

## MONOTONICITY PROPERTIES OF WEIGHTED GEOMETRIC SYMMETRIZATIONS

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*Abstract.* We prove new monotonicity properties for spectral radius, essential spectral radius, operator norm, Hausdorff measure of non-compactness and numerical radius of products and sums of weighted geometric symmetrizations of positive kernel operators on  $L^2$ . To our knowledge, several proved properties are new even in the finite dimensional case.

### 1. Introduction

Let  $A = [a_{ij}]$  be an entrywise nonnegative  $n \times n$  matrix and let  $S(A) = [\sqrt{a_{ij}a_{ji}}]$  be its geometric symmetrization. In [39], Schwenk proved the inequality

$$r(S(A)) \leq r(A), \quad (1)$$

for the spectral radius  $r(\cdot)$  by using graph-theoretical methods. In [16], Elsner, Johnson and Dias Da Silva proved that the inequality

$$r(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \dots \circ A_m^{(\alpha_m)}) \leq r(A_1)^{\alpha_1} r(A_2)^{\alpha_2} \dots r(A_m)^{\alpha_m} \quad (2)$$

for Hadamard weighted geometric mean holds for nonnegative  $n \times n$  matrices  $A_1, A_2, \dots, A_m$  and nonnegative numbers  $\alpha_1, \alpha_2, \dots, \alpha_m$  such that  $\sum_{j=1}^m \alpha_j \geq 1$ . Here  $A^{(\alpha)} = [a_{ij}^\alpha]$  denotes the Hadamard (Schur) power of  $A$  and  $A \circ B = [a_{ij}b_{ij}]$  denotes the Hadamard (Schur) product of matrices  $A$  and  $B$ . Clearly, (2) generalizes (1), since  $S(A) = A^{(\frac{1}{2})} \circ (A^T)^{(\frac{1}{2})}$ . Let us point out that inequality (2) can straightforwardly be deduced from an earlier result by Kingman [23] and that (1) is a special case of earlier results by Karlin and Ost [21, Theorem 2.1 and Remark 1] and that the case  $\sum_{j=1}^m \alpha_j = 1$  of (2) was already obtained in [21, Remark 1] (in [21] these results were applied in the context of finite stationary Markov chains). Since then inequalities and equalities on Hadamard weighted geometric means and weighted geometric symmetrizations received a lot of attention and have been applied in a variety of contexts (see e.g. [3, 5–8, 10–15, 17–19, 24, 27, 28, 30–38, 40, 42, 43]).

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In [11], Drnovšek proved that in the case when  $\sum_{j=1}^m \alpha_j = 1$ , inequality (2) holds also for positive compact operators on Banach function spaces. In [12], Drnovšek and the second author of the current article proved that the compactness assumption can be removed and that analogous results hold also for operator norm and also for numerical radius on  $L^2$ . Further they proved additional results for products of Hadamard weighted geometric means (see Theorem 1 below). In [33], the second author also showed that analogous results also hold for Hausdorff measure of non-compactness and for essential spectral radius on suitable Banach functions spaces (including  $L^2$ , see Theorem 1 below). In [12] and [33], also generalizations of inequality (1) for products and sums of geometric symmetrizations of positive kernel operators were proved (see inequalities (17) and (18) below).

In [40], Shen and Huang studied weighted geometric symmetrizations  $S_\alpha(A) = [a_{ij}^\alpha a_{ji}^{1-\alpha}]$  for  $\alpha \in [0, 1]$  and for nonnegative  $n \times n$  matrices. They showed that for a given square nonnegative matrix  $A$  the function  $\alpha \mapsto r(S_\alpha(A))$  is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$  ([40, Theorem 3.3]). They also proved an analogous result for the operator (largest singular value) norm ([40, Theorem 2.3]). In [6, Theorem 2.7], we obtained an analogous result for the spectral radius, essential spectral radius, operator norm, Hausdorff measure of non-compactness and numerical radius of weighted geometric symmetrizations of a given positive kernel operator on  $L^2$ . In the current article we further extend a technique of Shen and Huang to obtain additional results. For instance, as a special case of our results (see Corollary 2 below) we show that also the function  $\alpha \mapsto r(S_\alpha(A_1)S_\alpha(A_2))$  is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$ , where  $A_1$  and  $A_2$  are positive kernel operators on  $L^2$  (and that the analogue of this result holds also for the essential spectral radius).

The rest of the article is organized in the following way. In Section 2 we recall some definitions and results that will be needed in our proofs. In Section 3 we prove new monotonicity properties for spectral radius, essential spectral radius, operator norm, Hausdorff measure of non-compactness and numerical radius of products and sums of weighted geometric symmetrizations of positive kernel operators on  $L^2$ . The main results of this article are Theorems 3 and 5.

