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GENERAL NUMERICAL RADIUS INEQUALITIES FOR MATRICES OF HILBERT SPACE OPERATORS

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Abstract. We obtain new bounds for the numerical radius of certain $n \times n$ and general 2×2 operator matrices. We show that our bounds refines the bounds that are given in [1] and [3].

1. Introduction

Let $B(H)$ be the C^*-algebra of all bounded linear operators on complex Hilbert space H with the inner product $\langle .,. \rangle$. For $T \in B(H))$, let

$$w(T) = \sup_{||x||=1} |\langle Tx, x\rangle|,$$

$$m(T) = \inf_{||x||=1} |\langle Tx, x\rangle|,$$

$$\|T\| = \sup_{||x||=1} \sqrt{\langle Tx, Tx\rangle},$$

$$c(T) = \inf_{||x||=1} \sqrt{\langle Tx, Tx\rangle},$$

denote the numerical radius, the Crawford number of T, the usual operator norm and the minimum norm of T respectively. It is well-known that $w(.)$ and $||.||$ are equivalent norms on $B(H)$ and satisfying the following sharp inequality

$$\frac{1}{2}\|T\| \leqslant w(T) \leqslant \|T\| \quad \text{for every} \quad T \in B(H). \tag{1.1}$$

The upper bound of the inequality (1.1) has refined by Kittaneh [10] by showing that

$$w^2(T) \leqslant \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\| \quad \text{for every} \quad T \in B(H). \tag{1.2}$$

Over the years many authors have generalized and improved the above inequality, (see [6]–[14]).

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In this work, we obtain an improvement for the lower bound of the inequality (1.1), we prove that

$$w(T) \geqslant \max\left\{\|T\| + \frac{m(T^2)}{\|T\|}, \frac{c^2(T)+w(T^2)}{\|T\|}\right\},$$

for every non-zero operator $T \in B(H)$.

Let H be Hilbert space. Then the n-copies of H denoted by $H^{(n)} = H \oplus H \oplus \cdots \oplus H$. For $A_{ij} \in B(H)$, $(1 \leqslant i,j \leqslant n)$, the $n \times n$ operator matrix $T = [A_{ij}] \in B(H^{(n)})$ is defined by

$$Tx = \begin{bmatrix} \sum_{j=1}^{n} A_{1j}x_j \\ \sum_{j=1}^{n} A_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj}x_j \end{bmatrix}, \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in H^{(n)}.$$

In this work, we establish new lower and upper bounds for the numerical radius of certain $n \times n$ and 2×2 operator matrix. We show that the bounds obtained here improve on the existing bounds given in [1] and [3].

2. Numerical radius inequalities

In order to prove our results we need the following lemmas.

LEMMA 2.1. [4] *Let $T \in B(H)$. Then*

$$\|Tx\|^2 + |\langle T^2x,x\rangle| \leqslant 2w(T)\|Tx\|\|x\|, \tag{2.1}$$

for every $x \in H$.

LEMMA 2.2. [1] *Let $a,b,e \in H$ with $\|e\| = 1$. Then for every $p \in [0,1]$,*

$$|\langle a,e\rangle\langle e,b\rangle|^2 \leqslant \frac{1+p}{2}\|a\|^2\|b\|^2 + \frac{1-p}{2}\|a\|\|b\||\langle a,b\rangle|. \tag{2.2}$$

LEMMA 2.3. [11] *Let $T \in B(H)$ be a positive operator and $x \in H$ be any unit vector. Then for every $r \geqslant 1$,*

$$|\langle Tx,x\rangle|^r \leqslant |\langle T^r x,x\rangle|. \tag{2.3}$$

LEMMA 2.4. [2] *Let f be a non-negative convex function on $[0,\infty)$ and $A,B \in B(H)$ be positive operators. Then*

$$\left\|f\left(\frac{A+B}{2}\right)\right\| \leqslant \left\|\frac{f(A)+f(B)}{2}\right\|.$$

The first result in this paper which gives new lower bound for $n \times n$ off-diagonal operator matrix can be stated as follows.

