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ESTIMATES FOR p–ADIC HARDY OPERATORS ON VARIABLE EXPONENT MORREY–HERZ TYPE SPACES

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Abstract. In this paper, we establish some necessary and sufficient conditions for the boundedness of p-adic Hardy-Cesàro operators on central Morrey, Herz, and Morrey-Herz spaces with variable exponents. Finally, the boundedness of rough p-adic Hardy operators on variable exponent Morrey-Herz spaces is also discussed.

1. Introduction

Let f be a non-negative measurable function on $\mathbb{R}^+$. The one-dimensional Hardy operator was presented by Hardy [19]

$$H(f)(x) = \frac{1}{x}\int_0^x f(t)dt, \ \ x > 0.$$

The operator H is bounded on $L^q(\mathbb{R}^+)$ with $1 < q < \infty$. Furthermore, Hardy obtained

$$\|H(f)\|_{L^q(\mathbb{R}^+)} \leqslant \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \ \ \text{for all} \ \ f \in L^q(\mathbb{R}^+),$$

where $\frac{q}{q-1}$ is the sharpest constant. In 1995, Christ and Grafakos [6] studied the higher-dimensional Hardy operator

$$\mathscr{H}(f)(x) = \frac{1}{|x|^n}\int_{|t|<|x|} f(t)dt, \ \ x \in \mathbb{R}^n \setminus \{0\}.$$

The authors [6] stated as follows.

THEOREM 1. *If $q \in (1,\infty)$ then we have*

$$\|\mathscr{H}(f)\|_{L^q(\mathbb{R}^n)} \leqslant \frac{q.v_n}{q-1}\|f\|_{L^q(\mathbb{R}^n)}, \ \ \text{for all} \ \ f \in L^q(\mathbb{R}^n),$$

where $\frac{qv_n}{q-1}$ is the sharpest constant and $v_n = \pi^{n/2}/\Gamma(1+n/2)$.

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It is well known that the Hardy operators are concerned with the areas of mathematical inequalities, function theory, and operator theory (see [27, 28, 29, 31, 32]). In 2014, Chuong and Hung [9] researched the Hardy-Cesàro operator

$$U_{\psi,s}(f)(x) = \int\limits_{[0,1]^d} \psi(t) f(s(t)x) dt,$$

where f is a measurable complex-valued function on $\mathbb{R}^n$ and $s : [0,1]^d \to \mathbb{R}$ is a measurable function. It is clear to see that if we choose $d = n = 1$ and $\psi(t) = 1$, the Hardy-Cesàro operator $U_{\psi,s}$ reduces to the Hardy operator H.

On the other hand, Fu et al [14] introduced the rough Hardy operator, which is generalized of Hardy operator, defined as follows.

DEFINITION 1. Let f be a locally integrable function on $\mathbb{R}^n$ and $\beta \in \mathbb{R}$. The higher-dimensional rough Hardy operator is given by

$$\mathscr{H}_{\Omega,\beta}(f)(x) = \frac{1}{|x|^{n-\beta}} \int\limits_{|t|<|x|} \Omega(x-t) f(t) dt, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where Ω satisfies

$$\Omega(tx) = \Omega(x), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^n,$$

$$\Omega \in L^{\xi}(S^{n-1}), \quad \text{for some } \xi \geqslant 1.$$

The authors [14] obtained central BMO estimates for commutators of $\mathscr{H}_{\Omega,\beta}$.

Recently, the Hardy-Cesàro operator, rough Hardy operator, and their commutators have been extensively studied on real Euclidean spaces (see [15, 16, 21]). Moreover, the p-adic analysis has an important role in mathematical physics (see [24], [25], [26], [33]). Hence, more and more researchers are interested in harmonic analysis on p-adic fields (see [2, 17, 30, 34]). In particular, Hung [20] considered the Hardy-Cesàro operator in the p-adic case as follows.

DEFINITION 2. Let $\psi : \mathbb{Z}_p^* \to [0,\infty)$ and $s : \mathbb{Z}_p^* \to \mathbb{Q}_p$ be measurable functions, and f be a measurable complex-valued function on $\mathbb{Q}_p^n$. The p-adic Hardy-Cesàro operator along curve $s(t,x) = s(t)x$ is defined by

$$\mathscr{H}^p_{\psi,s}(f)(x) = \int_{\mathbb{Z}_p^*} \psi(t) f(s(t)x) dt. \tag{1}$$

From the results of Fu et al [14] and Hussain et al [18], we give the form of the rough Hardy operator in p-adic analysis.

DEFINITION 3. Let $f : \mathbb{Q}_p^n \to \mathbb{C}$ be a measurable function. The rough p-adic Hardy operator is determined by

$$\mathscr{H}^p_{\Omega}(f)(x) = \frac{1}{|x|_p^n} \int\limits_{|t|_p<|x|_p} \Omega(x-t) f(t) dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\}. \tag{2}$$

Here we are interested in the function Ω with

$$\Omega(p^k x) = \Omega(x), \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in S_0, \tag{3}$$

$$\Omega \in L^{\xi}(S_0), \quad \text{for some } \xi > 1. \tag{4}$$

The theory of function spaces with variable exponents is studied in the field of harmonic analysis, partial differential equations, and applied mathematics (see [4], [10], [11], [23]). It is well-known that many mathematicians are interested in the classical operators on some spaces with variable exponents. For example, on variable exponent Lebesgue spaces, the boundedness of the Hardy-type operators is discussed in the papers of Bandaliev [3], Diening et al [12], and Edmunds et al [13]. Besides, the authors [1] investigated the Herz spaces with two variable exponents and obtained the boundedness of sublinear operators on these spaces. In 2010, Izuki [22] defined the Herz-Morrey spaces with one variable exponent. By generalizing the above function spaces, Wu et al [35] considered the boundedness of the fractional Hardy operators. In 2020, Chuong et al [8] established the necessary and sufficient conditions for the boundedness of multilinear Hausdorff operators on the product of weighted Herz and Morrey-Herz spaces with two variable exponents. Recently, the authors [5] introduced the p-adic variable exponent Lebesgue spaces and proved many properties of these spaces.

Motivated by the above results, the main purpose of this paper is to investigate the necessary and sufficient conditions for the boundedness of the operator $\mathscr{H}^p_{\psi,s}$ on p-adic central Morrey spaces, p-adic Herz spaces, and p-adic Morrey-Herz spaces with variable exponents. Moreover, the boundedness of the operator $\mathscr{H}^p_{\Omega}$ with homogeneous kernel Ω on p-adic variable exponent Morrey-Herz spaces is also obtained.

Our paper is organized as follows. In Section 2, we present the necessary preliminaries on p-adic Lebesgue spaces, p-adic central Morrey spaces, p-adic Herz spaces, and p-adic Morrey-Herz spaces with variable exponents. Our main results are given and proved in Section 3.

2. Some notation and definitions

On the field of rational numbers $\mathbb{Q}$ with a prime number p, we define

$$|x|_p = \begin{cases} 0, & \text{if } x = 0, \\ p^{-\gamma}, & \text{otherwise } x = p^{\gamma}\dfrac{m}{n} \text{ with } m, n \not\vdots\, p, \gamma \in \mathbb{Z}. \end{cases}$$

The field $\mathbb{Q}_p$ arises as a result of the completion of the field $\mathbb{Q}$ with the norm $|\cdot|_p$. Then

(i) $|x|_p \geqslant 0$, for all $x \in \mathbb{Q}_p$;

(ii) $|x|_p = 0 \Leftrightarrow x = 0$;

(iii) $|xy|_p = |x|_p |y|_p$, for all $x, y \in \mathbb{Q}_p$;

(iv) $|x+y|_p \leqslant \max(|x|_p, |y|_p)$, for all $x, y \in \mathbb{Q}_p$, and $|x+y|_p = \max(|x|_p, |y|_p)$ with $|x|_p \neq |y|_p$.

