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ON C–HYPONORMAL OPERATORS

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Abstract. A bounded linear operator $T : \mathscr{H} \to \mathscr{H}$ is a C-*hyponormal* operator if $T^*T - CTT^*C \geqslant 0$ for a conjugation C on $\mathscr{H}$. In this paper, we study properties of C-hyponormal operators. Especially, we prove that for $\mathscr{M} \in Lat(T)$ and a conjugation $C = C_1 \oplus C_2$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$, if T is C-hyponormal, then $T|_{\mathscr{M}}$ is C_1-hyponormal. Moreover, we show that $T - \lambda I$ is C-hyponormal for all $\lambda \in \mathbb{C}$ if and only if T is a complex symmetric operator. Finally, we prove that if T^* is p-hyponormal for $0 < p \leqslant 1$ and C is a conjugation on $\mathscr{H}$, then T is C-hyponormal if and only if T is normal.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ be the set of all bounded linear operators on a separable (complex) Hilbert space $\mathscr{H}$. For $T \in \mathscr{L}(\mathscr{H})$, let T^*, $\ker(T)$, and $ran(T)$ denote the adjoint of T, the kernel, and range of T, respectively. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be an *isometry* if $T^*T = I$, *unitary* if $T^*T = TT^* = I$, *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geqslant TT^*$, and p-*hyponormal* operator if $(T^*T)^p \geqslant (TT^*)^p$ for $0 < p < \infty$, respectively. It is well known that

$$\text{hyponormal} \Rightarrow p\text{-hyponormal}\ (0 < p \leqslant 1).$$

A *conjugation* C on $\mathscr{H}$ is said to be an antilinear operator satisfying $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathscr{H}$ and $C^2 = I$. An operator $T \in \mathscr{L}(\mathscr{H})$ is called a *complex symmetric* operator if $T = CT^*C$ for a conjugation C on $\mathscr{H}$ (see [5]).

If T is an antilinear (or linear) operator, then a *Hermitian adjoint* operator of T on $\mathscr{H}$ is an antilinear operator $T^\# : \mathscr{H} \to \mathscr{H}$ given by

$$\langle Tx, y\rangle = \overline{\langle x, T^\# y\rangle} \tag{1}$$

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for all $x,y \in \mathcal{H}$. For a bounded antilinear operator T, the Hermitian adjoint of T exists and is unique by the Riesz representation theorem ([2, p. 90]). If T and R are antilinear operators, then it turns out, by (1), that

$$(T^{\#})^{\#} = T, \quad (T+R)^{\#} = T^{\#} + R^{\#}, \quad \text{and} \quad (TR)^{\#} = R^{\#}T^{\#}.$$

An operator T in $\mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker U = \ker|T| = \ker T$ and $\ker U^* = \ker T^*$. An operator $T \in \mathcal{L}(\mathcal{H})$ is *C-normal* if CT and $(CT)^{\#}$ commute where C is a conjugation on $\mathcal{H}$. Notice that by the definition of C-normal operators, $C|T|^2C = |T^*|^2 \Leftrightarrow C|T|C = |T^*|$ and hence T is C-normal if and only if so is T^*.

It is well known from [1, Theorem 4.1] that for a conjugation C on $\mathcal{H}$, the Hermitian adjoint of C is the conjugation C, i.e., $C^{\#} = C$. Note that if $A \geqslant 0$, then $CAC \geqslant 0$. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be *C-hyponormal* if there exists a conjugation C on $\mathcal{H}$ such that

$$[(CT)^{\#}, CT] = [T^*C, CT] = T^*T - CTT^*C \geqslant 0$$

where $[R,S] := RS - SR$, or equivalently, $\|Tx\| = \|CTx\| \geqslant \|T^*Cx\|$ for all $x \in \mathcal{H}$. From the definition of C-hyponormal operators, if $|T|^2 \geqslant C|T^*|^2C$ holds, then by Löwner's Lemma, we have

$$C|T^*|C \leqslant C(C|T|^2C)^{\frac{1}{2}}C = C(C|T|CC|T|C)^{\frac{1}{2}}C = |T|.$$

In 2020, the authors in [10] introduced the concept of C-normal operators. The C-symmetric operators and C-skew-symmetric operators are contained in the class of C-normal operators. Recently, C. Wang, J. Zhao, and S. Zhu [11] studied the structure of C-normal operators. Recently, we also studied properties of C-normal operators (see [8] and [9]). In this paper, we introduce the concept of C-hyponormal operators. To some extend, C-hyponormal operators are close to C-normal operators. Since the class of C-hyponormal operators contains C-normal operators as a subclass, we want to know the properties of C-hyponormal operators which are similar to those of C-normal operators. Moreover, we want to investigate the phenomena which only occur in the case of C-nonnormal operators.

