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A NEW HALF–DISCRETE MULTIDIMENSIONAL HILBERT–TYPE INEQUALITY INVOLVING ONE HIGHER–ORDER DERIVATIVE FUNCTION

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Abstract. This paper presents a new half-discrete multidimensional Hilbert-type inequality involving one higher-order derivative function utilizing transfer formula and Hermite–Hadamard's inequality. The inequality investigates a general intermediate variable in kernel $\frac{1}{(x+\|v(k)\|_\alpha)^{\lambda+m}}$ $(x,\lambda>0)$ than previous work. The research explores the best value related to certain parameters. Finally, the equivalence forms and operator expressions are also presented.

1. Introduction

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\geqslant 0$, $0<\sum_{m=1}^{\infty}a_m^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$, we have the following Hardy-Hilbert's inequality (cf. [3], Theorem 315):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{m+n}<\frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where, $\pi/\sin(\frac{\pi}{p})$ is the best value.

In 2006, Krnić (see. [10]) provided an extension of (1) below by using the Euler-Maclaurin's summation formula:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

where, $\lambda_i\in(0,2]$, $i=1,2$, $\lambda_1+\lambda_2=\lambda\in(0,4]$. The best value $B(\lambda_1,\lambda_2)$ is expressed by the beta function as follows:

$$B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt \quad (u,v>0). \tag{3}$$

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The half-discrete Hilbert-type inequality with a nonhomogeneous kernel was first described by Hardy et al. in 1934. (cf. [3], Theorem 351): If the function $K(t)$ is a decreasing, $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ 0<\phi(s)=\int_0^\infty K(x)x^{s-1}dx<\infty,\ f(x)\geqslant 0,\ 0<\int_0^\infty f^p(x)dx<\infty$, then we have

$$\sum_{m=1}^{\infty} n^{p-2}\left(\int_0^\infty K(nx)f(x)dx\right)^p < \phi^p\left(\frac{1}{q}\right)\int_0^\infty f^p(x)dx. \tag{4}$$

Based on (4), M. You obtained some special functions such as hyperbolic functions (cf. [20]), and the cotangent function (cf. [15]) in half-discrete Hilbert-type inequality. In a 2016 publication, Y. Hong [8] discussed the equivalent statements of these inequalities and explored the best values. Subsequently, Y. Hong [7] expanded his theoretical research on multiple Hilbert-type integral inequalities. The quasi-homogeneous kernel involving half-discrete Hilbert-type inequality was also discussed in [5]. According to this theory, some extension works about the equivalent statements on half-discrete inequalities were brought up by [4, 6, 13, 17, 19].

In addition, some applications about Hilbert-type inequality were obtained by [1, 2, 16, 18]. Recently, Y. Hong et al. [9] came up with the idea of weight functions and used the transfer formula and Hermite–Hadamard's inequality to derive a half-discrete multidimensional Hilbert-type inequality with a homogeneous kernel.

Utilizing the extension transfer formula and the approach from [9], this paper comes up with a new half-discrete multidimensional Hilbert-type inequality involving one higher-order derivative function. The inequality investigates a general intermediate variable in kernel $\frac{1}{(x+\|v(k)\|_\alpha)^{\lambda+m}}$ $(x,\lambda>0)$ than previous work [9]. The equivalent statements outline the comparable expressions for the optimal constant factor associated with certain parameters. Finally, the equivalent forms and the operator expressions are considered.

2. Some Lemmas

Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda>0,\ \lambda_1\in(0,\lambda),\ \lambda_2\in(0,\lambda)\cap(0,n],\ n\in\mathbf{N}=\{1,2,\ldots\}$, $m\in\mathbf{N}_0=\mathbf{N}\cup\{0\},\ \alpha\in(0,1],\ \xi\in[0,\frac{1}{2}],\ v(y)=(v_1(y_1),\cdots,v_n(y_n)),\ y\in A_\xi:=\{y=\{y_1,\cdots y_n\};\ y_i\in(\xi,\infty)\}$, such that $v_i(y_i)>0,\ v_i'(y_i)>0,\ v_i''(y_i)\leqslant 0,\ v_i'''(y_i)\geqslant 0$, $v_i(\xi^+)=0,\ v_i(\infty)=\infty\ \ (i=1,\cdots n)$. $\widetilde{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\ \widetilde{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. We also suppose that $f(x)\geqslant 0,\ f^{(m)}(x)\ (m\in\mathbf{N_0})$ is a nonnegative continuous function except at finite points in $\mathbf{R}_+:=(0,\infty)$, and

$$\begin{aligned}
&f^{(k-1)}(x)=o(e^{tx})\ \ (t>0;\ x\longrightarrow\infty),\\
&f^{(k-1)}(0^+)=0,\ \ k=1,\cdots,m\ \ (m\in\mathbf{N}),\\
&\qquad a_k=(a_{k_1},\cdots,a_{k_n})\geqslant 0\ \ (x\in\mathbf{R}_+,\ k=(k_1,\cdots,k_n)\in\mathbf{N}^n),
\end{aligned}$$

satisfying for $m \in \mathbf{N_0}$,

$$0 < \int_0^\infty x^{p(1-\widetilde{\lambda}_1)-1} (f^{(m)}(x))^p dx < \infty \quad \text{and} \quad 0 < \sum_k \frac{\|v(k)\|_\alpha^{q\left(n-\widetilde{\lambda}_2\right)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} a_k^q < \infty.$$

Assuming that $M > 0$, $\psi(u)$ $(u > 0)$ is a nonnegative measurable function, we have the following transfer formula (cf. [14]):

$$\int \cdots \int_{\{y \in \mathbf{R}_+^n; 0 < \sum_{i=1}^n \left(\frac{y_i}{M}\right)^\alpha \leqslant 1\}} \psi\left(\sum_{i=1}^n \left(\frac{y_i}{M}\right)^\alpha\right) dy_1 \cdots dy_n = \frac{M^n \Gamma^n(\frac{1}{\alpha})}{\alpha^n \Gamma(\frac{n}{\alpha})} \int_0^1 \psi(u) u^{\frac{n}{\alpha}-1} du. \tag{5}$$

Particularly, (i) for $\|y\|_\alpha = M[\sum_{k=1}^n \left(\frac{y_i}{M}\right)^\alpha]^{\frac{1}{\alpha}}$, $\psi(u) = \phi\left(Mu^{\frac{1}{\alpha}}\right)$, by (5), we derive

$$\begin{aligned}
&\int_{R_+^n} \phi(\|y\|_\alpha) dy \\
&= \lim_{M \longrightarrow \infty} \int \cdots \int_{\{y \in \mathbf{R}_+^n; 0 < \sum_{i=1}^n \left(\frac{y_i}{M}\right)^\alpha \leqslant 1\}} \phi\left(M\left[\sum_{k=1}^n (\frac{y_i}{M})^\alpha\right]^{\frac{1}{\alpha}}\right) dy_1 \cdots dy_n \\
&= \lim_{M \longrightarrow \infty} \frac{M^n \Gamma^n(\frac{1}{\alpha})}{\alpha^n \Gamma(\frac{n}{\alpha})} \int_0^1 \phi\left(Mu^{\frac{1}{\alpha}}\right) u^{\frac{n}{\alpha}-1} du \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1} \Gamma(\frac{n}{\alpha})} \int_0^\infty \phi(v) v^{n-1} dv \quad \left(v = Mu^{\frac{1}{\alpha}}\right).
\end{aligned} \tag{6}$$