For $n \in \mathbb{N}^*$, the space $\mathbb{Q}_p^n$ is defined as $\{x = (x_1, \ldots, x_n) : x_i \in \mathbb{Q}_p, i = 1, \ldots, n\}$ and equipped with the norm defined by

$$|x|_p = \max_{1 \leqslant i \leqslant n} |x_i|_p. \tag{5}$$

Let

$$B_k(a) = \left\{x \in \mathbb{Q}_p^n : |x - a|_p \leqslant p^k\right\}$$

be a ball of radius p^k with center at $a \in \mathbb{Q}_p^n$. Similarly, denote by

$$S_k(a) = \left\{x \in \mathbb{Q}_p^n : |x - a|_p = p^k\right\}$$

the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^k. Denote $B_k = B_k(0)$, $S_k = S_k(0)$, $\mathbb{Z}_p^* = B_0 \setminus \{0\}$, and χ_k be the characteristic function of the sphere S_k.

There exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to a positive constant multiple and is translation invariant. This measure is unique by normalizing dx such that

$$\int_{B_0} dx = |B_0| = 1,$$

where $|B|$ denotes the Haar measure of a measurable subset B of $\mathbb{Q}_p^n$. It is easy to obtain that

$$|B_k(a)| = p^{nk} \text{ and } |S_k(a)| = p^{nk}(1 - p^{-n}),$$

for any $a \in \mathbb{Q}_p^n$.

The Lebesgue space $L^q(\mathbb{Q}_p^n)$ $(0 < q < \infty)$ is defined to be the space of all measurable functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^q(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^q dx\right)^{1/q} < \infty.$$

The space $L^q_{\mathrm{loc}}(\mathbb{U})$ is defined as the set of all measurable functions f on $\mathbb{U}$ satisfying $\int_K |f(x)|^q dx < \infty$, for any compact subset K of $\mathbb{U}$. A function $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$ is called improperly integrable on $\mathbb{Q}_p^n$ if the exists

$$\lim_{\alpha \to \infty} \int_{B_\alpha} f(x)dx = \lim_{\alpha \to \infty} \sum_{-\infty < \gamma \leqslant \alpha} \int_{S_\gamma} f(x)dx.$$

The above limit is denoted by $\int_{\mathbb{Q}_p^n} f(x)dx$. In particular, if $f \in L^1(\mathbb{Q}_p^n)$, we have

$$\int_{\mathbb{Q}_p^n} f(x)dx = \sum_{\alpha=-\infty}^{\infty} \int_{S_\alpha} f(x)dx.$$

For a more complete introduction to the p-adic analysis, we refer the readers to [24, 33].

Let us write $\|\mathscr{H}\|_{X \to Y}$, the norm of $\mathscr{H}$ between two normed vector spaces X and Y. We also denote $u \lesssim v$ to mean that there is a positive constant C such that $u \leqslant Cv$. The symbol $u \simeq v$ means that $C^{-1}u \leqslant v \leqslant Cu$.

DEFINITION 4. [4, 5] Let $\mathscr{P}(\mathbb{Q}_p^n)$ be the set of all measurable functions $q(\cdot)$ from $\mathbb{Q}_p^n$ to $(1,\infty)$ such that

$$1 < q_- \leqslant q(x) \leqslant q_+ < \infty, \quad \text{for all} \quad x \in \mathbb{Q}_p^n,$$

where $q_- = \operatorname{ess\,inf}_{x\in\mathbb{Q}_p^n} q(x)$ and $q_+ = \operatorname{ess\,sup}_{x\in\mathbb{Q}_p^n} q(x)$.

For $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, the p-adic variable exponent Lebesgue space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is the set of all complex-valued measurable functions f defined on $\mathbb{Q}_p^n$ such that there exists a constant $\eta > 0$ satisfying

$$F_{q(\cdot)}(f/\eta) = \int_{\mathbb{Q}_p^n} \left(\frac{|f(x)|}{\eta}\right)^{q(x)} dx < \infty.$$

The p-adic variable exponent Lebesgue space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ then becomes a normed space equipped with a norm given by

$$\|f\|_{L^{q(\cdot)}} = \inf\left\{\eta > 0 : F_{q(\cdot)}\left(\frac{f}{\eta}\right) \leqslant 1\right\}.$$

For $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, for all $f \in L^{q(\cdot)}(\mathbb{Q}_p^n)$ we have

$$\begin{cases} \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leqslant \max\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}, & \text{if } F_{q(\cdot)}(f) \leqslant C, \\ \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \geqslant \min\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}, & \text{otherwise.} \end{cases} \tag{6}$$

For $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, we set $q'(\cdot)$ such that

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \quad \text{for all } x \in \mathbb{Q}_p^n.$$

Let $\mathbf{C}_0^{\log}(\mathbb{Q}_p^n)$ be the set of all log-Hölder continuous functions $\alpha(\cdot) : \mathbb{Q}_p^n \to \mathbb{R}$ satisfying at the origin,

$$|\alpha(x) - \alpha(0)| \leqslant \frac{C_0^\alpha}{\log\left(e + |x|_p^{-1}\right)}, \quad \text{for all } x \in \mathbb{Q}_p^n.$$

Let $\mathbf{C}_\infty^{\log}(\mathbb{Q}_p^n)$ be the set of all log-Hölder continuous functions $\alpha(\cdot) : \mathbb{Q}_p^n \to \mathbb{R}$ satisfying at infinity,

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{C_\infty^\alpha}{\log(e + |x|_p)}, \quad \text{for all } x \in \mathbb{Q}_p^n,$$

where $\lim_{|x|_p \to \infty} \alpha(x) = \alpha_\infty \in \mathbb{R}$.

We present the definitions of the p-adic variable exponent Herz space $\dot{K}_{\ell,q(\cdot)}^{\alpha(\cdot)}(\mathbb{Q}_p^n)$, the p-adic variable exponent Morrey-Herz space $M\dot{K}_{\ell,q(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{Q}_p^n)$, and p-adic variable exponent central Morrey space $\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)$.

DEFINITION 5. [1] Let $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot): \mathbb{Q}_p^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n)$, and $\ell \in (0,\infty)$. The p-adic variable exponent Herz space is defined as

$$\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n) = \Big\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{Q}_p^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} < \infty \Big\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} = \Big(\sum_{k=-\infty}^{\infty} \|p^{k\alpha(\cdot)} f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \Big)^{1/\ell}.$$

DEFINITION 6. [35] Let $\lambda \in [0,\infty), \ell \in (0,\infty), q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, and $\alpha(\cdot): \mathbb{Q}_p^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n)$. The p-adic variable exponent Morrey-Herz space is defined as

$$M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n) = \Big\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{Q}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} < \infty \Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \Big(\sum_{k=-\infty}^{k_0} \|p^{k\alpha(\cdot)} f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \Big)^{1/\ell}.$$

REMARK 1. If $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n) = \dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$. When both $\alpha(\cdot)$ and $q(\cdot)$ are constant, $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$ is just $M\dot{K}^{\alpha,\lambda}_{\ell,q}(\mathbb{Q}_p^n)$, which is introduced in [7].