## 2. Preliminaries

Let  $\mu$  be a  $\sigma$ -finite positive measure on a  $\sigma$ -algebra  $\mathcal{M}$  of subsets of a non-void set  $X$ . Let  $M(X, \mu)$  be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on  $X$ . A Banach space  $L \subseteq M(X, \mu)$  is called a *Banach function space* if  $f \in L$ ,  $g \in M(X, \mu)$ , and  $|g| \leq |f|$  imply that  $g \in L$  and  $\|g\| \leq \|f\|$ . Throughout the article, it is assumed that  $X$  is the carrier of  $L$ , that is, there is no subset  $Y$  of  $X$  of strictly positive measure with the property that  $f = 0$  a.e. on  $Y$  for all  $f \in L$  (see [41]).

Standard examples of Banach function spaces are Euclidean spaces,  $L^p(X, \mu)$  spaces for  $1 \leq p \leq \infty$ , the space  $c_0$  of all null convergent sequences (equipped with the usual norms and the counting measure) and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces (see

e.g. [4, 9, 22] and the references cited there), which are important e.g. in interpolation theory and in the theory of partial differential equations. Recall that the cartesian product  $L = E \times F$  of Banach function spaces is again a Banach function space, equipped with the norm  $\|(f, g)\|_L = \max\{\|f\|_E, \|g\|_F\}$ .

If  $\{f_n\}_{n \in \mathbb{N}} \subset M(X, \mu)$  is a real valued decreasing sequence and  $f = \inf\{f_n \in M(X, \mu) : n \in \mathbb{N}\}$ , then we write  $f_n \downarrow f$ . A Banach function space  $L$  has an *order continuous norm*, if  $0 \leq f_n \downarrow 0$  implies  $\|f_n\|_L \rightarrow 0$  as  $n \rightarrow \infty$ . It is well known that spaces  $L^p(X, \mu)$ ,  $1 \leq p < \infty$ , have order continuous norm. Moreover, the norm of any reflexive Banach function space is order continuous. In particular, we are interested in Banach function spaces  $L$  such that  $L$  and its Banach dual space  $L^*$  have order continuous norms. Examples of such spaces are  $L^p(X, \mu)$ ,  $1 < p < \infty$ , while the space  $L = c_0$  is an example of a non-reflexive Banach sequence space, such that  $L$  and  $L^* = l^1$  have order continuous norms.

By an *operator* on a Banach function space  $L$  we always mean a linear operator on  $L$ . An operator  $K$  on  $L$  is said to be *positive* if it maps nonnegative functions to nonnegative ones, i.e.,  $KL_+ \subset L_+$ , where  $L_+$  denotes the positive cone  $L_+ = \{f \in L : f \geq 0 \text{ a.e.}\}$ . Given operators  $K$  and  $H$  on  $L$ , we write  $K \geq H$  if the operator  $K - H$  is positive.

Recall that a positive operator  $K$  is always bounded, i.e., its operator norm

$$\|K\| = \sup\{\|Kf\|_L : f \in L, \|f\|_L \leq 1\} = \sup\{\|Kf\|_L : f \in L_+, \|f\|_L \leq 1\} \quad (3)$$

is finite (the second equality in (3) follows from  $|Kf| \leq K|f|$  for  $f \in L$ ). Also, its spectral radius  $r(K)$  is always contained in the spectrum.

In the special case  $L = L^2(X, \mu)$  we can define the *numerical radius*  $w(K)$  of a bounded operator  $K$  on  $L^2(X, \mu)$  by

$$w(K) = \sup\{|\langle Kf, f \rangle| : f \in L^2(X, \mu), \|f\|_2 = 1\}.$$

If, in addition,  $K$  is positive, then it is easy to prove that

$$w(K) = \sup\{\langle Kf, f \rangle : f \in L^2(X, \mu)_+, \|f\|_2 = 1\}.$$

From this it follows easily that  $w(K) \leq w(H)$  for all positive operators  $K$  and  $H$  on  $L^2(X, \mu)$  with  $K \leq H$ .

An operator  $K$  on a Banach function space  $L$  is called a *kernel operator* if there exists a  $\mu \times \mu$ -measurable function  $k(x, y)$  on  $X \times X$  such that, for all  $f \in L$  and for almost all  $x \in X$ ,

$$\int_X |k(x, y)f(y)| d\mu(y) < \infty \quad \text{and} \quad (Kf)(x) = \int_X k(x, y)f(y) d\mu(y).$$

One can check that a kernel operator  $K$  is positive iff its kernel  $k$  is non-negative almost everywhere.