(ii) For $\psi(u) = \phi\left(Mu^{\frac{1}{\alpha}}\right) = 0$, $0 < u < \frac{b^\alpha}{M^\alpha}$ $(b > 0)$, by (5), we derive

$$\begin{aligned}
&\int_{\{y \in \mathbf{R}_+^n, \|y\|_\alpha \geqslant b\}} \phi(\|y\|_\alpha) dy \\
&= \lim_{M \longrightarrow \infty} \frac{M^n \Gamma^n(\frac{1}{\alpha})}{\alpha^n \Gamma(\frac{n}{\alpha})} \int_{\frac{b^\alpha}{M^\alpha}}^1 \phi\left(Mu^{\frac{1}{\alpha}}\right) u^{\frac{n}{\alpha}-1} du \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1} \Gamma(\frac{n}{\alpha})} \int_b^\infty \phi(v) v^{n-1} dv.
\end{aligned} \tag{7}$$

LEMMA 2.1. *Suppose that $\lambda > 0$, $\alpha \in (0,1]$, define the following function*

$$h_x(y) := \frac{\|v(y)\|_\alpha^{\lambda_2-n} \prod_{i=1}^n v_i'(y_i)}{(x + \|v(y)\|_\alpha)^\lambda} \quad \left(x > 0,\ y \in A_\xi,\ \ y_i \in (\xi, \infty)\right).$$

Then $\frac{\partial}{\partial y_j} h_x(y) < 0$, $\frac{\partial^2}{\partial y_j^2} h_x(y) > 0$ $(y \in A_\xi;\ j = 1, \cdots, n)$.

Proof. Since $\lambda > 0,\ \alpha \in (0,1],\ \xi \in [0,\frac{1}{2}],\ y \in A_\xi$, we define

$$g_x(y) := \frac{1}{(x + \|v(y)\|_\alpha)^\lambda} = \frac{1}{[x + (\sum_{i=1}^n v_i^\alpha(y_i))^{\frac{1}{\alpha}}]^\lambda},$$
$$f_1(y) := \|v(y)\|_\alpha^{\lambda_2 - n} = \left(\sum_{i=1}^n v_i^\alpha(y_i)\right)^{\frac{\lambda_2 - n}{\alpha}},$$
$$f_2(y) := \prod_{i=1}^n v_i'(y_i).$$

Since $v_j'(y_j) > 0,\ v_j''(y_j) \leqslant 0,\ v_j'''(y_j) \geqslant 0$, we derive

$$\frac{\partial}{\partial y_j} g_x(y) = \frac{-\lambda(\sum_{i=1}^n v_i^\alpha(y_i))^{(1/\alpha)-1} v_j^{\alpha-1}(y_j) v_j'(y_j)}{[x + (\sum_{i=1}^n v_i^\alpha(y_i))^{1/\alpha}]^{\lambda+1}} < 0,$$
$$\frac{\partial}{\partial y_j} f_1(y) = (\lambda_2 - n)\left(\sum_{i=1}^n v_i^\alpha(y_i)\right)^{\frac{\lambda_2 - n}{\alpha} - 1} v_j^{\alpha-1}(y_j) v_j'(y_j) \leqslant 0,$$
$$\frac{\partial}{\partial y_j} f_2(y) = v_j''(y_j) \prod_{i=1,(i\neq j)}^n v_i'(y_i) \leqslant 0, \quad \frac{\partial^2}{\partial y_j^2} f_1(y) \geqslant 0, \quad \frac{\partial^2}{\partial y_j^2} f_2(y) \geqslant 0.$$

We still can find that $\frac{\partial^2}{\partial y_j^2} g_x(y) > 0$, and then in the same way,

$$\frac{\partial}{\partial y_j} h_x(y) = f_1(y) f_2(y) \frac{\partial}{\partial y_j} g_x(y) + g_x(y) \frac{\partial}{\partial y_j}(f_1(y) f_2(y)) < 0,$$
$$\frac{\partial^2}{\partial y_j^2} h_x(y) = \frac{\partial}{\partial y_j}\left[f_1(y) f_2(y) \frac{\partial}{\partial y_j} g_x(y)\right] + \frac{\partial}{\partial y_j}\left[g_x(y) \frac{\partial}{\partial y_j}(f_1(y) f_2(y))\right]$$
$$> 0 \ \ (y_j \in (\xi, \infty),\ j = 1, \cdots, n).$$

The lemma has been shown. □

LEMMA 2.2. *For* $n \in \mathbf{N},\ c > 0$,

$$b = \min_{1 \leqslant i \leqslant n}\{v_i(1)\}, \quad e = \max_{1 \leqslant i \leqslant n}\{v_i(1)\}(>0),$$

there exists a constant $a_n \in \mathbf{R}_+$, *such that the following inequalities hold:*

$$\frac{e^{-c}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left(\frac{1}{c} - \frac{n-1}{1+c}\right)$$
$$< \sum_k \|v(k)\|_\alpha^{-c-n} \prod_{i=1}^n v_i'(k_i) < a_n + \frac{b^{-c}\Gamma^n(\frac{1}{\alpha})}{c\alpha^{n-1}\Gamma(\frac{n}{\alpha})}, \tag{8}$$

where $a_1 := v^{-c-1}(1)v'(1),\ a_n := \sum_{i=1}^{n} M_i \in \mathbf{R}_+\ \ (n \in \mathbf{N}\setminus\{1\})$, *satisfying*

$$M_i = \sum_{k_1=1}^{\infty}\cdots\sum_{k_{i-1}=1}^{\infty}\sum_{k_{i+1}=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[v_1^{\alpha}(k_1)+\cdots v_{i-1}^{\alpha}(k_{i-1})+v_i^{\alpha}(1)\right.$$
$$\left.+v_{i+1}^{\alpha}(k_{1i+1})+\cdots+v_n^{\alpha}(k_n)\right]^{\frac{1}{\alpha}(-c-n)} v_i'(1)\prod_{j=1(j\neq i)}^{n} v_j'(k_j)\ \ (i=1,\cdots,n).$$