By using Theorem 3.8 [1], we prove the following norm equivalence:

THEOREM 2. *If* $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_0^{log}(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$, $\ell \in (0,\infty)$, *and* $\lambda \in [0,\infty)$, *then we obtain*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} \simeq \max\Big\{ \sup_{L \in \mathbb{Z}^- \cup \{0\}} \mathscr{F}_{1,L}, \sup_{L \in \mathbb{Z}^+} \big(\mathscr{F}_{2,L} + \mathscr{F}_{3,L}\big) \Big\}, \text{ for all } f \in M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n).$$

Here

$$\mathscr{F}_{1,L} = p^{-L\lambda} \Big(\sum_{k=-\infty}^{L} p^{k\alpha(0)\ell} \|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \Big)^{1/\ell},$$

$$\mathscr{F}_{2,L} = p^{-L\lambda} \Big(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell} \|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \Big)^{1/\ell},$$

$$\mathscr{F}_{3,L} = p^{-L\lambda} \Big(\sum_{k=0}^{L} p^{k\alpha_\infty\ell} \|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \Big)^{1/\ell}.$$

Proof. For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$, by the definition of the space $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$, we have

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} &= \sup_{L\in\mathbb{Z}} p^{-L\lambda}\Big(\sum_{k=-\infty}^{L}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\ &\simeq \sup_{L\in\mathbb{Z}^-\cup\{0\}} p^{-L\lambda}\Big(\sum_{k=-\infty}^{L}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\ &\quad+\sup_{L\in\mathbb{Z}^+} p^{-L\lambda}\Big(\sum_{k=-\infty}^{-1}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}+\sum_{k=0}^{L}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}.\end{aligned}$$

By $(|a|+|b|)^{1/\ell}\simeq |a|^{1/\ell}+|b|^{1/\ell}$, we get

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} \simeq &\sup_{L\in\mathbb{Z}^-\cup\{0\}} p^{-L\lambda}\Big(\sum_{k=-\infty}^{L}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\ &+\sup_{L\in\mathbb{Z}^+}\Big\{p^{-L\lambda}\Big(\sum_{k=-\infty}^{-1}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\ &+p^{-L\lambda}\Big(\sum_{k=0}^{L}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\Big\}.\end{aligned}\tag{7}$$

On the other hand, since $\alpha\in \mathbf{C}_0^{\log}(\mathbb{Q}_p^n)$, for any $k\in\mathbb{Z}^-\cup\{0\}$ and $x\in S_k$,

$$-k|\alpha(x)-\alpha(0)|\lesssim\frac{-k}{\log(e+1/|x|_p)}=\frac{\log_{\mathrm{p}}(1/|x|_p)}{\log(e+1/|x|_p)}\lesssim 1.$$

Hence

$$-C_0\leqslant k(\alpha(x)-\alpha(0))\leqslant C_0,$$

where the positive constant C_0 is independent of $k\in\mathbb{Z}^-\cup\{0\}$ and $x\in S_k$. This leads to that

$$p^{k\alpha(x)}\simeq p^{k\alpha(0)},\ \text{ for all } k\in\mathbb{Z}^-\cup\{0\} \text{ and } x\in S_k.$$

Consequently, by the definition of the norm of p-adic variable exponent Lebesgue space,

$$\|p^{k\alpha(\cdot)}f\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\simeq p^{k\alpha(0)}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)},\ \text{ for all } k\in\mathbb{Z}^-\cup\{0\}.\tag{8}$$

Then

$$p^{-L\lambda}\Big(\sum_{k=-\infty}^{L}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\simeq\mathscr{F}_{1,L},\ \text{ for all } L\in\mathbb{Z}^-\cup\{0\},$$

and

$$p^{-L\lambda}\Big(\sum_{k=-\infty}^{-1}\|p^{k\alpha(\cdot)}f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\simeq\mathscr{F}_{2,L},\ \text{ for all } L\in\mathbb{Z}^+.$$

Besides, by $\alpha \in \mathbf{C}_\infty^{\log}(\mathbb{Q}_p^n)$, for any $k \in \mathbb{Z}^+$ and $x \in S_k$,

$$k|\alpha(x)-\alpha_\infty| \lesssim \frac{k}{\log(e+|x|_p)} = \frac{\log_{\mathrm{p}}(|x|_p)}{\log(e+|x|_p)} \lesssim 1.$$

This gives

$$-C_\infty \leqslant k(\alpha(x)-\alpha_\infty) \leqslant C_\infty,$$

where the positive constant C_∞ is independent of $k \in \mathbb{Z}^+$ and $x \in S_k$. Clearly,

$$p^{k\alpha(x)} \simeq p^{k\alpha_\infty}, \quad \text{for all } \; k \in \mathbb{Z}^+ \; \text{ and } \; x \in S_k.$$

Thus, by the definition of the norm of p-adic variable exponent Lebesgue space,

$$\|p^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \simeq p^{k\alpha_\infty}\|f\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}, \; \text{ for all } \; k \in \mathbb{Z}^+. \tag{9}$$

Hence

$$p^{-L\lambda}\Big(\sum_{k=0}^{L}\|p^{k\alpha(\cdot)} f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell} \simeq \mathscr{F}_{3,L}, \; \text{ for all } \; L \in \mathbb{Z}^+.$$

By combining these with (7), we see that

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} &\simeq \sup_{L\in\mathbb{Z}^-\cup\{0\}} \mathscr{F}_{1,L} + \sup_{L\in\mathbb{Z}^+}\{\mathscr{F}_{2,L}+\mathscr{F}_{3,L}\} \\ &\simeq \max\Big\{\sup_{L\in\mathbb{Z}^-\cup\{0\}} \mathscr{F}_{1,L}, \sup_{L\in\mathbb{Z}^+}\Big(\mathscr{F}_{2,L}+\mathscr{F}_{3,L}\Big)\Big\}.\end{aligned}$$

Thus, we finish the proof of this theorem. □

THEOREM 3. *If $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_0^{log}(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$, $\ell \in (0,\infty)$, then we have*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} \simeq \Big(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell} + \Big(\sum_{k=0}^{\infty} p^{k\alpha_\infty\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}.$$

Proof. In view of $\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n) = M\dot{K}^{\alpha(\cdot),0}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$, $\max\{|a|,|b|\} \simeq |a|+|b|$, and Theorem 2, we deduce

$$\begin{aligned}\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} &\simeq \max\Big\{\sup_{L\in\mathbb{Z}^-\cup\{0\}} \mathscr{F}_{1,L}, \sup_{L\in\mathbb{Z}^+}\Big(\mathscr{F}_{2,L}+\mathscr{F}_{3,L}\Big)\Big\} \\ &\simeq \sup_{L\in\mathbb{Z}^-\cup\{0\}} \mathscr{F}_{1,L} + \sup_{L\in\mathbb{Z}^+}\Big(\mathscr{F}_{2,L}+\mathscr{F}_{3,L}\Big) \\ &= \Big(\sum_{k=-\infty}^{0} p^{k\alpha(0)\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell} + \Big(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell} \\ &\quad + \Big(\sum_{k=0}^{\infty} p^{k\alpha_\infty\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}.\end{aligned}$$

Next, by simple calculations, we get

$$\begin{aligned}
&\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}\\
&\simeq \Big(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}+\|f\chi_0\|_{L^{q(\cdot)}(\mathbb{Q}^n)}+\Big(\sum_{k=1}^{\infty} p^{k\alpha_\infty\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\
&\simeq \Big(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}+\Big(\sum_{k=0}^{\infty} p^{k\alpha_\infty\ell}\|f\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}.
\end{aligned}$$

Hence, we complete the proof of Theorem 3. □

From the definition of p-adic Morrey-Herz spaces with variable exponent, we estimate the following result.

LEMMA 1. *If $q(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{Q}_p^n)\cap\mathbf{C}_0^{log}(\mathbb{Q}_p^n)\cap\mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$, $\ell\in(0,\infty)$, and $\lambda\in[0,\infty)$, then*

$$\big\|f\chi_j\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\lesssim p^{j(\lambda-\alpha(0))}\big\|f\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)},\ \textit{for all}\ \ j\in\mathbb{Z}^-\cup\{0\},$$
$$\big\|f\chi_j\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\lesssim p^{j(\lambda-\alpha_\infty)}\big\|f\big\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)},\ \textit{for all}\ \ j\in\mathbb{Z}^+.$$

Proof. By using (8), for any $j\in\mathbb{Z}^-\cup\{0\}$, we get

$$\begin{aligned}
\big\|f\chi_j\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}&=p^{-j\alpha(0)}\Big(p^{j\alpha(0)\ell}\|f\chi_j\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\
&\leqslant p^{-j\alpha(0)}\Big(\sum_{i=-\infty}^{j} p^{i\alpha(0)\ell}\|f\chi_i\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\
&\lesssim p^{j(\lambda-\alpha(0))}\Big(p^{-j\lambda}\Big(\sum_{i=-\infty}^{j}\|p^{i\alpha(\cdot)}f\chi_i\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\Big)\\
&\leqslant p^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

By estimating as above and using (9), for any $j\in\mathbb{Z}^+$,

$$\big\|f\chi_j\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\lesssim p^{j(\lambda-\alpha_\infty)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.$$

Therefore, the proof of Lemma 1 is complete. □

DEFINITION 7. Assume that $r>0$, $q(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$. The p-adic variable exponent central Morrey space $\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)$ is defined by

$$\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)=\Big\{f\in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{Q}_p^n):\|f\|_{\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)}<\infty\Big\},$$

where

$$\|f\|_{\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)}=\sup_{k\in\mathbb{Z}}\frac{1}{|B_k|^r}\|f\|_{L^{q(\cdot)}(B_k)}.$$

REMARK 2. If $q(\cdot)$ is constant and $r = 1/q + \lambda$, then $\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n) = \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined in [36]. Moreover, it is not hard to see that $\dot{B}^{q,-1/q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$.