THEOREM 2.5. *Let* $T = \begin{bmatrix} 0 & 0 & A_1 \\ 0 & \cdot^{\cdot^{\cdot}} & 0 \\ A_n & 0 & 0 \end{bmatrix}$, *such that* $A_i \neq 0$ *for any* $i = 1,2,3,\ldots,n$. *Then*

$$w(A) \geqslant \frac{1}{2}\max\{\alpha, \beta\}, \tag{2.4}$$

where

$$\alpha = \max\left\{ \|A_{n-s+1}\| + \frac{m(A_s A_{n-s+1})}{\|A_{n-s+1}\|} : s = 1,2,\ldots,n \right\}$$

and

$$\beta = \max\left\{ \frac{c^2(A_{n-s+1}) + w(A_s A_{n-s+1})}{\|A_{n-s+1}\|} : s = 1,2,\ldots,n \right\}.$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in H^{(n)}$ be any unit vector. Then by Lemma 2.1 we have

$$\left\| \sum_{i=1}^{n} A_i x_{n-i+1} \right\|^2 + \left| \sum_{i=1}^{n} \langle A_i A_{n-i+1} x_i, x_i \rangle \right| \leqslant 2w(T) \left\| \sum_{i=1}^{n} A_i x_{n-i+1} \right\|. \tag{2.5}$$

For $s \in \{1,2,\ldots,n\}$, let $x_j = 0$ for all $j = 1,2,\ldots,s-1,s+1,\ldots,n$. then $\|x_s\| = 1$ and so by (2.5), we get

$$\|A_{n-s+1}x_s\|^2 + |\langle A_s A_{n-s+1} x_s, x_s \rangle| \leqslant 2w(T)\,\|A_{n-s+1}\|,$$

thus

$$\|A_{n-s+1}x_s\|^2 + m(A_s A_{n-s+1}) \leqslant 2w(T)\,\|A_{n-s+1}\|,$$

hence

$$w(T) \geqslant \frac{\|A_{n-s+1}x_s\|^2 + m(A_s A_{n-s+1})}{2\,\|A_{n-s+1}\|}.$$

Taking the supremum over all $x_s \in H$ with $\|x_s\| = 1$, we get

$$w(T) \geqslant \frac{1}{2}\left[\|A_{n-s+1}x_s\| + \frac{m(A_s A_{n-s+1})}{\|A_{n-s+1}x_s\|} \right].$$

Hence

$$w(T) \geqslant \frac{1}{2}\alpha. \tag{2.6}$$

Similarly from the inequality (2.5) we have

$$c^2(A_{n-s+1}) + |\langle A_s A_{n-s+1} x_s, x_s \rangle| \leqslant 2w(T)\,\|A_{n-s+1}\|,$$

so

$$w(T) \geqslant \frac{c^2(A_{n-s+1}) + |\langle A_s A_{n-s+1} x_s, x_s\rangle|}{2\,\|A_{n-s+1}\|}.$$

Taking the supremum over all $x_s \in H$,

$$w(T) \geqslant \frac{c^2(A_{n-s+1}) + w(A_s A_{n-s+1} x_s, x_s)}{2\,\|A_{n-s+1}\|}.$$

Thus

$$w(T) \geqslant \frac{1}{2}\beta. \tag{2.7}$$

By (2.6) and (2.7) we get the required result. □

The next result which improve the lower bound of the inequality (1.1) can be obtained directly from the above theorem.

COROLLARY 2.6. *Let $T \in B(H)$. Then*

$$w(T) \geqslant \frac{1}{2}\max\left\{\|T\| + \frac{m(T^2)}{\|T\|}, \frac{c^2(T) + w(T^2)}{\|T\|}\right\}. \tag{2.8}$$

Also, if $n = 2$ in Theorem 2.5, then we get ([5], Theorem 2).