Let  $L$  be a Banach function space such that  $L$  and  $L^*$  have order continuous norms and let  $K$  and  $H$  be positive kernel operators on  $L$ . By  $\gamma(K)$  we denote the Hausdorff measure of non-compactness of  $K$ , i.e.,

$$\gamma(K) = \inf\{\delta > 0 : \text{there is a finite } M \subset L \text{ such that } K(D_L) \subset M + \delta D_L\},$$

where  $D_L = \{f \in L : \|f\|_L \leq 1\}$ . Then  $\gamma(K) \leq \|K\|$ ,  $\gamma(K + H) \leq \gamma(K) + \gamma(H)$ ,  $\gamma(KH) \leq \gamma(K)\gamma(H)$  and  $\gamma(\alpha K) = \alpha\gamma(K)$  for  $\alpha \geq 0$ . Also  $0 \leq K \leq H$  implies  $\gamma(K) \leq \gamma(H)$  (see e.g. [25, Corollary 4.3.7 and Corollary 3.7.3]). Let  $r_{ess}(K)$  denote the essential spectral radius of  $K$ , i.e., the spectral radius of the Calkin image of  $K$  in the Calkin algebra. Then

$$r_{ess}(K) = \lim_{j \rightarrow \infty} \gamma(K^j)^{1/j} = \inf_{j \in \mathbb{N}} \gamma(K^j)^{1/j} \tag{4}$$

and  $r_{ess}(K) \leq \gamma(K)$ . Recall that if  $L = L^2(X, \mu)$ , then  $\gamma(K^*) = \gamma(K)$  and  $r_{ess}(K^*) = r_{ess}(K)$ , where  $K^*$  denotes the adjoint of  $K$  (see e.g. [25, Proposition 4.3.3, Theorems 4.3.6 and 4.3.13 and Corollary 3.7.3], [29, Theorem 1], [24]). Note that equalities (4) and  $r_{ess}(K^*) = r_{ess}(K)$  are valid for any bounded operator  $K$  on a given complex Banach space  $L$  (see e.g. [25, Theorem 4.3.13 and Proposition 4.3.11], [29, Theorem 1]).

It is well-known that kernel operators play a very important, often even central, role in a variety of applications from differential and integro-differential equations, problems from physics (in particular from thermodynamics), engineering, statistical and economic models, etc (see e.g. [20, 34] and the references cited there). For the theory of Banach function spaces and more general Banach lattices we refer the reader to the books [1, 2, 4, 25, 41].

Let  $K$  and  $H$  be positive kernel operators on a Banach function space  $L$  with kernels  $k$  and  $h$  respectively, and  $\alpha \geq 0$ . The *Hadamard (or Schur) product*  $K \circ H$  of  $K$  and  $H$  is the kernel operator with kernel equal to  $k(x, y)h(x, y)$  at point  $(x, y) \in X \times X$  which can be defined (in general) only on some order ideal of  $L$ . Similarly, the *Hadamard (or Schur) power*  $K^{(\alpha)}$  of  $K$  is the kernel operator with kernel equal to  $(k(x, y))^\alpha$  at point  $(x, y) \in X \times X$  which can be defined only on some order ideal of  $L$ .

Let  $K_1, \dots, K_m$  be positive kernel operators on a Banach function space  $L$ , and  $\alpha_1, \dots, \alpha_m$  nonnegative numbers such that  $\sum_{j=1}^m \alpha_j = 1$ . Then the *Hadamard weighted geometric mean*  $K = K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \dots \circ K_m^{(\alpha_m)}$  of the operators  $K_1, \dots, K_m$  is a positive kernel operator defined on the whole space  $L$ , since  $K \leq \alpha_1 K_1 + \alpha_2 K_2 + \dots + \alpha_m K_m$  by the inequality between the weighted arithmetic and geometric means.

Let us recall the following result, which was proved in [12, Theorem 2.2] and [33, Theorem 5.1 and Example 3.7] (see also e.g. [31, Theorem 2.1]).