DEFINITION 8. Let $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$. Then the Hardy-Littlewood maximal operator $\mathscr{M}$ is defined by

$$\mathscr{M}(f)(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{p^{\gamma n}} \int_{B_\gamma(x)} |f(y)|dy.$$

The set $\mathfrak{B}(\mathbb{Q}_p^n)$ consists of all measurable functions $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ satisfying that the operator $\mathscr{M}$ is bounded on $L^{q(\cdot)}(\mathbb{Q}_p^n)$.

By using Lemma 1 and Lemma 2 in the paper [22], we have the following result.

LEMMA 2. *Let* $q(\cdot) \in \mathfrak{B}(\mathbb{Q}_p^n)$.

(i) *Then we have a positive constant* $\delta \in (0,1)$ *and*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^\delta,$$

for all balls B *in* $\mathbb{Q}_p^n$ *and all measurable subsets* $S \subset B$.

(ii) *Then we obtain*

$$\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \simeq p^{kn},$$

for all $k \in \mathbb{Z}$.

3. The main results

Now, we state the first main result in this paper.

LEMMA 3. *Let* $\lambda \in (0,\infty)$, $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_0^{log}(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$, $\ell \in (1,\infty)$, *and either* $\lambda = \alpha(0)$ *or* $q_+ = q_-$. *Then,* $\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} \in (0,\infty)$.

Here

$$f_0(x) = |x|_p^{-\alpha(0) - \frac{n}{q(x)} + \lambda}.$$

Proof. It is clear to see that $\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} > 0$. Now, we prove that

$$\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} < \infty.$$

Indeed, we calculate

$$F_{q(\cdot)}(f_0\chi_k) = \int\limits_{S_k} |x|^{(\lambda-\alpha(0))q(x)-n} dx \lesssim p^{\max\{k(\lambda-\alpha(0))q_+,\, k(\lambda-\alpha(0))q_-\}}.$$

Thus, by the inequality (6),

$$\|f_0\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim p^{\max\{k(\lambda-\alpha(0))q_-/q_+,\,k(\lambda-\alpha(0))q_+/q_-\}}. \tag{10}$$

Besides, by Theorem 2,

$$\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \max\Big\{\sup_{k_0\in\mathbb{Z}^-\cup\{0\}} \mathscr{G}_1,\ \sup_{k_0\in\mathbb{Z}^+}(\mathscr{G}_2+\mathscr{G}_3)\Big\}, \tag{11}$$

with

$$\mathscr{G}_1 = p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\alpha(0)\ell}\|f_0\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell},$$

$$\mathscr{G}_2 = p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell}\|f_0\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell},$$

$$\mathscr{G}_3 = p^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} p^{k\alpha_\infty\ell}\|f_0\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}.$$

To present the next section, we set

$$\kappa := \alpha_\infty + \max\{(\lambda-\alpha(0))q_-/q_+,\,(\lambda-\alpha(0))q_+/q_-\}.$$

Notice that, by the conditions $\alpha(0)=\lambda>0$ or $q_+=q_-$,

$$\alpha(0)+\min\{(\lambda-\alpha(0))q_-/q_+,\,(\lambda-\alpha(0))q_+/q_-\}-\lambda = 0.$$

From this, by (10) and $\max\{k.a,k.b\}=k.\min\{a,b\}$ with $k\in\mathbb{Z}^-\cup\{0\}$, we infer

$$\begin{aligned}\mathscr{G}_1 &\lesssim p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\alpha(0)\ell+\max\{k(\lambda-\alpha(0))q_-/q_+,\,k(\lambda-\alpha(0))q_+/q_-\}\ell}\Big)^{1/\ell}\\ &\lesssim p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\ell\left(\alpha(0)+\min\{(\lambda-\alpha(0))q_-/q_+,\,(\lambda-\alpha(0))q_+/q_-\}\right)}\Big)^{1/\ell}\\ &\lesssim p^{k_0(\alpha(0)+\min\{(\lambda-\alpha(0))q_-/q_+,\,(\lambda-\alpha(0))q_+/q_-\}-\lambda)}\\ &= 1.\end{aligned} \tag{12}$$

Similarly, we also have

$$\begin{aligned}\mathscr{G}_2 &\lesssim p^{-k_0\lambda-(\alpha(0)+\min\{(\lambda-\alpha(0))q_-/q_+,\,(\lambda-\alpha(0))q_+/q_-\})}\\ &\lesssim p^{-k_0\lambda}.\end{aligned} \tag{13}$$

By using the inequality (10) and the definition of κ above,

$$\begin{aligned}\mathscr{G}_3 &\lesssim p^{-k_0\lambda}\Big(\sum_{k=1}^{k_0} p^{k\ell\kappa}\Big)^{1/\ell} \lesssim \begin{cases} p^{-k_0\lambda}(k_0^{1/\ell}+1), & \text{if } \kappa=0,\\ p^{-k_0\lambda}+p^{-k_0(\lambda-\kappa)}, & \text{otherwise,}\end{cases}\\ &\lesssim p^{-k\lambda}(k_0^{1/\ell}+1)+p^{-k_0(\lambda-\kappa)}.\end{aligned} \tag{14}$$

In view of the conditions $\lambda = \alpha(0) > 0$ or $q_+ = q_-$, we see that $\zeta = \lambda - \kappa \geqslant 0$. As a consequence, by (11)–(14),

$$\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \max\Big\{ \sup_{k_0\in\mathbb{Z}^-\cup\{0\}} 1,\ \sup_{k_0\in\mathbb{Z}^+} \big(p^{-k_0\lambda}(k_0^{1/\ell}+2)+p^{-k_0\zeta}\big)\Big\}$$
$$< \infty.$$

This finishes our proof. $\square$

THEOREM 4. *Let* $\lambda \in (0,\infty)$, $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ *such that* $q(s^{-1}(t)\cdot) = q(\cdot)$ *for almost everywhere* $t \in \operatorname{supp}(\psi)$, $\ell \in (1,\infty)$, *and* $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_0^{log}(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$ *with* $\alpha(0) - \alpha_\infty \geqslant 0$.

(i) *If*

$$\mathscr{L}_{1,\max} = \int\limits_{\mathbb{Z}_p^*} \psi(t) \times \max\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\} \times \max\{|s(t)|_p^{\lambda-\alpha(0)}, |s(t)|_p^{\lambda-\alpha_\infty}\}dt < \infty,$$

then $\mathscr{H}^p_{\psi,s}$ *is bounded from* $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$ *to* $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$.

