\leqslant 1$. Applying Rouché's theorem again, it follows from (4.4) that for every $|\gamma|\leqslant 1$, $|\beta|\leqslant 1$, $R>r$, $rR\geqslant k^2$, all the zeros of the polynomial

$$\begin{aligned} g(z) &:= P(Rz)-\beta P(rz)+\gamma\left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\}P(rz) \\ &= P(Rz)+\psi P(rz) \end{aligned}\tag{4.5}$$

lie in $|z|<1$. Using Lemma 3.2, we get for every $\alpha\in\mathbb{C}$ with $|\alpha|\geqslant 1$ and $|z|=1$,

$$2\left|zD_\alpha g(z)\right| \geqslant n(|\alpha|-1)\left|g(z)\right|.$$

Hence, for any complex number λ with $|\lambda|<1$, we have for $|z|=1$,

$$2\left|zD_\alpha g(z)\right| > n|\lambda|(|\alpha|-1)\left|g(z)\right|.$$

Therefore, it follows by Lemma 3.3 and Rouché's theorem that all the zeros of

$$\begin{aligned} W(z) &:= 2zD_\alpha g(z)+n\lambda(|\alpha|-1)g(z) \\ &= 2zD_\alpha P(Rz)+2z\psi D_\alpha P(rz)+n\lambda\big(|\alpha|-1\big)\big(P(Rz)+\psi P(rz)\big) \end{aligned}\tag{4.6}$$

lie in $|z|<1$.

Replacing $P(z)$ by $f(z)-\delta F(z)$ and using definition of polar derivative gives

$$\begin{aligned}
W(z) = 2z\Big[n\Big\{f(Rz)-\delta F(Rz)\Big\}+(\alpha-Rz)\Big\{f(Rz)-\delta F(Rz)\Big\}'\Big] \\
+2z\psi\Big[n\Big\{f(rz)-\delta F(rz)\Big\}+(\alpha-rz)\Big\{f(rz)-\delta F(rz)\Big\}'\Big] \\
+n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\},
\end{aligned}$$

which on simplification gives

$$\begin{aligned}
W(z) &= 2z\Big[(n-m)f(Rz)+mf(Rz)+(\alpha-Rz)\big(f(Rz)\big)' \\
&\qquad -\delta\Big\{nF(rz)+(\alpha-rz)\big(F(Rz)\big)'\Big\}\Big] \\
&\quad +2z\psi\Big[(n-m)f(rz)+mf(rz)+(\alpha-rz)\big(f(rz)\big)' \\
&\qquad -\delta\Big\{nF(rz)+(\alpha-rz)\big(F(rz)\big)'\Big\}\Big] \\
&\quad +n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\} \\
&= 2z\Big\{(n-m)f(Rz)+D_\alpha f(Rz)-\delta D_\alpha F(Rz)\Big\} \\
&\quad +2z\psi\Big\{(n-m)f(rz)+D_\alpha f(rz)-\delta D_\alpha F(rz)\Big\} \\
&\quad +n\lambda(|\alpha|-1)\Big\{f(Rz)-\delta F(Rz)\Big\}+n\lambda\psi(|\alpha|-1)\Big\{f(rz)-\delta F(rz)\Big\} \\
&= 2z\Big\{(n-m)f(Rz)+\psi(n-m)f(rz)+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big\} \\
&\quad +n\lambda\psi(|\alpha|-1)f(Rz)+n\lambda\psi(|\alpha|-1)f(rz)-\delta\Big\{2zD_\alpha F(Rz) \\
&\quad +2z\psi D_\alpha F(rz)+n\lambda(|\alpha|-1)F(Rz)+n\lambda\psi(|\alpha|-1)F(rz)\Big\}. \qquad (4.7)
\end{aligned}$$

Since by (4.6), $W(z)$ has all its zeros in $|z|<1$, therefore, by (4.7), we get for $|z|\geqslant 1$,

$$\begin{aligned}
&\Big|z\Big[(n-m)\Big\{f(Rz)+\psi f(rz)\Big\}+D_\alpha f(Rz)+\psi D_\alpha f(rz)\Big] \\
&\qquad +\frac{n\lambda}{2}(|\alpha|-1)\Big\{f(Rz)+\psi f(rz)\Big\}\Big| \\
&\leqslant \Big|z\Big\{D_\alpha F(Rz)+\psi D_\alpha F(rz)\Big\}+\frac{n\lambda}{2}(|\alpha|-1)\Big\{F(Rz)+\psi F(rz)\Big\}\Big|. \qquad (4.8)
\end{aligned}$$