The aim of this paper is to study several properties of C-hyponormal operators. Let $Lat(T)$ be the set of T-invariant subspaces of $\mathcal{H}$, which is called the invariant-subspace lattice of T. In particular, we show that for $\mathcal{M} \in Lat(T)$ and a conjugation $C = C_1 \oplus C_2$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, if T is C-hyponormal, then $T|_{\mathcal{M}}$ is C_1-hyponormal. Moreover, we demonstrate that $T - \lambda I$ is C-hyponormal for all $\lambda \in \mathbb{C}$ if and only if T is a complex symmetric operator. Finally, we show that if T^* is p-hyponormal for $0 < p \leqslant 1$ and C is a conjugation on $\mathcal{H}$, then T is C-hyponormal if and only if T is normal.

2. Main results

In this section we study various properties of C-hyponormal operators in $\mathcal{L}(\mathcal{H})$. Recall that $T \in \mathcal{L}(\mathcal{H})$ is *C-hyponormal* if

$$[(CT)^{\#}, CT] = [T^*C, CT] = T^*T - CTT^*C \geqslant 0,$$

for a conjugation C on $\mathcal{H}$, or equivalently, $\|Tx\| \geqslant \|T^*Cx\|$ for all $x \in \mathcal{H}$. We remark that there are examples of C-hyponormal operators which are not hyponormal.

EXAMPLE 2.1. (i) Suppose that $A \in \mathcal{L}(\mathcal{H})$ is normal and J is a conjugation on $\mathcal{H}$. Then A is J-normal from [10]. If $T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, then T is not hyponormal. Set $C = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$. Then C is a conjugation on $\mathcal{H} \oplus \mathcal{H}$. Since A is J-normal, it follows that

$$T^*T - CTT^*C = \begin{pmatrix} 0 & 0 \\ 0 & A^*A - JAA^*J \end{pmatrix} = 0.$$

Hence T is C-normal, and hence is C-hyponormal.

(ii) Assume that $S \in \mathcal{L}(\mathcal{H})$ is the unilateral shift given by $S(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$ on $\mathcal{H} = \ell^2(\mathbb{N})$ and J is a canonical conjugation on $\mathcal{H}$ given by $J(a_0, a_1, a_2, \ldots) = (\overline{a_0}, \overline{a_1}, \overline{a_2}, \ldots)$. Let $T = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix}$ and let $C = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$. Then C is clearly a conjugation on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$. Since $I - SS^* = e_0 \otimes e_0$ is positive and $SJ = JS$, we have

$$T^*T - CTT^*C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I - SS^* & 0 \\ 0 & 0 & I - SS^* \end{pmatrix} \geqslant 0.$$

Therefore T is C-hyponormal, but is not hyponormal. Furthermore, T is not C-normal.

Recall that the Hardy space H^2 consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc $\mathbb{D}$ so that $\|f\|_2 := (\sum_{n=0}^{\infty} |a_n|^2)^{\frac{1}{2}} < \infty$. Recall that for nonzero $u, v \in \mathcal{H}$, we write $u \otimes v$ for the *rank one* operator defined by

$$(u \otimes v)x = \langle x,\ v \rangle u, \quad x \in \mathcal{H}$$

where $\langle\ ,\ \rangle$ is the inner product in $\mathcal{H}$. We next consider an example of a C-hyponormal operator defined on a function space.