(ii) *Suppose that* $\mathscr{H}^p_{\psi,s}$ *is bounded from* $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$ *to* $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$, *and either* $\lambda = \alpha(0)$ *or* $q_+ = q_-$. *We have*

$$\mathscr{L}_{1,\min} = \int\limits_{\mathbb{Z}_p^*} \psi(t) \times \min\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\}|s(t)|_p^{\lambda-\alpha(0)}dt < \infty.$$

Moreover,

$$\mathscr{L}_{1,\min} \leqslant \|\mathscr{H}^p_{\psi,s}\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)\to M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.$$

Proof. First, we will prove (i). By the Minkowski inequality, we give

$$\|\mathscr{H}^p_{\psi,s}(f)\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \int\limits_{\mathbb{Z}_p^*} \psi(t)\|f(s(t)\cdot)\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}dt. \tag{15}$$

Next, for $\eta > 0$ and $t \in \mathbb{Z}_p^*$ such that $|s(t)|_p \neq 0$, we have

$$\int\limits_{\mathbb{Q}_p^n} \left(\frac{|f(s(t)x)|\chi_k(x)}{\eta}\right)^{q(x)} dx \tag{16}$$

where $\ell_0 = \ell_0(t) \in \mathbb{Z}$ such that $|s(t)|_p = p^{\ell_0}$. This leads to

$$\|f(s(t)\cdot)\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leqslant \max\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\}\|f\chi_{k+\ell_0}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

Hence, by (15), we estimate

$$\left\|\mathscr{H}^p_{\psi,s}(f)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{Q}^n_p)} \lesssim \int\limits_{\mathbb{Z}^*_p} \psi(t)\times \max\left\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\right\}\|f\chi_{k+\ell_0}\|_{L^{q(\cdot)}(\mathbb{Q}^n_p)}dt. \quad (17)$$

On the other hand, by using Lemma 1 and $|s(t)|_p = p^{\ell_0}$, we infer

$$\begin{aligned}&\left\|f\chi_{k+\ell}\right\|_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\\ &\lesssim p^{\max\{(k+\ell_0)(\lambda-\alpha(0)),(k+\ell_0)(\lambda-\alpha_\infty)\}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}\\ &= \max\left\{|s(t)|_p^{\lambda-\alpha(0)}, |s(t)|_p^{\lambda-\alpha_\infty}\right\}p^{\max\{k(\lambda-\alpha(0)),k(\lambda-\alpha_\infty)\}}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}.\end{aligned}$$

Then

$$\left\|\mathscr{H}^p_{\psi,s}(f)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{Q}^n_p)} \lesssim p^{\max\{k(\lambda-\alpha(0)),k(\lambda-\alpha_\infty)\}}\mathscr{L}_{1,\max}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}. \quad (18)$$

By using Theorem 2, we compose

$$\left\|\mathscr{H}^p_{\psi,s}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)} \lesssim \max\left\{\sup_{k_0\in\mathbb{Z}^-\cup\{0\}}\mathscr{T}_1,\ \sup_{k_0\in\mathbb{Z}^+}(\mathscr{T}_2+\mathscr{T}_3)\right\}. \quad (19)$$

Here

$$\begin{aligned}\mathscr{T}_1 &= p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha(0)\ell}\left\|\mathscr{H}^p_{\psi,s}(f)\chi_k\right\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\right)^{1/\ell},\\ \mathscr{T}_2 &= p^{-k_0\lambda}\left(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell}\left\|\mathscr{H}^p_{\psi,s}(f)\chi_k\right\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\right)^{1/\ell},\\ \mathscr{T}_3 &= p^{-k_0\lambda}\left(\sum_{k=0}^{k_0} p^{k\alpha_\infty\ell}\left\|\mathscr{H}^p_{\psi,s}(f)\chi_k\right\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\right)^{1/\ell}.\end{aligned}$$

Now, by applying the inequality (18), we have

$$\mathscr{T}_1 \leqslant \mathscr{L}_{1,\max}\times p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\ell\min\{\lambda,\lambda-\alpha_\infty+\alpha(0)\}}\right)^{1/\ell}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}.$$

Thus, by $\min\{\lambda,\lambda-\alpha_\infty+\alpha(0)\}>0$, $\alpha(0)-\alpha_\infty\geqslant 0$, and $k_0\in\mathbb{Z}^-\cup\{0\}$, we have

$$\begin{aligned}\mathscr{T}_1 &\lesssim \mathscr{L}_{1,\max}\times p^{k_0(\min\{\lambda,\lambda-\alpha_\infty+\alpha(0)\}-\lambda)}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}\\ &\lesssim \mathscr{L}_{1,\max}\times\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}.\end{aligned} \quad (20)$$

By estimating as $\mathscr{T}_1$, we also obtain

$$\mathscr{T}_2 \lesssim \mathscr{L}_{1,\max}\times p^{-k_0\lambda}\left\|f\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}. \quad (21)$$

From $\alpha(0)-\alpha_\infty \geqslant 0$, we see that $\max\{\lambda-\alpha(0)+\alpha_\infty,\lambda\}=\lambda$. Then, $\mathscr{T}_3$ is controlled as follows:

$$\begin{aligned}
\mathscr{T}_3 &\lesssim \mathscr{L}_{1,\max}\times p^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} p^{k\alpha_\infty\ell+\max\{k(\lambda-\alpha(0))\ell,k(\lambda-\alpha_\infty)\ell\}}\Big)^{1/\ell}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}\\
&= \mathscr{L}_{1,\max}\times p^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} p^{\max\{k\ell(\lambda-\alpha(0)+\alpha_\infty),k\ell\lambda\}}\Big)^{1/\ell}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}\\
&= \mathscr{L}_{1,\max}\times p^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} p^{k\ell\lambda}\Big)^{1/\ell}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim \mathscr{L}_{1,\max}\times p^{-k_0\lambda}\Big(p^{k_0\lambda}+1\Big)\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}. \qquad (22)
\end{aligned}$$

Hence, from (19)–(22), we obtain

$$\|\mathscr{H}^p_{\psi,s}(f)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \mathscr{L}_{1,\max}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)},$$

which concludes the proof of case (i) of Theorem 4. We will consider the proof for case (ii). Assume that $\mathscr{H}^p_{\psi,s}$ is a bounded operator on $M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$. Let us choose the function f_0 as in Lemma 3. Then

$$\mathscr{H}^p_{\psi,s}(f_0)(x)=\Big(\int_{\mathbb{Z}_p^*}\psi(t)\times|s(t)|_p^{-\alpha(0)-\frac{n}{q(x)}+\lambda}dt\Big)f_0(x)\geqslant \mathscr{L}_{1,\min}\times f_0(x).$$

By Lemma 3, we have $\|f_0\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}\in(0,\infty)$. Hence,

$$\mathscr{L}_{1,\min}\leqslant \|\mathscr{H}^p_{\psi,s}\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)\to M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}<\infty.$$

This completes our proof. □

Let us next give the necessary and sufficient conditions for the boundedness of Hardy-Cesàro operators on variable exponent Herz spaces and central Morrey spaces.

THEOREM 5. *Let the assumptions of Theorem 4 be fulfilled, and* $\alpha(0)=\alpha_\infty$.
(i) *If*

$$\mathscr{L}_{2,\max}=\int_{\mathbb{Z}_p^*}\psi(t)\times\max\Big\{|s(t)|_p^{\frac{-n}{q_+}},|s(t)|_p^{\frac{-n}{q_-}}\Big\}|s(t)|_p^{-\alpha(0)}dt<\infty,$$

then $\mathscr{H}^p_{\psi,s}$ *is a bounded operator from* $\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$ *to itself.*

(ii) *Suppose that* $\mathscr{H}^p_{\psi,s}$ *is bounded from* $\dot{K}^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{Q}_p^n)$ *to itself, and either* $\alpha_\infty=0$ *or* $q_+=q_-$. *Then we obtain*

$$\mathscr{L}_{2,\min}=\int_{\mathbb{Z}_p^*}\psi(t)\times\min\Big\{|s(t)|_p^{\frac{-n}{q_+}},|s(t)|_p^{\frac{-n}{q_-}}\Big\}|s(t)|_p^{-\alpha(0)}dt<\infty.$$

Moreover,

$$\mathscr{L}_{2,\min} \lesssim \|\mathscr{H}^p_{\psi,s}\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)\to\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}.$$