THEOREM 2.7. *Let $A, B, C, D \in B(H)$ and let $0 \leqslant p \leqslant 1$. Then*

$$\begin{aligned} w^4\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant\ & 8\max\{w^4(A), w^4(D)\} + (2+2p)\max\{\|\,|C|^4 + |B^*|^4\,\|, \|\,|B|^4 + |C^*|^4\,\|\} \\ & + (2-2p)\max\{\|\,|C|^2 + |B^*|^2\,\|, \|\,|B|^2 + |C^*|^2\,\|\} \\ & \times \max\{w(BC), w(CB)\}. \end{aligned} \tag{2.9}$$

Proof. Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $T_1 = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Then for any unit vector $x \in H^{(2)}$, we get

$$\begin{aligned} |\langle Tx, x\rangle|^4 &= |\langle T_1 x, x\rangle + \langle T_2 x, x\rangle|^4 \\ &\leqslant (|\langle T_1 x, x\rangle| + |\langle T_2 x, x\rangle|)^4 \\ &= 8(|\langle T_1 x, x\rangle|^4 + |\langle T_2 x, x\rangle|^4) \qquad \text{(by convixity of } f(t) = t^4) \\ &= 8|\langle T_1 x, x\rangle|^4 + 8|\langle T_2 x, x\rangle\langle x, T_2^* x\rangle|^2 \\ &\leqslant 8|\langle T_1 x, x\rangle|^4 + (4+4p)\|T_2 x\|^2\|T_2^* x\|^2 + (4-4p)\|T_2 x\|\|T_2^* x\||\langle T_2 x, T_2^* x\rangle| \\ &\qquad \text{(by Lemma 2.2)} \\ &= 8|\langle T_1 x, x\rangle|^4 + (4+4p)\langle |T_2|^2 x, x\rangle\langle |T_2^*|^2 x, x\rangle \\ &\quad + (4-4p)\sqrt{\langle |T_2|^2 x, x\rangle}\sqrt{\langle |T_2^*|^2 x, x\rangle}|\langle T_2^2 x, x\rangle| \end{aligned}$$

$$
\begin{aligned}
&\leqslant 8|\langle T_1x,x\rangle|^4+(2+2p)(\langle|T_2|^2x,x\rangle^2+\langle|T_2^*|^2x,x\rangle^2)\\
&\quad+(2-2p)\langle(|T_2|^2x,+|T_2^*|^2)x,x\rangle|\langle T_2^2x,x\rangle|\\
&\leqslant 8|\langle T_1x,x\rangle|^4+(2+2p)\langle(|T_2|^4+|T_2^*|^4)x,x\rangle\\
&\quad+(2-2p)\langle(|T_2|^2x,+|T_2^*|^2)x,x\rangle|\langle T_2^2x,x\rangle| \qquad \text{(by Lemma 2.3)}\\
&\leqslant 8\left|\left\langle\begin{bmatrix}A&0\\0&D\end{bmatrix}x,x\right\rangle\right|^4+(2+2p)\left\langle\begin{bmatrix}|C|^4+|B^*|^4&0\\0&|B|^4+|C^*|^4\end{bmatrix}x,x\right\rangle\\
&\quad+(2-2p)\left\langle\begin{bmatrix}|C|^2+|B^*|^2&0\\0&|B|^2+|C^*|^2\end{bmatrix}x,x\right\rangle\left\langle\begin{bmatrix}BC&0\\0&CB\end{bmatrix}x,x\right\rangle.
\end{aligned}
$$

Now, by taking the supremum over all unit vectors $x\in H^{(2)}$ we get the required bound. □

REMARK 2.8. The upper bound in Theorem 2.7 is less than the upper bound in the inequality [[3], Theorem 3.1] for $p=\frac{1}{3}$. That is,

$$
\begin{aligned}
w^4\left(\begin{bmatrix}A&B\\C&D\end{bmatrix}\right)&\leqslant 8\max\{w^4(A),w^4(D)\}+\frac{8}{3}\max\{\|\,|C|^4+|B^*|^4\,\|,\|\,|B|^4+|C^*|^4\,\|\}\\
&\quad+\frac{4}{3}\max\{\|\,|C|^2+|B^*|^2\,\|,\,\|\,|B|^2+|C^*|^2\,\|\}\max\{w(BC),w(CB)\}\\
&\leqslant 8\max\{w^4(A),w^4(D)\}+3\max\{\|\,|C|^4+|B^*|^4\,\|,\|\,|B|^4+|C^*|^4\,\|\}\\
&\quad+\max\{\|\,|C|^2+|B^*|^2\,\|,\,\|\,|B|^2+|C^*|^2\,\|\}\max\{w(BC),w(CB)\}.
\end{aligned}
\tag{2.10}
$$

Which is clear since

$$
\begin{aligned}
&\max\{\|\,|C|^2+|B^*|^2\,\|,\,\|\,|B|^2+|C^*|^2\,\|\}\max\{w(BC),w(CB)\}\\
&\leqslant \max\{\|\,|C|^4+|B^*|^4\,\|,\|\,|B|^4+|C^*|^4\,\|\}.
\end{aligned}
$$

The next result is obtained by letting $A=D$ and $B=C$ in Theorem 2.7.