**THEOREM 1.** *Let  $\{K_{ij}\}_{i=1, j=1}^{n, m}$  be positive kernel operators on a Banach function space  $L$  and  $\alpha_1, \alpha_2, \dots, \alpha_m$  nonnegative numbers.*

*If  $\sum_{j=1}^m \alpha_j = 1$ , then the positive kernel operator*

$$K := \left( K_{11}^{(\alpha_1)} \circ \dots \circ K_{1m}^{(\alpha_m)} \right) \dots \left( K_{n1}^{(\alpha_1)} \circ \dots \circ K_{nm}^{(\alpha_m)} \right) \tag{5}$$

*satisfies the following inequalities*

$$K \leq (K_{11} \dots K_{n1})^{(\alpha_1)} \circ \dots \circ (K_{1m} \dots K_{nm})^{(\alpha_m)}, \tag{6}$$

$$\begin{aligned} \|K\| &\leq \left\| (K_{11} \dots K_{n1})^{(\alpha_1)} \circ \dots \circ (K_{1m} \dots K_{nm})^{(\alpha_m)} \right\| \\ &\leq \|K_{11} \dots K_{n1}\|^{\alpha_1} \dots \|K_{1m} \dots K_{nm}\|^{\alpha_m} \end{aligned} \tag{7}$$

$$\begin{aligned}
 r(K) &\leq r\left((K_{11} \cdots K_{n1})^{(\alpha_1)} \circ \cdots \circ (K_{1m} \cdots K_{nm})^{(\alpha_m)}\right) \\
 &\leq r(K_{11} \cdots K_{n1})^{\alpha_1} \cdots r(K_{1m} \cdots K_{nm})^{\alpha_m}.
 \end{aligned}
 \tag{8}$$

If, in addition,  $L$  and  $L^*$  have order continuous norms, then

$$\begin{aligned}
 \gamma(K) &\leq \gamma\left((K_{11} \cdots K_{n1})^{(\alpha_1)} \circ \cdots \circ (K_{1m} \cdots K_{nm})^{(\alpha_m)}\right) \\
 &\leq \gamma(K_{11} \cdots K_{n1})^{\alpha_1} \cdots \gamma(K_{1m} \cdots K_{nm})^{\alpha_m},
 \end{aligned}
 \tag{9}$$

$$\begin{aligned}
 r_{ess}(K) &\leq r_{ess}\left((K_{11} \cdots K_{n1})^{(\alpha_1)} \circ \cdots \circ (K_{1m} \cdots K_{nm})^{(\alpha_m)}\right) \\
 &\leq r_{ess}(K_{11} \cdots K_{n1})^{\alpha_1} \cdots r_{ess}(K_{1m} \cdots K_{nm})^{\alpha_m}.
 \end{aligned}
 \tag{10}$$

The following result is a special case of Theorem 1.

**THEOREM 2.** *Let  $K_1, \dots, K_m$  be positive kernel operators on a Banach function space  $L$  and  $\alpha_1, \dots, \alpha_m$  nonnegative numbers.*

*If  $\sum_{j=1}^m \alpha_j = 1$ , then*

$$\|K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}\| \leq \|K_1\|^{\alpha_1} \|K_2\|^{\alpha_2} \cdots \|K_m\|^{\alpha_m}
 \tag{11}$$

and

$$r(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}) \leq r(K_1)^{\alpha_1} r(K_2)^{\alpha_2} \cdots r(K_m)^{\alpha_m}.
 \tag{12}$$

If, in addition,  $L$  and  $L^*$  have order continuous norms, then

$$\gamma(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}) \leq \gamma(K_1)^{\alpha_1} \gamma(K_2)^{\alpha_2} \cdots \gamma(K_m)^{\alpha_m}
 \tag{13}$$

and

$$r_{ess}(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}) \leq r_{ess}(K_1)^{\alpha_1} r_{ess}(K_2)^{\alpha_2} \cdots r_{ess}(K_m)^{\alpha_m}.
 \tag{14}$$

We will need the following well-known inequalities (see e.g. [26]). For nonnegative measurable functions and for nonnegative numbers  $\alpha$  and  $\beta$  such that  $\alpha + \beta \geq 1$  we have

$$f_1^\alpha g_1^\beta + \cdots + f_m^\alpha g_m^\beta \leq (f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^\beta.
 \tag{15}$$

More generally, for nonnegative measurable functions  $\{f_{ij}\}_{i=1, j=1}^{n, m}$  and for nonnegative numbers  $\alpha_j, j = 1, \dots, m$ , such that  $\sum_{j=1}^m \alpha_j \geq 1$  we have

$$(f_{11}^{\alpha_1} \cdots f_{1m}^{\alpha_m}) + \cdots + (f_{n1}^{\alpha_1} \cdots f_{nm}^{\alpha_m}) \leq (f_{11} + \cdots + f_{n1})^{\alpha_1} \cdots (f_{1m} + \cdots + f_{nm})^{\alpha_m}.
 \tag{16}$$