To see that the inequality (4.8) holds, note that if the inequality (4.8) is not true, then there is a point $z = z_0$ with $|z_0| \geqslant 1$, such that

$$\Big| z_0 \Big[(n-m)\Big\{ f(Rz_0) + \psi f(rz_0) \Big\} + D_\alpha f(Rz_0) + \psi D_\alpha f(rz_0) \Big] + \frac{n\lambda}{2}(|\alpha|-1)\Big\{ f(Rz_0) + \psi f(rz_0) \Big\} \Big|$$
$$> \Big| z_0 \Big\{ D_\alpha F(Rz_0) + \psi D_\alpha F(rz_0) \Big\} + \frac{n\lambda}{2}(|\alpha|-1)\Big\{ F(Rz_0) + \psi F(rz_0) \Big\} \Big|. \qquad (4.9)$$

Now, because by hypothesis all the zeros of $F(z)$ lie in $|z| \leqslant k$, the polynomial $F(Rz)$ has all its zeros in $|z| \leqslant \frac{k}{R} < 1$, and therefore, if we use Rouché's theorem and Lemmas 3.1 and 3.3 and argument similar to the above, we will get that all the zeros of

$$z\Big(D_\alpha F(Rz) + \psi D_\alpha F(rz) \Big) + \frac{n\lambda}{2}(|\alpha|-1)\Big\{ F(Rz) + \psi F(rz) \Big\}$$

lie in $|z| < 1$ for every $|\alpha| \geqslant 1$, $|\lambda| < 1$ and $R > r$, $rR \geqslant k^2$, that is,

$$z\Big(D_\alpha F(Rz_0) + \psi D_\alpha F(rz_0) \Big) + \frac{n\lambda}{2}(|\alpha|-1)\Big\{ F(Rz_0) + \psi F(rz_0) \Big\} \neq 0$$

for every z_0 with $|z_0| \geqslant 1$.

Therefore, if we take

$$\delta = \frac{A+B}{C},$$

where

$$A = z_0 \Big[(n-m)\Big\{ f(Rz_0) + \psi f(rz_0) \Big\} + D_\alpha f(Rz_0) + \psi D_\alpha f(rz_0) \Big]$$
$$B = \frac{n\lambda}{2}(|\alpha|-1)\Big\{ f(Rz_0) + \psi f(rz_0) \Big\}$$

and

$$C = z_0\Big(D_\alpha F(Rz_0) + \psi F(rz_0) D_\alpha \Big) + \frac{n\lambda}{2}(|\alpha|-1)\Big\{ F(Rz_0) + \psi F(rz_0) \Big\},$$

then δ is a well-defined complex number, and in view of (4.9), we also have $|\delta| > 1$. Hence, with this choice of δ, we get from (4.7) that $W(z_0) = 0$ for some z_0, satisfying $|z_0| \geqslant 1$, which is clearly a contradiction to the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus for every $R > r$, $rR \geqslant k^2$, $|\alpha| \geqslant 1$, $|\lambda| < 1$ and $|z| \geqslant 1$, inequality (4.8) holds and this completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Since all the zeros of $F(z)$ lie in $|z| \leqslant k$, $k > 0$, for $R > r$, $rR \geqslant k^2$, $|\beta| \leqslant 1$, $|\gamma| \leqslant 1$, it follows as in the proof of Theorem 2.1, that all the zeros of

$$h(z) := F(Rz) - \beta F(rz) + \gamma\left\{\left(\frac{R+k}{r+k}\right)^n - |\beta|\right\}F(rz)$$
$$= F(Rz) + \psi F(rz)$$

lie in $|z| < 1$. Hence by Lemma 3.2, we get for $|\alpha| \geqslant 1$,

$$2\left|zD_\alpha h(z)\right| \geqslant n(|\alpha|-1)\left|h(z)\right|, \text{ for } |z| \geqslant 1.$$

This gives for every λ with $|\lambda| < 1$,

$$\left|z\left\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\right\}\right| - \frac{n|\lambda|}{2}(|\alpha|-1)\left|F(Rz) + \psi F(rz)\right| \geqslant 0, \quad \text{for } |z| \geqslant 1.$$