Proof. Since $c>0,\ \alpha\in(0,1],\ v_i'(y_i)>0,\ v_i''(y_i)\leqslant 0,\ v_i'''(y_i)\geqslant 0,$ for $j=1,\cdots,n,$ similar to the proof of Lemma 2.1, we can derive

$$\frac{\partial}{\partial y_j}\left[\|v(y)\|_{\alpha}^{-c-n}\prod_{i=1}^{n}v_i'(y_i)\right]<0,$$
$$\frac{\partial^2}{\partial y_j^2}\left[\|v(y)\|_{\alpha}^{-c-n}\prod_{i=1}^{n}v_i'(y_i)\right]>0\ \ (j=1,\cdots,n).$$

Supposed that $f(x):=(x+d)^{\frac{1}{\alpha}}-x^{\frac{1}{\alpha}}-d^{\frac{1}{\alpha}}\ \ (x,d\geqslant 0)$, we have

$$f'(x):=\frac{1}{\alpha}[(x+d)^{\frac{1}{\alpha}-1}-x^{\frac{1}{\alpha}-1}]\geqslant 0\ \ (\alpha\in(0,1]).$$

For $f(0)=0$, we find $(x+d)^{\frac{1}{\alpha}}\geqslant x^{\frac{1}{\alpha}}+d^{\frac{1}{\alpha}}$. Then for $n=1$, we find $a_1=v^{-c-1}(1)v'(1)\in\mathbf{R}_+$; for $n\in\mathbf{N}\setminus\{1\}$, by (6) and the above inequality, we obtain

$$\begin{aligned}
0<M_n&<\int_{\{y_i\geqslant\frac{1}{2},i=1,\cdots,n-1\}}\left(\sum_{j=1}^{n-1}v_j^{\alpha}(y_j)+v_n^{\alpha}(1)\right)^{-\frac{1}{\alpha}(c+n)}\\
&\quad\times v_n'(1)\prod_{i=1}^{n-1}v_i'(y_i)dy\\
&\overset{u=v(y)}{=} v_n'(1)\int_{\{u_i\geqslant v_i(\frac{1}{2}),i=1,\cdots,n-1\}}\left(\sum_{j=1}^{n-1}u_j^{\alpha}+v_n^{\alpha}(1)\right)^{-\frac{1}{\alpha}(c+n)}du\\
&\leqslant v_n'(1)\int_{\mathbf{R}_+^{n-1}}\left[\left(\sum_{j=1}^{n-1}u_j^{\alpha}\right)^{\frac{1}{\alpha}}+v_n(1)\right]^{-c-n}du\\
&=v_n'(1)\frac{\Gamma^{n-1}(\frac{1}{\alpha})}{\alpha^{n-2}\Gamma(\frac{n-1}{\alpha})}\int_0^{\infty}(x+v_n(1))^{-c-n}x^{(n-1)-1}dx\\
&=\frac{\Gamma^{n-1}(\frac{1}{\alpha})}{\alpha^{n-2}\Gamma(\frac{n-1}{\alpha})}\frac{v_n'(1)}{v_n^{c+1}(1)}\int_0^{\infty}\frac{y^{n-2}}{(1+y)^{c+n}}dy\\
&=\frac{\Gamma^{n-1}(\frac{1}{\alpha})}{\alpha^{n-2}\Gamma(\frac{n-1}{\alpha})}\frac{v_n'(1)}{v_n^{c+1}(1)}B(n-1,c+1)<\infty.
\end{aligned}$$

Above all, $M_i \ (i=1,\cdots,n)$ is a positive constant, then, $a_n \ (n\in\mathbf{N})$ is a positive constant.

For $n\in\mathbf{N}\setminus\{1\}$, setting $k'=(k'_1,\cdots,k'_n),\ (k'_i\in\{2,3,\cdots\},\ i=1,\cdots,n)$, by (7), we obtain

$$\begin{aligned}
\sum_k \|v(k)\|_\alpha^{-c-n}\prod_{i=1}^n v_i'(k_i) &\leqslant a_n+\sum_{k'}\left\|v(k')\right\|_\alpha^{-c-n}\prod_{i=1}^n v_i'(k_i')\\
&< a_n+\int_{\left\{y\in\mathbf{R}_+^n;y_i\geqslant 1\right\}}\|v(y)\|_\alpha^{-c-n}\prod_{i=1}^n v_i'(y_i)dy\\
&= a_n+\int_{\left\{u\in\mathbf{R}_+^n;u_i\geqslant v_i(1)\right\}}\|u\|_\alpha^{-c-n}du\\
&\leqslant a_n+\int_{\left\{u\in\mathbf{R}_+^n;u_i\geqslant b\right\}}\|u\|_\alpha^{-c-n}du\\
&\leqslant a_n+\int_{\left\{u\in\mathbf{R}_+^n;\|u\|_\alpha\geqslant b\right\}}\|u\|_\alpha^{-c-n}du\\
&= a_n+\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_b^\infty x^{-c-n}x^{n-1}dx\\
&= a_n+\frac{b^{-c}\Gamma^n(\frac{1}{\alpha})}{c\alpha^{n-1}\Gamma(\frac{n}{\alpha})}.
\end{aligned}$$

We cand find that the above result is satisfied for $n=1$.

On the other hand, we obtain

$$\begin{aligned}
&\sum_k \|v(k)\|_\alpha^{-c-n}\prod_{i=1}^n v_i'(k_i)\\
&> \int_{\left\{y\in\mathbf{R}_+^n;y_i\geqslant 1\right\}}\|v(y)\|_\alpha^{-c-n}\prod_{i=1}^n v_i'(y_i)dy=\int_{\left\{u\in\mathbf{R}_+^n;u_i\geqslant v_i(1)\right\}}\|u\|_\alpha^{-c-n}du\\
&\geqslant \int_{\left\{u\in\mathbf{R}_+^n;u_i\geqslant e\right\}}\|u\|_\alpha^{-c-n}du \overset{w=u-e}{=} \int_{\mathbf{R}_+^n}\|w+e\|_\alpha^{-c-n}dw.
\end{aligned}$$

Setting $\phi(x):=x^{-c-n}\ \ (x>0)$, by (6), we have

$$\begin{aligned}
&\int_{\mathbf{R}_+^n}\|w+e\|_\alpha^{-c-n}dw=\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \phi(x+e)x^{n-1}dx\\
&=\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left[\int_0^\infty\frac{(x+e)^{n-1}dx}{(x+e)^{c+n}}-\int_0^\infty\frac{(x+e)^{n-1}-x^{n-1}}{(x+e)^{c+n}}dx\right]\\
&=\frac{e^{-c}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left(\frac{1}{c}-A_n(c)\right),
\end{aligned}$$

where we indicate that

$$A_n(c):=e^c\int_0^\infty\frac{(x+e)^{n-1}-x^{n-1}}{(x+e)^{c+n}}dx.$$

Fot $n=1,$ we find $A_1(c)=0;$ for $n\in\mathbf{N}\setminus\{1\},$ by the mid-value theorem, we have

$$A_n(c) = (n-1)e^{1+c}\int_0^\infty \frac{(x+\theta_x e)^{n-2}dx}{(x+e)^{c+n}} \quad (\theta_x\in(0,1))$$
$$\leqslant (n-1)e^{1+c}\int_0^\infty \frac{(x+e)^{n-2}}{(x+e)^{c+n}}dx = \frac{n-1}{1+c}.$$