### 3. New results

Let  $K$  be a positive kernel operator on  $L = L^2(X, \mu)$  with a kernel  $k$  and let  $\alpha \in [0, 1]$ . Denote by  $S_\alpha(K) = K^{(\alpha)} \circ (K^*)^{(1-\alpha)}$  a positive kernel operator on  $L$  with a kernel  $s_\alpha(k)(x, y) = k^\alpha(x, y)k^{1-\alpha}(y, x)$ . Note that  $S(K) = S_{\frac{1}{2}}(K)$  is a geometric symmetrization of  $K$ , which is a selfadjoint and positive kernel operator on  $L^2(X, \mu)$  with

a kernel  $\sqrt{k(x,y)k(y,x)}$ . Let  $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$ . It was proved in [6, Proposition 2.2 (19), (20)] that

$$\begin{aligned} \rho(S_\alpha(K_1) \cdots S_\alpha(K_n)) &\leq \rho\left((K_1 \cdots K_n)^{(\alpha)} \circ ((K_n \cdots K_1)^*)^{(1-\alpha)}\right) \\ &\leq \rho(K_1 \cdots K_n)^\alpha \rho(K_n \cdots K_1)^{1-\alpha} \end{aligned} \quad (17)$$

and

$$\rho(S_\alpha(K_1) + \dots + S_\alpha(K_n)) \leq \rho(K_1 + \dots + K_n). \quad (18)$$

The following result generalizes [6, Theorem 2.7] by extending a technique of Shen and Huang [40].

**THEOREM 3.** *Let  $K_1, \dots, K_n$  be positive kernel operators on  $L = L^2(X, \mu)$ . For  $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$  define  $\rho_n : [0, 1] \rightarrow [0, \infty)$  by*

$$\rho_n(\alpha) = \sqrt{\rho(S_\alpha(K_1)S_\alpha(K_2) \cdots S_\alpha(K_n))\rho(S_\alpha(K_n)S_\alpha(K_{n-1}) \cdots S_\alpha(K_1))}.$$

*Then  $\rho_n$  is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$ .*

*In particular,  $\rho_n(\alpha) \geq \rho_n(\frac{1}{2})$  for each  $\alpha \in [0, 1]$ .*

*Proof.* Assume  $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$  and let  $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$ . Then  $\alpha \in (0, 1)$  and for every positive kernel operator  $K$  on  $L$  we have  $S_{\alpha_2}(K) = S_\alpha(S_{\alpha_1}(K))$ . Indeed, the kernel of the operator  $S_\alpha(S_{\alpha_1}(K))$  is equal to

$$\begin{aligned} &(s_{\alpha_1}(k)(x,y))^\alpha (s_{\alpha_1}(k)(y,x))^{1-\alpha} \\ &= (k(x,y)^{\alpha_1} k(y,x)^{1-\alpha_1})^\alpha (k(y,x)^{\alpha_1} k(x,y)^{1-\alpha_1})^{1-\alpha} \\ &= k(x,y)^{\alpha_1 \alpha + (1-\alpha_1)(1-\alpha)} k(y,x)^{\alpha(1-\alpha_1) + \alpha_1(1-\alpha)} \\ &= k(x,y)^{\alpha_2} k(y,x)^{1-\alpha_2}, \end{aligned}$$

which is a kernel of the operator  $S_{\alpha_2}(K)$  since

$$\alpha_1 \alpha + (1 - \alpha_1)(1 - \alpha) = \alpha(2\alpha_1 - 1) + 1 - \alpha_1 = \alpha_1 + \alpha_2 - 1 + 1 - \alpha_1 = \alpha_2$$

and

$$\alpha(1 - \alpha_1) + \alpha_1(1 - \alpha) = \alpha(1 - 2\alpha_1) + \alpha_1 = 1 - \alpha_1 - \alpha_2 + \alpha_1 = 1 - \alpha_2.$$

It follows from (17) that

$$\begin{aligned} \rho_n(\alpha_2) &= \sqrt{\rho(S_{\alpha_2}(K_1)S_{\alpha_2}(K_2) \cdots S_{\alpha_2}(K_n))\rho(S_{\alpha_2}(K_n)S_{\alpha_2}(K_{n-1}) \cdots S_{\alpha_2}(K_1))} \\ &= \sqrt{\rho(S_\alpha(S_{\alpha_1}(K_1)) \cdots S_\alpha(S_{\alpha_1}(K_n)))\rho(S_\alpha(S_{\alpha_1}(K_n)) \cdots S_\alpha(S_{\alpha_1}(K_1)))} \\ &\leq \sqrt{\rho(S_{\alpha_1}(K_1) \cdots S_{\alpha_1}(K_n))^\alpha \rho(S_{\alpha_1}(K_n) \cdots S_{\alpha_1}(K_1))^{1-\alpha}} \\ &\quad \times \sqrt{\rho(S_{\alpha_1}(K_n) \cdots S_{\alpha_1}(K_1))^\alpha \rho(S_{\alpha_1}(K_1) \cdots S_{\alpha_1}(K_n))^{1-\alpha}} \\ &= \sqrt{\rho(S_{\alpha_1}(K_1) \cdots S_{\alpha_1}(K_n))\rho(S_{\alpha_1}(K_n) \cdots S_{\alpha_1}(K_1))} = \rho_n(\alpha_1), \end{aligned}$$