Proof. Now, we will present the proof of the case (i). Let us give the function f in the Herz space $\dot{K}^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{Q}^n_p)$. From Theorem 3 and the condition $\alpha(0)=\alpha_\infty$, we get

$$\begin{aligned}\big\|\mathscr{H}^p_{\psi,s}(f)\big\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)} &\lesssim \Big(\sum_{k=-\infty}^{\infty} p^{k\alpha(0)\ell}\big\|\mathscr{H}^p_{\psi,s}(f)\chi_k\big\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\Big)^{1/\ell}\\ &:=\mathscr{U}.\end{aligned}\tag{23}$$

By (17) and the Minkowski inequality, one has

$$\mathscr{U}\lesssim\int\limits_{\mathbb{Z}^*_p}\psi(t)\times\max\Big\{|s(t)|_p^{\frac{-n}{q_+}},|s(t)|_p^{\frac{-n}{q_-}}\Big\}\Big\{\sum_{k=-\infty}^{\infty}p^{k\alpha(0)\ell}\big\|f\chi_{k+\ell_0}\big\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\Big\}^{1/\ell}dt.\tag{24}$$

On the other hand, by $|s(t)|_p=p^{\ell_0(t)}$, we estimate

$$\begin{aligned}\Big(\sum_{k=-\infty}^{\infty}p^{k\alpha(0)\ell}\big\|f\chi_{k+\ell_0}\big\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\Big)^{1/\ell} &= \Big(\sum_{m=-\infty}^{\infty}p^{(m-\ell_0)\alpha(0)\ell}\|f\chi_m\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}^n_p)}\Big)^{1/\ell}\\ &= p^{-\ell_0\alpha(0)}\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}\\ &= |s(t)|_p^{-\alpha(0)}\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}.\end{aligned}$$

Thus, by (23) and (24), we obtain

$$\big\|\mathscr{H}^p_{\psi,s}(f)\big\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}\lesssim\mathscr{L}_{2,\max}\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}.$$

Hence, the proof of case (i) is finished.

Next step, we will prove the case (ii). For each $r\in\mathbb{Z}^+$, let us determine the function f_r as follows

$$f_r(x)=\begin{cases}0, & \text{if } |x|_p\leqslant 1,\\ |x|_p^{-\alpha_\infty-\frac{n}{q(x)}-p^{-r}}, & \text{otherwise.}\end{cases}$$

It is clear to see that $\|f_r\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}^n_p)}>0$. Moreover,

$$F_{q(\cdot)}(f_r\chi_k)=\int\limits_{S_k}|x|_p^{(-\alpha_\infty-p^{-r})q(x)-n}dx=\int\limits_{S_0}p^{k\{(-\alpha_\infty-p^{-r})q(p^{-k}z)-n\}}p^{kn}dz.$$

Thus

$$p^{\min\{a_r,b_r\}}\lesssim F_{q(\cdot)}(f_r\chi_k)\lesssim p^{\max\{a_r,b_r\}},\ \text{ for all } k\in\mathbb{Z}^+,$$

where $a_r := -k(\alpha_\infty + p^{-r})q_-$ and $b_r := -k(\alpha_\infty + p^{-r})q_+$. Combining this with (6), we get

$$p^{\min\{a_r/q_+, b_r/q_-\}} \lesssim \|f_r\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim p^{\max\{a_r/q_+, b_r/q_-\}}, \text{ for all } k \in \mathbb{Z}^+.$$

Let us set

$$\zeta_r = \alpha_\infty - \min\{(\alpha_\infty + p^{-r})q_-/q_+, (\alpha_\infty + p^{-r})q_+/q_-\},$$
$$\beta_r = \alpha_\infty - \max\{(\alpha_\infty + p^{-r})q_-/q_+, (\alpha_\infty + p^{-r})q_+/q_-\}.$$

By using either $\alpha_\infty = 0$ or $q_+ = q_-$, we see that $\zeta_r, \beta_r < 0$, and

$$\lim_{r\to\infty} \zeta_r = 0, \ \lim_{r\to\infty} \beta_r = 0, \ \lim_{r\to\infty} r\beta_r = 0 \text{ and } \lim_{r\to\infty} \frac{\beta_r}{\zeta_r} = \begin{cases} 1, & \text{if } q_+ = q_-, \\ q_+^2/q_-^2, & \text{if } \alpha_\infty = 0. \end{cases} \tag{25}$$

From these, by Theorem 3, we estimate

$$\begin{aligned} \|f_r\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \Big(\sum_{k=1}^{\infty} p^{k\alpha_\infty\ell} \|f_r\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell} \\ &\lesssim \Big(\sum_{k=1}^{\infty} p^{k\ell\zeta_r}\Big)^{1/\ell} \lesssim \frac{p^{\zeta_r}}{\big(1-p^{\zeta_r\ell}\big)^{1/\ell}}, \end{aligned} \tag{26}$$

which leads that the function f_r belongs to the Herz space $\dot{K}^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{Q}_p^n)$. On the other hand, for any $d \in \mathbb{Z}^+$, we have

$$\Big(\sum_{k=d}^{\infty} p^{k\alpha_\infty\ell} \|f_r\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell} \gtrsim \Big(\sum_{k=d}^{\infty} p^{k\ell\beta_r}\Big)^{1/\ell} = \frac{p^{d\beta_r}}{\big(1-p^{\beta_r\ell}\big)^{1/\ell}}. \tag{27}$$

Let us denote $V_{s,x} = \{t \in \mathbb{Z}_p^* : |s(t)x|_p \geqslant p\}$ and $U_{s,r} = \{t \in \mathbb{Z}_p^* : |s(t)|_p \geqslant p^{-r}\}$. This leads to

$$U_{s,r} \subset V_{s,x}, \text{ for all } x \in \mathbb{Q}_p^n \setminus B_r.$$

In addition, for $t \in U_{r,s}$, we have $|s(t)|_p^{-\frac{n}{q(\cdot)}} \geqslant \min\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\}$. Therefore, by assuming $\alpha(0) = \alpha_\infty$,

$$\begin{aligned} \mathscr{H}^p_{\psi,s}(f)(x) &\geqslant \int_{V_{s,x}} \psi(t) \times |s(t)x|_p^{-\alpha_\infty - \frac{n}{q(x)} - p^{-r}} dt \\ &\gtrsim \Big(\int_{U_{s,r}} \psi(t) \times |s(t)|_p^{-\alpha_\infty - \frac{n}{q(x)} - p^{-r}} dt\Big) f_r(x)\chi_{\mathbb{Q}_p^n\setminus B_r}(x) \\ &\geqslant \Big(\int_{U_{s,r}} \psi(t) \times \min\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\} |s(t)|_p^{-\alpha(0)-p^{-r}} dt\Big) f_r(x)\chi_{\mathbb{Q}_p^n\setminus B_r}(x). \end{aligned}$$

For all $r \in \mathbb{Z}^+$ and $t \in \mathbb{Z}_p^*$, we set

$$g_r(t) = \psi(t) \times \min\Big\{|s(t)|_p^{\frac{-n}{q+}}, |s(t)|_p^{\frac{-n}{q-}}\Big\}|s(t)|_p^{-\alpha(0)}\Big(|s(t)|_p^{-1}p^{-r}\Big)^{p^{-r}}\chi_{U_{s,r}}(t)$$

and

$$g(t) = \psi(t) \times \min\Big\{|s(t)|_p^{\frac{-n}{q+}}, |s(t)|_p^{\frac{-n}{q-}}\Big\}|s(t)|_p^{-\alpha(0)}.$$

Consequently, by Theorem 3 and (26)–(27) with $r \in \mathbb{Z}^+$, we get

$$\begin{aligned}\|\mathscr{H}^p_{\psi,s}(f)\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} &\gtrsim \Big(\int\limits_{U_{s,r}} \psi(t)\times \min\Big\{|s(t)|_p^{\frac{-n}{q+}}, |s(t)|_p^{\frac{-n}{q-}}\Big\}|s(t)|_p^{-\alpha(0)-p^{-r}}dt\Big)\\ &\quad\times\Big(\sum_{k=r}^{\infty} p^{k\alpha_\infty \ell}\|f_r\chi_k\|^{\ell}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\ &\gtrsim \frac{(1-p^{\zeta_r\ell})^{1/\ell}p^{r\beta_r}}{(1-p^{\beta_r\ell})^{1/\ell}p^{\zeta_r}}p^{rp^{-r}}\times\Big(\int_{\mathbb{Z}_p^*} g_r(t)dt\Big)\|f\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}. \end{aligned}\tag{28}$$