Hence, it follows that

$$\sum_k \|v(k)\|_\alpha^{-c-n}\prod_{i=1}^n v_i'(k_i) > \int_{\mathbf{R}_+^n}\|w+e\|_\alpha^{-c-n}dw$$
$$\geqslant \frac{e^{-c}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left(\frac{1}{c}-\frac{n-1}{1+c}\right).$$

Inequalities (8) are valid.
This proves the lemma. □

LEMMA 2.3. *Define the following weight functions:*

$$\widetilde{\omega}_\lambda(\lambda_2,x) := x^{\lambda-\lambda_2}\sum_k \frac{\|v(k)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(k_i)}{(x+\|v(k)\|_\alpha)^\lambda} \quad (x\in\mathbf{R}_+), \tag{9}$$

$$\omega_\lambda(\lambda_1,k) := \|v(k)\|_\alpha^{\lambda-\lambda_1}\int_0^\infty \frac{x^{\lambda_1-1}}{(x+\|v(k)\|_\alpha)^\lambda}dx \quad (k\in\mathbf{N}^n). \tag{10}$$

(i) For $\lambda_2\leqslant n,\ 0<\lambda_2<\lambda,$ the following inequality holds:

$$\widetilde{\omega}_\lambda(\lambda_2,x) < \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2),\ \ x\in\mathbf{R}_+. \tag{11}$$

(ii) For $0<\lambda_1<\lambda,$ the following expression holds:

$$\omega_\lambda(\lambda_1,k) = B(\lambda_1,\lambda-\lambda_1),\ \ k\in\mathbf{N}^n. \tag{12}$$

Proof. (i) For $\lambda_2\leqslant n,\ 0<\lambda_2<\lambda,$ by employing Lemma 2.1 and Hermite-Hadamard's inequality (cf. [11]), putting $u=v(y),\ du=\prod_{i=1}^n v_i'(y_i)dy,$ we have

$$\widetilde{\omega}_\lambda(\lambda_2,x) < x^{\lambda-\lambda_2}\int_{A_{1/2}}\frac{\|v(y)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(y_i)}{(x+\|v(y)\|_\alpha)^\lambda}dy$$
$$\leqslant x^{\lambda-\lambda_2}\int_{A_\xi}\frac{\|v(y)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(y_i)}{(x+\|v(y)\|_\alpha)^\lambda}dy$$
$$\overset{u=v(y)}{=} x^{\lambda-\lambda_2}\int_{\mathbf{R}_+^n}\frac{\|u\|_\alpha^{\lambda_2-n}}{(x+\|u\|_\alpha)^\lambda}du.$$

Setting $\phi(s) := \frac{s^{\lambda_2-n}}{(x+s)^{\lambda}}$, by (6), we obtain

$$\begin{aligned}
\widetilde{\omega}_{\lambda}(\lambda_2,x) &< x^{\lambda-\lambda_2}\int_{\mathbf{R}_+^n}\phi(\|u\|_{\alpha})du \\
&= x^{\lambda-\lambda_2}\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^{\infty}\phi(s)s^{n-1}ds \\
&= x^{\lambda-\lambda_2}\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^{\infty}\frac{s^{\lambda_2-1}}{(x+s)^{\lambda}}ds \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^{\infty}\frac{t^{\lambda_2-1}}{(1+t)^{\lambda}}dt \quad (t = s/x) \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2).
\end{aligned}$$

Then (11) is proved.

(ii) Setting $s = \frac{x}{\|v(k)\|_{\alpha}}$, by (10), we obtain

$$\begin{aligned}
\omega_{\lambda}(\lambda_1,k) &= \|v(k)\|_{\alpha}^{\lambda-\lambda_1}\int_0^{\infty}\frac{(s\|v(k)\|_{\alpha})^{\lambda_1-1}\|v(k)\|_{\alpha}}{(s\|v(k)\|_{\alpha}+\|v(k)\|_{\alpha})^{\lambda}}ds \\
&= \int_0^{\infty}\frac{s^{\lambda_1-1}}{(s+1)^{\lambda}}ds = B(\lambda_1,\lambda-\lambda_1).
\end{aligned}$$

Then (12) is proved.

The lemma has been shown. □

LEMMA 2.4. *For $t > 0,\ m \in \mathbf{N}_0$, the following expression holds (cf. [9]):*

$$\int_0^{\infty}e^{-tx}f(x)dx = t^{-m}\int_0^{\infty}e^{-tx}f^{(m)}(x)dx. \tag{13}$$

LEMMA 2.5. *For $m \in \mathbf{N}_0$, the following inequality holds:*

$$\begin{aligned}
I_{\lambda} &:= \sum_k\int_0^{\infty}\frac{f^{(m)}(x)a_k}{(x+\|v(k)\|_{\alpha})^{\lambda}}dx \\
&< \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\
&\quad\times\left[\int_0^{\infty}x^{p(1-\widetilde{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_k\frac{\|v(k)\|_{\alpha}^{q(n-\widetilde{\lambda}_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}a_k^q\right]^{\frac{1}{q}}.
\end{aligned}\tag{14}$$

Proof. By employing Hölder inequality (cf. [11]), we obtain

$$\begin{aligned}
I_\lambda &= \sum_k \int_0^\infty \frac{1}{(x+\|v(k)\|_\alpha)^\lambda} \left[\frac{\|v(k)\|_\alpha^{(\lambda_2-n)/p} (\prod_{i=1}^n v_i'(k_i))^{1/p}}{x^{(\lambda_1-1)/q}} f^{(m)}(x) \right] \\
&\quad \times \left[\frac{x^{(\lambda_1-1)/q}}{\|v(k)\|_\alpha^{(\lambda_2-n)/p} (\prod_{i=1}^n v_i'(k_i))^{1/p}} a_k \right] dx \\
&\leqslant \left\{ \int_0^\infty \left[\sum_k \frac{1}{(x+\|v(k)\|_\alpha)^\lambda} \frac{\|v(k)\|_\alpha^{\lambda_2-n} \prod_{i=1}^n v_i'(k_i)}{x^{(\lambda_1-1)(p-1)}} \left(f^{(m)}(x)\right)^p \right] dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_k \left[\int_0^\infty \frac{\|v(k)\|_\alpha^{(\lambda_2-n)(1-q)}}{(x+\|v(k)\|_\alpha)^\lambda} \frac{x^{\lambda_1-1} dx}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} \right] a_k^q \right\}^{\frac{1}{q}} \\
&= \left[\int_0^\infty \widetilde{\omega}_\lambda(\lambda_2, x) x^{p(1-\widetilde{\lambda}_1)-1} \left(f^{(m)}(x)\right)^p dx \right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_k \omega_\lambda(\lambda_1, k) \frac{\|v(k)\|_\alpha^{q\left(n-\widetilde{\lambda}_2\right)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} a_k^q \right]^{\frac{1}{q}}.
\end{aligned}$$

By (11) and (12), (14) follows.