COROLLARY 2.9. *Let $A,B\in B(H)$ and $p\in[0,1]$. Then*

$$
w^4\left(\begin{bmatrix}A&B\\B&A\end{bmatrix}\right)\leqslant 8w^4(A)+(2+2p)\|\,|B|^4+|B^*|^4\,\|+(2-2p)\|\,|B|^2+|B^*|^2\,\|w(B^2).
\tag{2.11}
$$

The next result which was obtained in [1] by Aldolat and Jaradat is a direct consequence of corollary 2.9 by letting $A=B$.

COROLLARY 2.10. *Let $A\in B(H)$ and $p\in[0,1]$. Then*

$$
w^4(A)\leqslant\frac{1+p}{4}\|\,|A|^4+|A^*|^4\,\|+\frac{1-p}{4}\|\,|A|^2+|A^*|^2\,\|w(A^2).
\tag{2.12}
$$

REMARK 2.11. Let $p=\frac{1}{3}$ in Corollary 2.10 . Then we get a refinement for the upper bound [[3], Remark 3.2]

$$\begin{aligned} w^4(A) &\leqslant \frac{1}{3}\| |A|^4+|A^*|^4 \| + \frac{1}{6}\| |A|^2+|A^*|^2 \| w(A^2) \\ &\leqslant \frac{3}{8}\| |A|^4+|A^*|^4 \| + \frac{1}{8}\| |A|^2+|A^*|^2 \| w(A^2). \end{aligned} \tag{2.13}$$

To explain this, note that

$$\begin{aligned} &\frac{1}{3}\| |A|^4+|A^*|^4 \| + \frac{1}{6}\| |A|^2+|A^*|^2 \| w(A^2) \\ &= \frac{1}{3}\| |A|^4+|A^*|^4 \| + \frac{1}{24}\| |A|^2+|A^*|^2 \| w(A^2) + \frac{1}{8}\| |A|^2+|A^*|^2 \| w(A^2) \\ &\leqslant \frac{1}{3}\| |A|^4+|A^*|^4 \| + \frac{1}{24}\| |A|^2+|A^*|^2 \| w^2(A) + \frac{1}{8}\| |A|^2+|A^*|^2 \| w(A^2) \\ &\leqslant \frac{1}{3}\| |A|^4+|A^*|^4 \| + \frac{1}{48}\| |A|^2+|A^*|^2 \|^2 + \frac{1}{8}\| |A|^2+|A^*|^2 \| w(A^2) \\ &\qquad \text{(by inequality (1.2))} \\ &\leqslant \frac{1}{3}\| |A|^4+|A^*|^4 \| + \frac{1}{48}\| (|A|^2+|A^*|^2)^2 \| + \frac{1}{8}\| |A|^2+|A^*|^2 \| w(A^2) \\ &\qquad \text{(by Lemma (2.4))} \\ &\leqslant \frac{1}{3}\| |A|^4+|A^*|^4 \| + \frac{1}{24}\| |A|^4+|A^*|^4 \| + \frac{1}{8}\| |A|^2+|A^*|^2 \| w(A^2) \\ &= \frac{3}{8}\| |A|^4+|A^*|^4 \| + \frac{1}{8}\| |A|^2+|A^*|^2 \| w(A^2). \end{aligned} \tag{2.14}$$