Besides, by (25), it's not hard to show that

$$\lim_{r\to\infty}\frac{(1-p^{\zeta_r\ell})^{1/\ell}p^{r\beta_r}}{(1-p^{\beta_r\ell})^{1/\ell}p^{\zeta_r}}p^{rp^{-r}} = \begin{cases}1, & \text{if } q_+ = q_-,\\ (q_+^2/q_-^2)^{1/\ell}, & \text{if } \alpha_\infty = 0.\end{cases}.$$

Thus, by (28) and the boundedness of $\mathscr{H}^p_{\psi,s}$ from $\dot{K}^{\alpha(\cdot)}_{p,q(\cdot)}(\mathbb{Q}_p^n)$ to itself,

$$\sup_{r\in\mathbb{Z}^+}\int_{\mathbb{Z}_p^*} g_r(t)dt < C.\|\mathscr{H}^p_{\psi,s}\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)\to\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} < \infty,$$

where the positive constant C depends on α_∞, q_+, q_-.

Combining this with $g_r(\cdot) \geqslant 0$, for all $r \in \mathbb{Z}^+$, $\lim\limits_{r\to\infty} g_r(\cdot) = g(\cdot)$, and Fatou's lemma, we infer

$$\int_{\mathbb{Z}_p^*} g(t)dt \leqslant \liminf_{r\to\infty}\int_{\mathbb{Z}_p^*} g_r(t)dt \leqslant C.\|\mathscr{H}^p_{\psi,s}\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)\to\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.$$

Hence

$$\mathscr{L}_{2,\min} \lesssim \|\mathscr{H}^p_{\psi,s}\|_{\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)\to\dot{K}^{\alpha(\cdot)}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} < \infty.$$

This gives the proof of this theorem. □

If $q(\cdot)$ is constant, we obtain the following result.

LEMMA 4. *If $r \in (0,\infty)$ and $q \in (1,\infty)$, then $\|h_0\|_{\dot{M}^{r,q}(\mathbb{Q}_p^n)} \simeq 1$. Here*

$$h_0(x) = |x|_p^{nr-\frac{n}{q}}.$$

Proof. By simple calculation,

$$\|h_0\|_{L^q(B_k)} = \Big(\int\limits_{B_k} |x|_p^{nrq-n} dx\Big)^{1/q} \simeq p^{knr}.$$

Then

$$\|h_0\|_{\dot{M}^{r,q}(\mathbb{Q}_p^n)} = \sup_{k\in\mathbb{Z}} \frac{1}{|B_k|^r}\|h_0\|_{L^q(B_k)} \simeq 1.$$

This completes our proof. □

THEOREM 6. *Let $r\in(0,\infty)$, $q(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$ such that $q(s^{-1}(t)\cdot) = q(\cdot)$ for almost everywhere $t\in \operatorname{supp}(\psi)$.*

(i) *If*

$$\mathscr{L}_{3,\max} = \int_{\mathbb{Z}_p^*} \psi(t)\times|s(t)|_p^{nr}\times\max\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\}dt < \infty,$$

then the operator $\mathscr{H}_{\psi,s}^p$ is bounded from $\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)$ to itself.

(ii) *If $\mathscr{H}_{\psi,s}^p$ is bounded from $\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)$ to itself and $q_+ = q_-$, we then have*

$$\mathscr{L}_3 = \int_{\mathbb{Z}_p^*} \psi(t)\times|s(t)|_p^{nr-\frac{n}{q}} dt < \infty,$$

and

$$\|\mathscr{H}_{\psi,s}^p\|_{\dot{M}^{r,q}(\mathbb{Q}_p^n)\to\dot{M}^{r,q}(\mathbb{Q}_p^n)} \simeq \mathscr{L}_3.$$

Proof. (i) By estimating as (17) above, we have

$$\big\|\mathscr{H}_{\psi,s}^p(f)\big\|_{L^{q(\cdot)}(B_k)} \lesssim \int\limits_{\mathbb{Z}_p^*} \psi(t)\times\max\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\}\big\|f\big\|_{L^{q(\cdot)}(B_{k+\ell_0})}dt, \tag{29}$$

where $\ell_0 = \ell_0(t)\in\mathbb{Z}$ such that $|s(t)|_p = p^{\ell_0}$. Hence, we get

$$\begin{aligned} \|\mathscr{H}_{\psi,s}^p(f)\|_{\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)} &= \sup_{k\in\mathbb{Z}} \frac{1}{|B_k|^r}\|\mathscr{H}_{\psi,s}^p(f)\|_{L^{q(\cdot)}(B_k)} \\ &\lesssim \sup_{k\in\mathbb{Z}}\Big(\int\limits_{\mathbb{Z}_p^*}\psi(t)\times\max\Big\{|s(t)|_p^{\frac{-n}{q_+}}, |s(t)|_p^{\frac{-n}{q_-}}\Big\}\frac{|B_{k+\ell_0}|^r}{|B_k|^r}dt\Big)\|f\|_{\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)} \\ &= \mathscr{L}_{3,\max}\|f\|_{\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)}. \end{aligned} \tag{30}$$

Thus the case (i) is proved.

(ii) Suppose that $\mathscr{H}^p_{\psi,s}$ is a bounded operator from $\dot{M}^{r,q(\cdot)}(\mathbb{Q}_p^n)$ to itself and the relation $q_+ = q_-$. We choose the function h_0 as in Lemma 4. Then,

$$\mathscr{H}^p_{\psi,s}(h_0)(x) = \Big(\int_{\mathbb{Z}_p^*} \psi(t) \times |s(t)|_p^{nr-\frac{n}{q}} dt\Big)|x|_p^{nr-\frac{n}{q}} = \mathscr{L}_3 \times h_0(x).$$

By Lemma 4,

$$\|\mathscr{H}^p_{\psi,s}(h_0)\|_{\dot{M}^{r,q}(\mathbb{Q}_p^n)} = \mathscr{L}_3 \|h_0\|_{\dot{M}^{r,q}(\mathbb{Q}_p^n)} \simeq \mathscr{L}_3.$$

As a consequence, we get

$$\mathscr{L}_3 \leqslant \|\mathscr{H}^p_{\psi,s}\|_{\dot{M}^{r,q}(\mathbb{Q}_p^n)\to \dot{M}^{r,q}(\mathbb{Q}_p^n)} < \infty.$$

Therefore, the proof of the theorem is ended. □

Finally, by using Theorem 2 again, we also establish the following useful result.

THEOREM 7. *Let $\xi, \ell \in (1,\infty)$, $\lambda \in (0,\infty)$, $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, and $s(\cdot) \in \mathfrak{B}(\mathbb{Q}_p^n)$. Also, let $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_0^{log}(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$ with $\lambda - \alpha(0) \geqslant 0$ and $\lambda - \alpha_\infty \geqslant 0$. Assume that $\Omega \in L^\xi(S_0)$ satisfies (4) and the following condition holds:*

$$\frac{1}{\xi} + \frac{1}{q(\cdot)} = \frac{1}{s(\cdot)}. \tag{31}$$

Then we have

$$\|\mathscr{H}^p_\Omega(f) \times |\cdot|_p^{-n/\xi}\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,s(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|\Omega\|_{L^\xi(S_0)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}, \text{ for all } f \in M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n).$$

Proof. For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)$, we compose

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) := \sum_{j=-\infty}^{\infty} f_j(x).$$

From (31) and the Hölder inequality, we have

$$\begin{aligned} |\mathscr{H}^p_\Omega(f)(x)\chi_k(x)| &\leqslant \frac{1}{|x|_p^n}\Big(\int_{B_k} |\Omega(x-t)| \times |f(t)|dt\Big)\chi_k(x) \\ &\leqslant p^{-kn} \sum_{j=-\infty}^{k} \|\Omega(x-\cdot)\|_{L^\xi(S_j)} \|f_j\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|\chi_j\|_{L^{s'(\cdot)}(\mathbb{Q}_p^n)} \chi_k(x). \end{aligned} \tag{32}$$