The lemma has been shown. □

3. Main results

THEOREM 3.1. *A new half-discrete multidimensional Hilbert-type inequality involving one derivative function of m-order holds as follows:*

$$\begin{aligned}
I &:= \sum_k \int_0^\infty \frac{f(x) a_k}{(x+\|v(k)\|_\alpha)^{\lambda+m}} dx \\
&< \left[\prod_{i=0}^{m-1} (\lambda+i) \right]^{-1} \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda-\lambda_2) \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \\
&\quad \times \left[\int_0^\infty x^{p(1-\widetilde{\lambda}_1)-1} (f^{(m)}(x))^p dx \right]^{\frac{1}{p}} \left[\sum_k \frac{\|v(k)\|_\alpha^{q\left(n-\widetilde{\lambda}_2\right)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} a_k^q \right]^{\frac{1}{q}}, \qquad (15)
\end{aligned}$$

where, for $m = 0$, we denote $\prod_{i=0}^{m-1} (\lambda+i) = 1$.

In particular, for $\lambda_1+\lambda_2=\lambda$, we obtain

$$I<\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)$$
$$\times\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_k\frac{\|v(k)\|_\alpha^{q(n-\lambda_2)-n}a_k^q}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}\right]^{\frac{1}{q}}, \tag{16}$$

where the value

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)$$

is the best.

Proof. For $\lambda,x>0$, we have

$$\frac{1}{(x+\|v(k)\|_\alpha)^{\lambda+m}}=\frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}e^{-(x+\|v(k)\|_\alpha)t}dt.$$

By employing (13) and Lebesgue term by term integration theorem (cf. [12]), we obtain

$$\begin{aligned}
I&=\frac{1}{\Gamma(\lambda+m)}\sum_k\int_0^\infty f(x)a_k\left[\int_0^\infty t^{\lambda+m-1}e^{-(x+\|v(k)\|_\alpha)t}dt\right]dx\\
&=\frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}\left(\int_0^\infty e^{-xt}f(x)dx\right)\sum_k e^{-\|v(k)\|_\alpha t}a_k dt\\
&=\frac{1}{\Gamma(\lambda+m)}\int_0^\infty t^{\lambda+m-1}\left(t^{-m}\int_0^\infty e^{-xt}f^{(m)}(x)dx\right)\sum_k e^{-\|v(k)\|_\alpha t}a_k dt\\
&=\frac{1}{\Gamma(\lambda+m)}\sum_k\int_0^\infty f^{(m)}(x)a_k\left[\int_0^\infty t^{\lambda-1}e^{-(x+\|v(k)\|_\alpha)t}dt\right]dx\\
&=\frac{\Gamma(\lambda)}{\Gamma(\lambda+m)}\sum_k\int_0^\infty\frac{f^{(m)}(x)a_k dx}{(x+\|v(k)\|_\alpha)^\lambda}=\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}I_\lambda.
\end{aligned}$$

Then by (14), we obtain (15). Particularly, for $\lambda_1+\lambda_2=\lambda$, we obtain (16).

For any $0<\varepsilon<p\lambda_1$, we set $\widehat{a}_k:=\|v(k)\|_\alpha^{\lambda_2-\frac{\varepsilon}{q}-n}\prod_{i=1}^n v_i'(k_i)$ $(k\in\mathbf{N}^n)$, and

$$\widehat{f}^{(m)}(x)=\begin{cases}0, & 0<x\leqslant 1,\\ \prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})x^{\lambda_1-\frac{\varepsilon}{p}-1}, & x>1,\end{cases}$$
$$\widehat{f}^{(m-k)}(x)=\int_0^x\left(\int_0^{t_k}\cdots\int_0^{t_2}\widehat{f}^{(m)}(t_1)dt_1\cdots dt_{k-1}\right)dt_k$$
$$\geqslant 0\ \ (k=1,\cdots,m).$$

Then, $\widehat{f}(x):=0,\ 0<x<1,$ and

$$\begin{aligned}\widehat{f}(x) &:= \prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\int_1^x\left(\int_1^{t_m}\cdots\int_1^{t_2} t_1^{\lambda_1-\frac{\varepsilon}{p}-1}dt_1\cdots dt_{m-1}\right)dt_m \\ &= x^{\lambda_1-\frac{\varepsilon}{p}+m-1}-p_{m-1}(x),\ \ x\geqslant 1,\end{aligned}$$

where, for $m\in\mathbf{N},\ p_{m-1}(x)$ is a nonnegative polynomial of $(m-1)$-order with $p_{m-1}(1)=1$; for $m=0,\ p_{m-1}(x):=0.$ We observe that for $m\in\mathbf{N}$,

$$\widehat{f}^{(k-1)}(x)=o(e^{tx})\ \ (t>0;\ x\longrightarrow\infty),\quad \widehat{f}^{(k-1)}(0^+)=0\ \ (k=1,\cdots,m).$$

If there exists a positive constant

$$M\leqslant\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2),$$

such that (16) is vaild when we replace

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)$$

by $M,$ then in particular, we have

$$\begin{aligned}\widehat{I} &:= \sum_k\int_0^\infty\frac{\widehat{f}(x)\widehat{a}_k}{(x+\|v(k)\|_\alpha)^{\lambda+m}}dx \\ &< M\left[\int_0^\infty x^{p(1-\lambda_1)-1}(\widehat{f}^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_k\frac{\|v(k)\|_\alpha^{q(n-\lambda_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}\widehat{a}_k^q\right]^{\frac{1}{q}}.\end{aligned}\tag{17}$$

By (8), we obtain

$$\begin{aligned}\widehat{J} &= \left[\int_0^\infty x^{p(1-\lambda_1)-1}(\widehat{f}^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_k\frac{\|v(k)\|_\alpha^{q(n-\lambda_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}\widehat{a}_k^q\right]^{\frac{1}{q}} \\ &= \prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left(\sum_k\|v(k)\|_\alpha^{-\varepsilon-n}\prod_{i=1}^n v_i'(k_i)\right)^{\frac{1}{q}} \\ &< \frac{1}{\varepsilon}\prod_{i=0}^{m-1}(\lambda_1+i-\frac{\varepsilon}{p})\left(\varepsilon a_n+\frac{b^{-\varepsilon}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}}.\end{aligned}$$