THEOREM 2.12. *Let $A,B,C,D \in B(H)$ and let $p \in [0,1]$. Then*

$$\begin{aligned} w^4\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\leqslant \max\{w^4(A), w^4(D)\} + \frac{1+p}{4}\max\{\| |C|^4+|B^*|^4 \|, \| |B|^4+|C^*|^4 \|\} \\ &+ \frac{1-p}{4}\max\{\| |C|^2+|B^*|^2 \|, \| |B|^2+|C^*|^2 \|\}\max\{w(BC), w(CB)\} \\ &+ 3\max\{w^2(A), w^2(D)\}\Big[\frac{1}{2}\max\{\| |C|^2+|B^*|^2 \|, \| |B|^2+|C^*|^2 \|\} \\ &+ \max\{w(BC), w(CB)\}\Big] \\ &+ 2\left[\frac{1}{2}\max\{\| |A|^2+|B^*|^2 \|, \| |D|^2+|C^*|^2 \|\} + w\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right)\right] \\ &\times \left[\max\{w^2(A), w^2(D)\} + w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right]. \end{aligned}$$

Proof. Let $T=\begin{bmatrix} A & B\\ C & D\end{bmatrix}$, $T_1=\begin{bmatrix} A & 0\\ 0 & D\end{bmatrix}$ and $T_2=\begin{bmatrix} 0 & B\\ C & 0\end{bmatrix}$. Then for any unit vector $x\in H^{(2)}$, we have

$$\begin{aligned}
|\langle Tx,x\rangle|^4 &\leqslant (|\langle T_1x,x\rangle|+|\langle T_2x,x\rangle|)^4\\
&= |\langle T_1x,x\rangle|^4+|\langle T_2x,x\rangle\langle x,T_2^*x\rangle|^2+6|\langle T_1x,x\rangle|^2|\langle T_2x,x\rangle\langle x,T_2^*x\rangle|\\
&\quad +4|\langle T_1x,x\rangle|^2|\langle T_1x,x\rangle\langle x,T_2^*x\rangle|+4|\langle T_2x,x\rangle|^2|\langle T_1x,x\rangle\langle x,T_2^*x\rangle|\\
&\leqslant |\langle T_1x,x\rangle|^4+\frac{1+p}{2}\|T_2x\|^2\|T_2^*x\|^2+\frac{1-p}{2}\|T_2x\|\|T_2^*x\||\langle T_2x,T_2^*x\rangle|\\
&\quad +3|\langle T_1x,x\rangle|^2\left(\|T_2x\|\|T_2^*x\||+|\langle T_2x,T_2^*x\rangle|\right)\\
&\quad +2\left[\|T_1x\|\|T_2^*x\|+|\langle T_1x,T_2^*x\rangle|\right]\left[|\langle T_1x,x\rangle|^2+|\langle T_2x,x\rangle|^2\right]\\
&\leqslant |\langle T_1x,x\rangle|^4+\frac{1+p}{4}\left\langle(|T_2|^4+|T_2^*|^4)x,x\right\rangle\\
&\quad +\frac{1-p}{4}\left\langle(|T_2|^2+|T_2^*|^2)x,x\right\rangle|\langle T_2^2x,x\rangle|\\
&\quad +3|\langle T_1x,x\rangle|^2\left(\left\langle\frac{(|T_2|^2+|T_2^*|^2)}{2}x,x\right\rangle+\langle T_2^2x,x\rangle\right)\\
&\quad +2\left[\left\langle\frac{(|T_1|^2+|T_2^*|^2)}{2}x,x\right\rangle+|\langle T_2T_1x,x\rangle|\right]\left[|\langle T_1x,x\rangle|^2+|\langle T_2x,x\rangle|^2\right]\\
&\leqslant |\langle T_1x,x\rangle|^4+\frac{1+p}{4}\left\langle\begin{bmatrix}|C|^4+|B^*|^4 & 0\\ 0 & |B|^4+|C^*|^4\end{bmatrix}x,x\right\rangle\\
&\quad +\frac{1-p}{4}\left\langle\begin{bmatrix}|C|^2+|B^*|^2 & 0\\ 0 & |B|^2+|C^*|^2\end{bmatrix}x,x\right\rangle\left|\left\langle\begin{bmatrix}BC & 0\\ 0 & CB\end{bmatrix}x,x\right\rangle\right|\\
&\quad +3|\langle T_1x,x\rangle|^2\left(\frac{1}{2}\left\langle\begin{bmatrix}|C|^2+|B^*|^2 & 0\\ 0 & |B|^2+|C^*|^2\end{bmatrix}x,x\right\rangle+\left|\left\langle\begin{bmatrix}BC & 0\\ 0 & CB\end{bmatrix}x,x\right\rangle\right|\right)\\
&\quad +2\left[\frac{1}{2}\left\langle\begin{bmatrix}|A|^2+|B^*|^2 & 0\\ 0 & |D|^2+|C^*|^2\end{bmatrix}x,x\right\rangle+\left|\left\langle\begin{bmatrix}0 & BD\\ CD & 0\end{bmatrix}x,x\right\rangle\right|\right]\\
&\quad \times\left[|\langle T_1x,x\rangle|^2+|\langle T_2x,x\rangle|^2\right].
\end{aligned}$$