which proves that  $\rho_n$  is decreasing on  $[0, \frac{1}{2}]$ .

Similarly, in the case  $\frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1$  let  $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_2 - 1}$ . It follows that  $\alpha \in (0, 1)$  and  $S_{\alpha_1}(K) = S_\alpha(S_{\alpha_2}(K))$  for every positive kernel operator on  $L$ . Similarly as before this holds since the kernel of  $S_\alpha(S_{\alpha_2}(K))$  equals

$$\begin{aligned} & (s_{\alpha_2}(k)(x, y))^\alpha (s_{\alpha_2}(k)(y, x))^{1-\alpha} \\ &= (k(x, y)^{\alpha_2} k(y, x)^{1-\alpha_2})^\alpha (k(y, x)^{\alpha_2} k(x, y)^{1-\alpha_2})^{1-\alpha} \\ &= k(x, y)^{\alpha_2 \alpha + (1-\alpha_2)(1-\alpha)} k(y, x)^{\alpha(1-\alpha_2) + \alpha_2(1-\alpha)} \\ &= k(x, y)^{\alpha_1} k(y, x)^{1-\alpha_1}, \end{aligned}$$

which is a kernel of the operator  $S_{\alpha_1}(K)$  since  $\alpha_2 \alpha + (1 - \alpha_2)(1 - \alpha) = \alpha_1$  and  $\alpha(1 - \alpha_2) + \alpha_2(1 - \alpha) = 1 - \alpha_1$ . From (17) we obtain

$$\begin{aligned} \rho_n(\alpha_1) &= \sqrt{\rho(S_{\alpha_1}(K_1) \cdots S_{\alpha_1}(K_n)) \rho(S_{\alpha_1}(K_n) \cdots S_{\alpha_1}(K_1))} \\ &= \sqrt{\rho(S_\alpha(S_{\alpha_2}(K_1)) \cdots S_\alpha(S_{\alpha_2}(K_n))) \rho(S_\alpha(S_{\alpha_2}(K_n)) \cdots S_\alpha(S_{\alpha_2}(K_1)))} \\ &\leq \sqrt{\rho(S_{\alpha_2}(K_1) \cdots S_{\alpha_2}(K_n))^\alpha \rho(S_{\alpha_2}(K_n) \cdots S_{\alpha_2}(K_1))^{1-\alpha}} \\ &\quad \times \sqrt{\rho(S_{\alpha_2}(K_n) \cdots S_{\alpha_2}(K_1))^\alpha \rho(S_{\alpha_2}(K_1) \cdots S_{\alpha_2}(K_n))^{1-\alpha}} \\ &= \sqrt{\rho(S_{\alpha_2}(K_1) \cdots S_{\alpha_2}(K_n)) \rho(S_{\alpha_2}(K_n) \cdots S_{\alpha_2}(K_1))} = \rho_n(\alpha_2), \end{aligned}$$

which proves that  $\rho_n$  is increasing on  $[\frac{1}{2}, 1]$ .  $\square$

In the case  $n = 1$  we obtain the result from [6, Theorem 2.7].

**COROLLARY 1.** *Let  $K$  be a positive kernel operator on  $L^2(X, \mu)$  and  $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$ . Then a function  $\rho_1 : [0, 1] \rightarrow [0, \infty)$ , defined by  $\rho_1(\alpha) = \rho(S_\alpha(K))$ , is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$ .*

*In particular,  $\rho(S_\alpha(K)) \geq \rho(S(K))$  for each  $\alpha \in [0, 1]$ .*

**COROLLARY 2.** *Let  $K_1$  and  $K_2$  be positive kernel operators on  $L^2(X, \mu)$  and  $\rho \in \{r, r_{ess}\}$ . Then  $\rho_2(\alpha) = \rho(S_\alpha(K_1)S_\alpha(K_2))$  is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$ .*

*In particular,  $\rho(S_\alpha(K_1)S_\alpha(K_2)) \geq \rho(S(K_1)S(K_2))$  for each  $\alpha \in [0, 1]$ .*