Therefore, it is possible to choose the argument of λ in the right hand side of (4.8) such that for $|z| \geqslant 1$,

$$\begin{aligned} &\left|z\left\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\right\} + \frac{n\lambda}{2}(|\alpha|-1)\left\{F(Rz) + \psi F(rz)\right\}\right| \\ &= \left|z\left\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\right\}\right| - \frac{n|\lambda|}{2}(|\alpha|-1)\left|F(Rz) + \psi F(rz)\right|. \end{aligned} \tag{4.10}$$

Hence from (4.8), we get by using (4.10), for $|z| \geqslant 1$,

$$\begin{aligned} &\left|z\left[(n-m)\left\{f(Rz) + \psi f(rz)\right\} + D_\alpha f(Rz) + \psi D_\alpha f(rz)\right]\right| \\ &\qquad - \frac{n|\lambda|}{2}(|\alpha|-1)\left|f(Rz) + \psi f(rz)\right| \\ &\leqslant \left|z\left\{D_\alpha F(Rz) + \psi D_\alpha F(rz)\right\}\right| - \frac{n|\lambda|}{2}(|\alpha|-1)\left|F(Rz) + \psi F(rz)\right|. \end{aligned} \tag{4.11}$$

Letting $|\lambda| \to 1$ in (4.11), we immediately get (2.7) and this proves Theorem 2.2 completely. □

REFERENCES

[1] N. C. ANKENY AND T. J. RIVLIN, *On a theorem of S. Bernstein*, Pacific J. Math., **5** (1955), 849–852.

[2] S. BERNSTEIN, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mem. Acad. R. Belg., **4** (1912), 1–103.

[3] S. BERNSTEIN, *Sur la limitation des dérivées des polynômes*, Comptes Rendus de l' Academic des Sciences, (Paris) **190** (1930), 338–340.

[4] N. K. GOVIL, M. A. QAZI AND Q. I. RAHMAN, *Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius*, Math. Inequal. Appl., **6** (2003), 453–467.

[5] N. K. GOVIL, A. LIMAN AND W. M. SHAH, *Some inequalities concerning derivative and maximum modulus of polynomials*, Austr. J. Math. Anal. Appl., **8** (2011), 1–8.

[6] V. K. JAIN, *Inequalities for a polynomial and its derivative*, Proc. Indian Acad. Sci. (Math. Sci.), **110** (2000), 137–146.

[7] P. N. KUMAR, *On the generalization of polynomial inequalities in the complex domain*, J. Contemporary Math. Anal., **50** (2015), 14–21.

[8] A. LIMAN, R. N. MOHAPATRA AND W. M. SHAH, *Inequalities for polynomials not vanishing in a disk*, Appl. Math. Comp., **218** (2011), 949–955.

[9] A. LIMAN, I. Q. PEER AND W. M. SHAH, *On some inequalities concerning the polar derivative of a polynomial*, Ramanujan J., **38** (2015), 349–360.

[10] P. D. LAX, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509–513.

[11] M. MARDEN, *Geometry of Polynomials*, Math. Surveys, **3** (1966).

[12] H. A. S. MEZERJI, M. A. BASERI, M. BIDHKAM AND A. ZIREH, *Generalization of certain inequalities for a polynomial and its derivative*, Lob. J. Math., **33** (2012), 68–74.

[13] A. MIR, *On an operator preserving inequalities between polynomials*, Ukrainian Math. J., **69** (2018), 1234–1247.

[14] A. MIR AND I. HUSSAIN, *On the Erdös-Lax inequality concerning polynomials*, C. R. Acad. Sci. Paris Ser. I, **355** (2017), 1055–1062.

[15] A. MIR AND M. I. SHEIKH, *Some upper bound estimates for the maximal modulus of the polar derivative of a polynomial*, J. Contemporary Math. Anal., **55** (2020), 189–195.

[16] A. MIR AND A. WANI, *A note on two recent results about polynomials with restricted zeros*, J. Math. Inequal., **14** (2020), 47–52.

[17] Q. I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Oxford University Press, (2002).

[18] W. M. SHAH, *A generalization of a theorem Paul Turán*, J. Ramanujan Math. Soc., **1** (1996), 67–72.

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