On the other hand, having $j \leqslant k$, $x \in S_k$, and $t \in S_j$, we learn from the ultra triangle inequality that

$$|x-t|_p \leqslant \max\{|x|_p, |t|_p\} = p^k.$$

Note that Ω is homogenous of degree zero and $\Omega \in L^{\xi}(S_0)$. Hence we have

$$\begin{aligned}\|\Omega(x-\cdot)\|_{L^{\xi}(S_j)} &= \Big(\int\limits_{S_j} |\Omega(x-t)|^{\xi} dt\Big)^{1/\xi} \leqslant \Big(\int\limits_{B_k} |\Omega(u)|^{\xi} du\Big)^{1/\xi}\\ &= \Big(\sum_{\theta=-\infty}^{k} \int\limits_{S_\theta} |\Omega(u)|^{\xi} du\Big)^{1/\xi} = \Big(\sum_{\theta=-\infty}^{k} \int\limits_{S_0} |\Omega(p^{-\theta}v)|^{\xi} p^{\theta n} dv\Big)^{1/\xi}\\ &= \|\Omega\|_{L^{\xi}(S_0)} \Big(\sum_{\theta=-\infty}^{k} p^{\theta n}\Big)^{1/\xi} \lesssim p^{nk/\xi} \|\Omega\|_{L^{\xi}(S_0)}.\end{aligned}$$

On the other hand, by $s(\cdot) \in \mathfrak{B}(\mathbb{Q}_p^n)$, we infer $s'(\cdot) \in \mathfrak{B}(\mathbb{Q}_p^n)$. By combining this with Lemma 2, one has

$$p^{-kn} \|\chi_{B_k}\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|\chi_{B_k}\|_{L^{s'(\cdot)}(\mathbb{Q}_p^n)} \quad \text{and} \quad \frac{\|\chi_j\|_{L^{s'(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_k}\|_{L^{s'(\cdot)}(\mathbb{Q}_p^n)}} \lesssim p^{(j-k)n\delta},$$

for some $\delta \in (0,1)$.

From these, by (32), we immediately have

$$\begin{aligned}&\|\mathscr{H}_{\Omega}^{p}(f) \times |\cdot|_p^{-n/\xi}\|_{L^{s(\cdot)}(S_k)}\\ &\lesssim p^{-kn} \|\Omega\|_{L^{\xi}(S_0)} \Big(\sum_{j=-\infty}^{k} \|f_j\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|\chi_j\|_{L^{s'(\cdot)}(\mathbb{Q}_p^n)}\Big) \|\chi_{B_k}\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|\Omega\|_{L^{\xi}(S_0)} \Big(\sum_{j=-\infty}^{k} p^{(j-k)n\delta} \|f_j\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big).\end{aligned}$$

By Theorem 2, we deduce

$$\|\mathscr{H}_{\Omega}^{p}(f) \times |\cdot|_p^{-n/\xi}\|_{M\dot{K}_{\ell,s(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{Q}_p^n)} \lesssim \max\Big\{\sup_{k_0 \in \mathbb{Z}^- \cup \{0\}} \mathscr{E}_1, \sup_{k_0 \in \mathbb{Z}^+} (\mathscr{E}_2 + \mathscr{E}_3)\Big\}. \tag{33}$$

Here

$$\begin{aligned}\mathscr{E}_1 &= p^{-k_0\lambda} \Big(\sum_{k=-\infty}^{k_0} p^{k\alpha(0)\ell} \|\mathscr{H}_{\Omega}^{p}(f) \times |\cdot|_p^{-n/\xi}\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)}^{\ell}\Big)^{1/\ell},\\ \mathscr{E}_2 &= p^{-k_0\lambda} \Big(\sum_{k=-\infty}^{-1} p^{k\alpha(0)\ell} \|\mathscr{H}_{\Omega}^{p}(f) \times |\cdot|_p^{-n/\xi}\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)}^{\ell}\Big)^{1/\ell},\\ \mathscr{E}_3 &= p^{-k_0\lambda} \Big(\sum_{k=0}^{k_0} p^{k\alpha_\infty\ell} \|\mathscr{H}_{\Omega}^{p}(f) \times |\cdot|_p^{-n/\xi}\|_{L^{s(\cdot)}(\mathbb{Q}_p^n)}^{\ell}\Big)^{1/\ell}.\end{aligned}$$

By using Lemma 1 with both $\lambda-\alpha(0)\geqslant 0$ and $\lambda>0$, we infer

$$\begin{aligned}
\mathscr{E}_1 &\lesssim p^{-k_0\lambda}\Big\{\sum_{k=-\infty}^{k_0} p^{k\alpha(0)\ell}\Big(\|\Omega\|_{L^{\xi}(S_0)}\sum_{j=-\infty}^{k} p^{(j-k)n\delta}\|f_j\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{\ell}\Big\}^{1/\ell}\\
&\lesssim \|\Omega\|_{L^{\xi}(S_0)} p^{-k_0\lambda}\Big\{\sum_{k=-\infty}^{k_0} p^{k\alpha(0)\ell}\Big(\sum_{j=-\infty}^{k} p^{(j-k)n\delta+j(\lambda-\alpha(0))}\Big)^{\ell}\Big\}^{1/\ell}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim \|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} p^{-k_0\lambda}\Big\{\sum_{k=-\infty}^{k_0} p^{k\lambda\ell}\Big(\sum_{j=-\infty}^{k} p^{(j-k)(n\delta+\lambda-\alpha(0))}\Big)^{\ell}\Big\}^{1/\ell}\\
&\lesssim \|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\lambda\ell}\Big)^{1/\ell}.
\end{aligned}$$

This gives

$$\mathscr{E}_1 \lesssim \|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}. \tag{34}$$

If we argue similarly as above, then

$$\begin{aligned}
\mathscr{E}_2 &\lesssim \|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{-1} p^{k\lambda\ell}\Big)^{1/\ell}\\
&\lesssim p^{-k_0\lambda}\|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{35}$$

Next, by applying Lemma 1 with both $\lambda-\alpha_\infty\geqslant 0$ and $\lambda>0$, we get

$$\begin{aligned}
\mathscr{E}_3 &\lesssim \|\Omega\|_{L^{\xi}(S_0)} p^{-k_0\lambda}\Big\{\sum_{k=0}^{k_0} p^{k\alpha_\infty\ell}\Big(\sum_{j=-\infty}^{k} p^{(j-k)n\delta}\|f_j\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\Big)^{\ell}\Big\}^{1/\ell}\\
&\lesssim \|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} p^{-k_0\lambda}\Big\{\sum_{k=0}^{k_0} p^{k\lambda\ell}\Big(\sum_{j=-\infty}^{k} p^{(j-k)(n\delta_2+\lambda-\alpha_\infty)}\Big)^{\ell}\Big\}^{1/\ell}\\
&\lesssim \|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)} p^{-k_0\lambda}\Big(\sum_{k=0}^{k_0} p^{k\lambda\ell}\Big)^{1/\ell}\\
&\lesssim p^{-k_0\lambda}\{p^{(k_0+1)\lambda}+1\}\|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Thus

$$\mathscr{E}_3 \lesssim \big(p^{\lambda}+p^{-k_0\lambda}\big)\|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.$$

As a consequence, by (33)–(35), we obtain

$$\begin{aligned}
&\|\mathscr{H}^p_\Omega(f)\times|\cdot|_p^{-n/\xi}\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,s(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim \max\Big\{\sup_{k_0\in\mathbb{Z}_-\cup\{0\}} 1,\ \sup_{k_0\in\mathbb{Z}_+}(2p^{-k_0\lambda}+p^{\lambda})\Big\}\|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}\\
&\lesssim \|\Omega\|_{L^{\xi}(S_0)}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{\ell,q(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Therefore, the proof of the theorem is completed. □

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