*Proof.* The statement follows from Theorem 3 since  $\rho(AB) = \rho(BA)$  for all bounded operators  $A$  and  $B$  on  $L^2(X, \mu)$ .  $\square$

**PROPOSITION 4.** *Let  $K_1, \dots, K_n$  be positive kernel operators on  $L = L^2(X, \mu)$  and  $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$ . Then a function  $\tilde{\rho}_n : [0, 1] \rightarrow [0, \infty)$ , defined by  $\tilde{\rho}_n(\alpha) = \rho(S_\alpha(K_1) + \cdots + S_\alpha(K_n))$ , is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$ .*

*Proof.* Let  $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$ . For  $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$  we have  $\alpha \in (0, 1)$  and  $S_{\alpha_2}(K) = S_\alpha(S_{\alpha_1}(K))$  (see the proof of Theorem 3). By applying (18) we obtain

$$\begin{aligned} \tilde{\rho}_n(\alpha_2) &= \rho(S_{\alpha_2}(K_1) + \dots + S_{\alpha_2}(K_n)) \\ &= \rho(S_\alpha(S_{\alpha_1}(K_1)) + \dots + S_\alpha(S_{\alpha_1}(K_n))) \\ &\leq \rho(S_{\alpha_1}(K_1) + \dots + S_{\alpha_1}(K_n)) \\ &= \tilde{\rho}_n(\alpha_1), \end{aligned}$$

which proves that  $\tilde{\rho}_n$  is decreasing on  $[0, \frac{1}{2}]$ .

For  $\frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1$  let  $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_2 - 1}$ . Then  $\alpha \in (0, 1)$  and  $S_{\alpha_1}(K) = S_\alpha(S_{\alpha_2}(K))$  (see the proof of Theorem 3). By (18) it follows that

$$\begin{aligned} \tilde{\rho}_n(\alpha_1) &= \rho(S_{\alpha_1}(K_1) + \dots + S_{\alpha_1}(K_n)) \\ &= \rho(S_\alpha(S_{\alpha_2}(K_1)) + \dots + S_\alpha(S_{\alpha_2}(K_n))) \\ &\leq \rho(S_{\alpha_2}(K_1) + \dots + S_{\alpha_2}(K_n)) = \tilde{\rho}_n(\alpha_2), \end{aligned}$$

which proves that  $\tilde{\rho}_n$  is increasing on  $[\frac{1}{2}, 1]$ .  $\square$

By applying (15) also the following more general result follows.

**THEOREM 5.** *Let  $K_{ij}$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$  be positive kernel operators on  $L = L^2(X, \mu)$ . For  $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$  define  $\bar{\rho}_n : [0, 1] \rightarrow [0, \infty)$  by*

$$\begin{aligned} \bar{\rho}_n(\alpha) &= (\rho((S_\alpha(K_{11}) + \dots + S_\alpha(K_{1m})) \cdots (S_\alpha(K_{n1}) + \dots + S_\alpha(K_{nm}))))^{\frac{1}{2}} \\ &\quad \times (\rho((S_\alpha(K_{n1}) + \dots + S_\alpha(K_{nm})) \cdots (S_\alpha(K_{11}) + \dots + S_\alpha(K_{1m}))))^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{aligned} \bar{\rho}_n(\alpha) &\leq \rho((K_{11} + \dots + K_{1m}) \cdots (K_{n1} + \dots + K_{nm}))^{\frac{1}{2}} \\ &\quad \times \rho((K_{n1} + \dots + K_{nm}) \cdots (K_{11} + \dots + K_{1m}))^{\frac{1}{2}} \end{aligned} \tag{19}$$

for each  $\alpha \in [0, 1]$ .

Moreover,  $\bar{\rho}_n$  is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$ .

In particular,  $\bar{\rho}_n(\alpha) \geq \bar{\rho}_n(\frac{1}{2})$  for each  $\alpha \in [0, 1]$ .