We also derive

$$\begin{aligned}\widehat{I} &= \sum_k\int_1^\infty\frac{x^{\lambda_1-\frac{\varepsilon}{p}+m-1}-O(x^{m-1})}{(x+\|v(k)\|_\alpha)^{\lambda+m}}\|v(k)\|_\alpha^{\lambda_2-\frac{\varepsilon}{q}-n}\prod_{i=1}^n v_i'(k_i)dx \\ &= I_1-I_2,\end{aligned}$$

where, we indicate that

$$I_1 := \sum_k \int_1^\infty \frac{x^{\lambda_1 - \frac{\varepsilon}{p} + m - 1}}{(x + \|v(k)\|_\alpha)^{\lambda + m}} \|v(k)\|_\alpha^{\lambda_2 - \frac{\varepsilon}{q} - n} \prod_{i=1}^n v_i'(k_i) dx,$$

$$I_2 := \sum_k \int_1^\infty \frac{O(x^{m-1})}{(x + \|v(k)\|_\alpha)^{\lambda + m}} \|v(k)\|_\alpha^{\lambda_2 - \frac{\varepsilon}{q} - n} \prod_{i=1}^n v_i'(k_i) dx.$$

By (8), we derive

$$\frac{1}{c - \varepsilon} \sum_k \|v(k)\|_\alpha^{-c-n} \prod_{i=1}^n v_i'(k_i) = O(1) \quad \left(c = \lambda_1 + m + \frac{\varepsilon}{q}\right).$$

Replacing λ (λ_1) by $\lambda + m$ $(\lambda_1 - \frac{\varepsilon}{p} + m)$ in (10) and (12), by (8), we obtain

$$\begin{aligned}
I_1 &= \sum_k \|v(k)\|_\alpha^{-\varepsilon-n} \prod_{i=1}^n v_i'(k_i) \left[\|v(k)\|_\alpha^{\lambda_2 + \frac{\varepsilon}{p}} \int_1^\infty \frac{x^{\lambda_1 + m - \frac{\varepsilon}{p} - 1} dx}{(x + \|v(k)\|_\alpha)^{\lambda + m}} \right] \\
&= \sum_k \|v(k)\|_\alpha^{-\varepsilon-n} \prod_{i=1}^n v_i'(k_i) \left[\|v(k)\|_\alpha^{\lambda_2 + \frac{\varepsilon}{p}} \int_0^\infty \frac{x^{\lambda_1 + m - \frac{\varepsilon}{p} - 1} dx}{(x + \|v(k)\|_\alpha)^{\lambda + m}} \right. \\
&\quad \left. - \|v(k)\|_\alpha^{\lambda_2 + \frac{\varepsilon}{p}} \int_0^1 \frac{x^{\lambda_1 + m - \frac{\varepsilon}{p} - 1}}{(x + \|v(k)\|_\alpha)^{\lambda + m}} dx \right] \\
&\geqslant \sum_k \|v(k)\|_\alpha^{-\varepsilon-n} \prod_{i=1}^n v_i'(k_i) \\
&\quad \times \left[\omega_{\lambda+m}\left(\lambda_1 + m - \frac{\varepsilon}{p}, k\right) - \|v(k)\|_\alpha^{\lambda_2 + \frac{\varepsilon}{p}} \int_0^1 \frac{x^{\lambda_1 + m - \frac{\varepsilon}{p} - 1} dx}{\|v(k)\|_\alpha^{\lambda + m}} \right] \\
&= \sum_k \|v(k)\|_\alpha^{-\varepsilon-n} \prod_{i=1}^n v_i'(k_i) \omega_{\lambda+m}\left(\lambda_1 + m - \frac{\varepsilon}{p}, k\right) \\
&\quad - \frac{1}{\lambda_1 + m - \frac{\varepsilon}{p}} \sum_k \|v(k)\|_\alpha^{-(\lambda_1 + m + \frac{\varepsilon}{q}) - n} \prod_{i=1}^n v_i'(k_i) \\
&= B\left(\lambda_1 + m - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) \sum_k \|v(k)\|_\alpha^{-\varepsilon-n} \prod_{i=1}^n v_i'(k_i) - O(1) \\
&> B\left(\lambda_1 + m - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) \frac{e^{-\varepsilon} \Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1} \Gamma(\frac{n}{\alpha})} \left(\frac{1}{\varepsilon} - \frac{n-1}{1+\varepsilon}\right) - O(1).
\end{aligned}$$

Furthermore, for $m = 0$, $I_2 = 0$; for $m \in \mathbf{N}$, we have

$$\begin{aligned}
0 < I_2 &= \sum_k \frac{\|v(k)\|_\alpha^{\lambda_2 - \frac{\varepsilon}{q} - n} \prod_{i=1}^n v_i'(k_i)}{(x + \|v(k)\|_\alpha)^{\lambda_2 + \frac{\lambda_1}{2}}} \int_1^\infty \frac{O(x^{m-1})}{(x + \|v(k)\|_\alpha)^{\frac{\lambda_1}{2} + m}} dx \\
&\leqslant \sum_k \frac{\|v(k)\|_\alpha^{\lambda_2 - \frac{\varepsilon}{q} - n} \prod_{i=1}^n v_i'(k_i)}{\|v(k)\|_\alpha^{\lambda_2 + \frac{\lambda_1}{2}}} \int_1^\infty \frac{O(x^{m-1})}{x^{\frac{\lambda_1}{2} + m}} dx \leqslant C < \infty.
\end{aligned}$$

Thus, by (17), we derive

$$B\left(\lambda_1+m-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right)\frac{e^{-\varepsilon}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left(1-\varepsilon\frac{n-1}{1+\varepsilon}\right)$$
$$-\varepsilon O(1)-\varepsilon I_2<\varepsilon\widehat{I}<\varepsilon M\widehat{J}$$
$$\leqslant M\prod_{i=0}^{m-1}\left(\lambda_1+i-\frac{\varepsilon}{p}\right)\left(\varepsilon a_n+\frac{b^{-\varepsilon}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}}.$$

For $\varepsilon\longrightarrow 0^+$, since the beta function is continuous, we derive

$$B(\lambda_1+m,\lambda_2)\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\leqslant M\prod_{i=0}^{m-1}(\lambda_1+i)\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}},$$

namely,

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)\leqslant M,$$

then

$$M=\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)$$

is the best value in (16).