Now, the proof is complete by taking the supremum over all unit vectors $x\in H^{(2)}$. □

COROLLARY 2.13. *Let $A,B\in B(H)$ and $p\in[0,1]$. Then*

$$\begin{aligned}
w^4\left(\begin{bmatrix} A & B\\ B & A\end{bmatrix}\right) &= \max\{w^4(A+B),w^4(A-B)\}\\
&\leqslant w^4(A)+\frac{1+p}{4}\|\,|B|^4+|B^*|^4\,\|+\frac{1-p}{4}\|\,|B|^2+|B^*|^2\,\|w(B^2)\\
&\quad +3w^2(A)\left(\frac{1}{2}\|\,|B|^2+|B^*|^2\,\|+w(B^2)\right)\\
&\quad +\left(\|\,|A|^2+|B^*|^2\,\|+2w(BA)\right)\left(w(A^2)+w(B^2)\right).
\end{aligned}$$

COROLLARY 2.14. *Let $A \in B(H)$ and $p \in [0,1]$. Then*

$$\begin{aligned} w^4(A) &\leqslant \frac{1+p}{60}\| |A|^4 + |A^*|^4 \| + \frac{1-p}{60}\| |A|^2 + |A^*|^2 \| w(A^2) \\ &\quad + \frac{7}{15} w^2(A)\left(\frac{1}{2}\| |A|^2 + |A^*|^2 \| + w(A^2)\right) \\ &\leqslant \frac{1}{2}\| |A|^4 + |A^*|^4 \|. \end{aligned}$$

Proof.

$$\begin{aligned} w^4(A) &\leqslant \frac{1+p}{60}\| |A|^4 + |A^*|^4 \| + \frac{1-p}{60}\| |A|^2 + |A^*|^2 \| w(A^2) \\ &\quad + \frac{7}{15} w^2(A)\left(\frac{1}{2}\| |A|^2 + |A^*|^2 \| + w(A^2)\right) \quad \text{(take } A = B \text{ in Corollary 2.13)} \\ &= \frac{1+p}{60}\| |A|^4 + |A^*|^4 \| + \frac{1-p}{60}\| |A|^2 + |A^*|^2 \| w(A^2) \\ &\quad + \frac{7}{30} w^2(A)\| |A|^2 + |A^*|^2 \| + + \frac{7}{15} w^2(A) w(A^2) \\ &\leqslant \frac{1+p}{60}\| |A|^4 + |A^*|^4 \| + \frac{1-p}{60}\| |A|^2 + |A^*|^2 \| w^2(A) \\ &\quad + \frac{7}{30} w^2(A)\| |A|^2 + |A^*|^2 \| + \frac{7}{15} w^4(A) \quad \text{(by the power inequality)} \\ &\leqslant \frac{1+p}{60}\| |A|^4 + |A^*|^4 \| + \frac{15-p}{120}\| |A|^2 + |A^*|^2 \|^2 + \frac{7}{60}\| |A|^2 + |A^*|^2 \|^2 \\ &= \frac{1+p}{60}\| |A|^4 + |A^*|^4 \| + \frac{29-p}{120}\| (|A|^2 + |A^*|^2)^2 \| \\ &\leqslant \frac{1+p}{60}\| |A|^4 + |A^*|^4 \| + \frac{29-p}{60}\| |A|^4 + |A^*|^4 \| \\ &= \frac{1}{2}\| |A|^4 + |A^*|^4 \|. \quad \square \end{aligned}$$

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