*Proof.* Let  $\alpha \in [0, 1]$ . By (15) it follows that

$$S_\alpha(K_{i1}) + \dots + S_\alpha(K_{im}) \leq S_\alpha(K_{i1} + \dots + K_{im}) \tag{20}$$



holds for all  $i = 1, \dots, n$ . From (20) and (17) it follows that

$$\begin{aligned} \bar{\rho}_n(\alpha) &\leq \rho(S_\alpha(K_{11} + \dots + K_{1m}) \cdots S_\alpha(K_{n1} + \dots + K_{nm}))^{\frac{1}{2}} \\ &\quad \times \rho(S_\alpha(K_{n1} + \dots + K_{nm}) \cdots S_\alpha(K_{11} + \dots + K_{1m}))^{\frac{1}{2}} \\ &\leq \rho((K_{11} + \dots + K_{1m}) \cdots (K_{n1} + \dots + K_{nm}))^{\frac{\alpha}{2}} \\ &\quad \times \rho((K_{n1} + \dots + K_{nm}) \cdots (K_{11} + \dots + K_{1m}))^{\frac{1-\alpha}{2}} \\ &\quad \times \rho((K_{n1} + \dots + K_{nm}) \cdots (K_{11} + \dots + K_{1m}))^{\frac{\alpha}{2}} \\ &\quad \times \rho((K_{11} + \dots + K_{1m}) \cdots (K_{n1} + \dots + K_{nm}))^{\frac{1-\alpha}{2}} \\ &= \rho((K_{11} + \dots + K_{1m}) \cdots (K_{n1} + \dots + K_{nm}))^{\frac{1}{2}} \\ &\quad \times \rho((K_{n1} + \dots + K_{nm}) \cdots (K_{11} + \dots + K_{1m}))^{\frac{1}{2}}, \end{aligned}$$

which proves (19).

To prove that  $\bar{\rho}_n$  is decreasing on  $[0, \frac{1}{2}]$  let  $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$ . For  $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$  we have  $\alpha \in (0, 1)$  and  $S_{\alpha_2}(K) = S_\alpha(S_{\alpha_1}(K))$ . Then by (19)

$$\begin{aligned} &\bar{\rho}_n(\alpha_2) \\ &= (\rho((S_\alpha(S_{\alpha_1}(K_{11}))) + \dots + S_\alpha(S_{\alpha_1}(K_{1m}))) \cdots (S_\alpha(S_{\alpha_1}(K_{n1})) + \dots + S_\alpha(S_{\alpha_1}(K_{nm}))))^{\frac{1}{2}} \\ &\quad \times (\rho((S_\alpha(S_{\alpha_1}(K_{n1}))) + \dots + S_\alpha(S_{\alpha_1}(K_{nm}))) \cdots (S_\alpha(S_{\alpha_1}(K_{11})) + \dots + S_\alpha(S_{\alpha_1}(K_{1m}))))^{\frac{1}{2}} \\ &\leq (\rho((S_{\alpha_1}(K_{11}) + \dots + S_{\alpha_1}(K_{1m})) \cdots (S_{\alpha_1}(K_{n1}) + \dots + S_{\alpha_1}(K_{nm}))))^{\frac{1}{2}} \\ &\quad \times (\rho((S_{\alpha_1}(K_{n1}) + \dots + S_{\alpha_1}(K_{nm})) \cdots (S_{\alpha_1}(K_{11}) + \dots + S_{\alpha_1}(K_{1m}))))^{\frac{1}{2}} \\ &= \bar{\rho}_n(\alpha_1), \end{aligned}$$

which establishes that  $\bar{\rho}_n$  is decreasing on  $[0, \frac{1}{2}]$ .

To prove that  $\bar{\rho}_n$  is increasing on  $[\frac{1}{2}, 1]$  let  $\frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1$ . For  $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_2 - 1}$  we have  $\alpha \in (0, 1)$  and  $S_{\alpha_1}(K) = S_\alpha(S_{\alpha_2}(K))$ . Similarly as above it follows from (19) that  $\bar{\rho}_n(\alpha_1) \leq \bar{\rho}_n(\alpha_2)$ , which completes the proof.  $\square$

**COROLLARY 3.** *Let  $K_1, \dots, K_m$  and  $H_1, \dots, H_m$  be positive kernel operators on  $L = L^2(X, \mu)$ . For  $\rho \in \{r, r_{ess}\}$  a function  $\bar{\rho}_2 : [0, 1] \rightarrow [0, \infty)$ , defined by*

$$\bar{\rho}_2(\alpha) = \rho((S_\alpha(K_1) + \dots + S_\alpha(K_m))(S_\alpha(H_1) + \dots + S_\alpha(H_m))),$$

satisfies

$$\bar{\rho}_n(\alpha) \leq \rho((K_1 + \dots + K_m)(H_1 + \dots + H_m)) \tag{21}$$

for each  $\alpha \in [0, 1]$ .

Moreover,  $\bar{\rho}_2$  is decreasing on  $[0, \frac{1}{2}]$  and increasing on  $[\frac{1}{2}, 1]$ .

In particular,  $\bar{\rho}_2(\alpha) \geq \bar{\rho}_2(\frac{1}{2})$  for each  $\alpha \in [0, 1]$ .

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