The theorem has been proved. □

REMARK 3.1. For $\widetilde{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widetilde{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda_2+\frac{\lambda-\lambda_1-\lambda_2}{q}$, then $\widetilde{\lambda}_1+\widetilde{\lambda}_2=\lambda$. Since $0<\lambda_1,\lambda_2<\lambda$, we have

$$0<\widetilde{\lambda}_1,\widetilde{\lambda}_2<\lambda,\quad\text{and}\quad B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)\in\mathbf{R}_+.$$

For $\lambda-\lambda_1-\lambda_2\leqslant q(n-\lambda_2)$, we obtain $\widetilde{\lambda}_2\leqslant n$. Then (16) is rewritten as follows:

$$I<\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\widetilde{\lambda}_1,\widetilde{\lambda}_2)$$
$$\times\left[\int_0^\infty x^{p(1-\widetilde{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_k\frac{\|v(k)\|_\alpha^{q\left(n-\widetilde{\lambda}_2\right)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}a_k^q\right]^{\frac{1}{q}}.\tag{18}$$

THEOREM 3.2. *For $\lambda-\lambda_1-\lambda_2\leqslant q(n-\lambda_2)$, if*

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

is the best value in (15), then $\lambda_1+\lambda_2=\lambda$.

Proof. By Hölder inequality (cf. [11]), we derive

$$
\begin{aligned}
B(\widetilde{\lambda}_1,\widetilde{\lambda}_2) &= \int_0^\infty \frac{u^{\widetilde{\lambda}_1-1}}{(1+u)^\lambda}du \\
&= \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}}{(1+u)^\lambda}du \\
&= \int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda-\lambda_2-1}{p}}u^{\frac{\lambda_1-1}{q}}du \\
&\leqslant \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda}du\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda}du\right]^{\frac{1}{q}} \\
&= B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1).
\end{aligned}
\tag{19}
$$

Comparing with the values in (15) and (18), by the assumption, the following inequality holds:

$$
\begin{aligned}
&\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\
\leqslant &\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\widetilde{\lambda}_1,\widetilde{\lambda}_2),
\end{aligned}
$$

namely, $B(\widetilde{\lambda}_1,\widetilde{\lambda}_2) \geqslant B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$, which means that (19) is an equality. We observe that (19) is an equality if and only if there exist two constants P and Q, such that they are not both zero and (cf. [11]), satisfying $Pu^{\lambda-\lambda_2-1} = Qu^{\lambda_1-1}$ a.e. in $\mathbf{R}_+$. Assuming that $P \neq 0$, then $u^{\lambda-\lambda_1-\lambda_2} = \frac{Q}{P}$ a.e. in $\mathbf{R}_+$, namely, $\lambda-\lambda_1-\lambda_2 = 0$, that is, $\lambda_1+\lambda_2 = \lambda$.

The theorem has been proved. □

4. Equivalent forms and operator expressions

THEOREM 4.1. *The following inequality is equivalent to inequality* (15)*:*

$$
\begin{aligned}
J &:= \left\{\sum_k \|v(k)\|_\alpha^{p\widetilde{\lambda}_2-n}\prod_{i=1}^n v_i'(k_i)\left[\int_0^\infty \frac{f(x)dx}{(x+\|v(k)\|_\alpha)^{\lambda+m}}\right]^p\right\}^{\frac{1}{p}} \\
&< \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\
&\quad\times\left[\int_0^\infty x^{p(1-\widetilde{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}.
\end{aligned}
\tag{20}
$$

In particular, for $\lambda_1+\lambda_2=\lambda$, *we obtain the following inequality equivalent to* (16)*:*

$$\begin{aligned}&\left\{\sum_k \|v(k)\|_\alpha^{p\lambda_2-n}\prod_{i=1}^n v_i'(k_i)\left[\int_0^\infty \frac{f(x)dx}{(x+\|v(k)\|_\alpha)^{\lambda+m}}\right]^p\right\}^{\frac{1}{p}}\\ &<\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)\\ &\times\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}. \qquad (21)\end{aligned}$$

Proof. Suppose that (20) is true, we have

$$\begin{aligned}I&=\sum_k\left[\|v(k)\|_\alpha^{\widetilde{\lambda}_2-\frac{n}{p}}\left(\prod_{i=1}^n v_i'(k_i)\right)^{\frac{1}{p}}\int_0^\infty\frac{f(x)}{(x+\|v(k)\|_\alpha)^{\lambda+m}}dx\right]\\ &\times\left[\frac{\|v(k)\|_\alpha^{\frac{n}{p}-\widetilde{\lambda}_2}a_k}{(\prod_{i=1}^n v_i'(k_i))^{\frac{1}{p}}}\right]\\ &\leqslant J\left[\sum_k\frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}a_k^q\right]^{\frac{1}{q}}. \qquad (22)\end{aligned}$$

Then by (20), we obtain (15).

In contrast, suppose that (15) is valid, we set

$$a_k=\|v(k)\|_\alpha^{p\widetilde{\lambda}_2-n}\prod_{i=1}^n v_i'(k_i)\left[\int_0^\infty\frac{f(x)dx}{(x+\|v(k)\|_\alpha)^{\lambda+m}}\right]^{p-1},\quad k\in\mathbf{N}^n.$$

If $J=0$, then (20) is true; if $J=\infty$, then (20) is not true, implying $J<\infty$. For $0<J<\infty$, by (15), we obtain

$$\begin{aligned}&\sum_k\frac{\|v(k)\|_\alpha^{q\left(n-\widetilde{\lambda}_2\right)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}a_k^q=J^p=I\\ &<\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}\\ &\times B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left[\int_0^\infty x^{p(1-\widetilde{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}J^{p-1},\end{aligned}$$

$$\left[\sum_k \frac{\|v(k)\|_\alpha^{q\left(n-\widetilde{\lambda}_2\right)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} a_k^q\right]^{\frac{1}{p}}$$

$$= J < \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}$$

$$\times B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left[\int_0^\infty x^{p(1-\widetilde{\lambda}_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}.$$

Thus, (20) is the equivalent form of (15).

The theorem is proved. □

THEOREM 4.2. *If* $\lambda_1+\lambda_2=\lambda$, *then the value*

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

in (20) *is the best. On contrast, if the same value in* (20) *is the best, then for* $\lambda-\lambda_1-\lambda_2 \leqslant q(n-\lambda_2)$, *we have* $\lambda_1+\lambda_2=\lambda$.

Proof. According to Theorem 3.2, for $\lambda_1+\lambda_2=\lambda$,

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

in (15) is the best value. The same value in (20) remains the best one. Alternatively, according to (22), it is a contradiction that the value in (15) is not the best.

In addition, if the value in (20) is the best, then using the equivalence between (20) and (15), and by considering $J^p = I$ as outlined in the proof of Theorem 4.1, the value in (15) is the best. Based on the assumption and Theorem 3.2, we have $\lambda_1+\lambda_2=\lambda$.

The theorem has been shown. □

Setting functions $\phi(x) := x^{p(1-\widetilde{\lambda})-1}$, $\psi(k) := \frac{\|v(k)\|_\alpha^{q\left(n-\widetilde{\lambda}_2\right)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}$, we have

$$\psi^{1-p}(k) = \|v(k)\|_\alpha^{p\widetilde{\lambda}_2-n}\prod_{i=1}^n v_i'(k_i), \quad (x\in\mathbf{R}_+,\ k\in\mathbf{N}^n).$$

We define the real normed spaces as follows:

$$L_{p,\phi}(\mathbf{R}_+) := \left\{ f = f(x); \|f\|_{p,\phi} := \left(\int_0^\infty \phi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ a = \{a_{k_1,\cdots k_n}\}; \|a\|_{q,\psi} := \left(\sum_k \psi(k) |a_k|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ b = \{b_{k_1,\cdots k_n}\}; \|b\|_{q,\psi} := \left(\sum_k \psi^{1-p}(k) |b_k|^p \right)^{\frac{1}{p}} < \infty \right\},$$

and $\widehat{L}(\mathbf{R}_+) := \{ f \in L_{p,\phi}(\mathbf{R}_+) \}$; $f^{(m)}(x)$ is a nonnegative continuous function except at finite points in $\mathbf{R}_+$, for $m \in \mathbf{N}$, $f^{(k-1)}(x) = o(e^{tx})$ $(t > 0;\ x \longrightarrow \infty)$, $f^{(k-1)}(0^+) = 0$, $(k = 1, \cdots, m)\}$.

For any $f \in \widehat{L}(\mathbf{R}_+)$, we set $b_k := \int_0^\infty \frac{f(x)}{(x+\|v(k)\|_\alpha)^{\lambda+m}} dx$, $k \in \mathbf{N}^n$. Then (20) is rewritten as follows:

$$\|b\|_{p,\psi^{1-p}} \leqslant \left[\prod_{i=0}^{m-1} (\lambda + i) \right]^{-1} \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda - \lambda_2) \right)^{\frac{1}{p}}$$
$$\times B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left\| f^{(m)} \right\|_{p,\phi} < \infty,$$

namely, $b \in l_{p,\psi^{1-p}}$.

DEFINITION 4.3. Define a half-discrete multidimensional Hilbert–type operator

$$T : \widehat{L}(\mathbf{R}_+) \longrightarrow l_{p,\psi^{1-p}}$$

as follows: For any $f \in \widehat{L}(\mathbf{R}_+)$, there exists a unique representation $b = Tf \in l_{p,\psi^{1-p}}$, such that for any $k \in \mathbf{N}^n$, $Tf(k) = b_k$. Define the formal inner product of Tf and $a \in l_{q,\psi}$, and the norm of T as follows:

$$(Tf, a) := \sum_k a_k \int_0^\infty \frac{f(x)dx}{(x + \|v(k)\|_\alpha)^{\lambda+m}} = I,$$

$$\|T\| := \sup_{f^{(m)}(\neq 0) \in \widehat{L}(\mathbf{R}_+)} \frac{\|Tf\|_{p,\psi^{1-p}}}{\left\| f^{(m)} \right\|_{p,\phi}}.$$

By Theorem 3.1, 3.2, 4.1, and 4.2, we have

THEOREM 4.4. *If* $f \in \widehat{L}(\mathbf{R}_+)$, $a(\geqslant 0) \in l_{q,\psi}$, $\|a\|_{q,\psi} > 0$, $\left\| f^{(m)} \right\|_{p,\varphi} > 0$, *then* *the following equivalent inequalities are hold:*

$$(Tf,a) < \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}} \times B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left\|f^{(m)}\right\|_{p,\phi}\|a\|_{q,\psi}, \tag{23}$$

$$\|Tf\|_{p,\psi^{1-p}} < \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}} \times B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left\|f^{(m)}\right\|_{p,\phi}. \tag{24}$$

Futhermore, if $\lambda_1+\lambda_2=\lambda$, then the value

$$\left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

in (23) and (24) is the best, that is,

$$\|T\| = \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2).$$

In contrast, if the value in (23) or (24) is the best, then for $\lambda-\lambda_1-\lambda_2 \leqslant q(n-\lambda_2)$, we have $\lambda_1+\lambda_2=\lambda$.

REMARK 4.1. (i) For $v_\xi(k)=k-\xi,\ \xi\in[0,\frac{1}{2}],\ k\in\mathbf{N}^n$, by (16) and (21), we obtain the following equivalent inequalities [cf. [9]]:

$$\sum_k\int_0^\infty\frac{f(x)a_k}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}dx < \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2) \times\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}\left[\sum_k\|k-\xi\|_\alpha^{q(n-\lambda_2)-n}a_k^q\right]^{\frac{1}{q}}, \tag{25}$$

$$\left\{\sum_k\|k-\xi\|_\alpha^{p\lambda_2-n}\left[\int_0^\infty\frac{f(x)dx}{(x+\|k-\xi\|_\alpha)^{\lambda+m}}\right]^p\right\}^{\frac{1}{p}} < \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2) \times\left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}. \tag{26}$$

(ii) For $v_\xi(k) = \|\ln(k+1-\xi)\|_\alpha$, $\xi \in [0, \frac{1}{2}]$, $k \in \mathbf{N}^n$, by (16) and (21), we obtain the following equivalent inequalities:

$$\sum_k \int_0^\infty \frac{f(x)a_k}{(x+\|\ln(k+1-\xi)\|_\alpha)^{\lambda+m}}dx$$
$$< \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1} \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B(\lambda_1,\lambda_2) \left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}$$
$$\times \left[\sum_k \|\ln(k+1-\xi)\|_\alpha^{q(n-\lambda_2)-n} \left(\prod_{i=1}^n (k_i+1-\xi)\right)^{q-1} a_k^q\right]^{\frac{1}{q}}, \tag{27}$$

$$\left\{\sum_k \frac{\|\ln(k+1-\xi)\|_\alpha^{p\lambda_2-n}}{\prod_{i=1}^n (k_i+1-\xi)} \left[\int_0^\infty \frac{f(x)dx}{(x+\|\ln(k+1-\xi)\|_\alpha)^{\lambda+m}}\right]^p\right\}^{\frac{1}{p}}$$
$$< \left[\prod_{i=0}^{m-1}(\lambda+i)\right]^{-1} \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B(\lambda_1,\lambda_2)$$
$$\times \left[\int_0^\infty x^{p(1-\lambda_1)-1}(f^{(m)}(x))^p dx\right]^{\frac{1}{p}}. \tag{28}$$

The value in the above inequalities is the best.

5. Conclusions

This paper uses the transfer formula and weight functions to develop a half-discrete multidimensional Hilbert-type inequality. It involves a $m-$order derivative function and a general intermediate variable in the kernel as

$$\frac{1}{(x+\|v(k)\|_\alpha)^{\lambda+m}} \quad (x,\lambda > 0)$$

in Theorem 3.1. Theorem 3.2 focuses on the equivalence statements of the best value linked to some parameters. Additionally, Theorem 4.1, Theorem 4.2, and Theorem 4.3 explore the equivalent forms and operator expressions.

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