EXAMPLE 2.2. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of H^2 and let $\mathcal{C} = J \oplus J$ where J is a conjugation defined by $Jf(z) = \overline{f(\overline{z})}$. Assume that

$$T = \begin{pmatrix} S & e_0 \otimes e_0 \\ 0 & I \end{pmatrix} \in \mathcal{L}(H^2 \oplus H^2)$$

where S is the unilateral shift given by $Se_n = e_{n+1}$. Then T is C-hyponormal. Indeed, since for $h = f \oplus g \in H^2 \oplus H^2$,

$$Th = \begin{pmatrix} S & e_0 \otimes e_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} Sf + \langle g, e_0 \rangle e_0 \\ g \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \hat{f}(n) e_{n+1} + \hat{g}(0) e_0 \\ \sum_{n=0}^{\infty} \hat{g}(n) e_n \end{pmatrix},$$

we have $\|Th\| = \sqrt{\sum_{n=0}^{\infty} |\hat{f}(n)|^2 + \sum_{n=0}^{\infty} |\hat{g}(n)|^2 + |\hat{g}(0)|^2}$. On the other hand, we get that

$$\begin{aligned} T^*Ch &= \begin{pmatrix} S^* & 0 \\ e_0 \otimes e_0 & I \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \\ &= \begin{pmatrix} S^*J & 0 \\ (e_0 \otimes e_0)J & J \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \\ &= \begin{pmatrix} S^*Jf \\ (e_0 \otimes e_0)Jf + Jg \end{pmatrix} \\ &= \begin{pmatrix} S^* \overline{f(\overline{z})} \\ (e_0 \otimes e_0) \overline{f(\overline{z})} + \overline{g(\overline{z})} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{n=0}^{\infty} \overline{\hat{f}(n)} e_{n-1} \\ \overline{\hat{f}(0)} e_0 + \sum_{n=0}^{\infty} \overline{\hat{g}(n)} e_n \end{pmatrix}. \end{aligned}$$

From this, $\|T^*Ch\| = \sqrt{\sum_{n=1}^{\infty} |\hat{f}(n)|^2 + \sum_{n=1}^{\infty} |\hat{g}(n)|^2 + |\hat{f}(0)|^2 + |\hat{g}(0)|^2}$. Thus

$$\begin{aligned} \|T^*Ch\| &= \sqrt{\sum_{n=0}^{\infty} |\hat{f}(n)|^2 + \sum_{n=0}^{\infty} |\hat{g}(n)|^2} \\ &\leqslant \sqrt{\sum_{n=0}^{\infty} |\hat{f}(n)|^2 + \sum_{n=0}^{\infty} |\hat{g}(n)|^2 + |\hat{g}(0)|^2} = \|Th\| \end{aligned}$$

for each $h \in H^2 \oplus H^2$. Hence T is C-hyponormal.

We next study the structure of C-hyponormal operators.

THEOREM 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be C-hyponormal with a conjugation C. If $\mathcal{M} \in Lat(T)$ and $C = C_1 \oplus C_2$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, then the following statements hold.*

(i) *$T|_{\mathcal{M}}$ is C_1-hyponormal.*

(ii) *If $T|_{\mathcal{M}}$ is C_1-normal, then $\mathcal{M}$ reduces T.*

Proof. (i) Let $\mathcal{M} \in Lat(T)$ and let $C = C_1 \oplus C_2$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Set

$$T := \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathcal{L}(\mathcal{M} \oplus \mathcal{M}^{\perp}).$$

Since T is C-hyponormal, we have

$$[(CT)^{\#}, CT] = \begin{pmatrix} A^*A - C_1AA^*C_1 - C_1BB^*C_1 & * \\ * & * \end{pmatrix} \geqslant 0$$

where $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$ on $\mathscr{M} \oplus \mathscr{M}^{\perp}$. This gives from [4] that

$$A^*A - C_1AA^*C_1 - C_1BB^*C_1 \geqslant 0.$$

Therefore $A^*A - C_1AA^*C_1 \geqslant C_1BB^*C_1 \geqslant 0$ which means that A is C_1-hyponormal. Hence $T|_{\mathscr{M}}$ is C_1-hyponormal.

(ii) Suppose that $T = \begin{pmatrix} T|_{\mathscr{M}} & X \\ 0 & Y \end{pmatrix} \in \mathscr{L}(\mathscr{M} \oplus \mathscr{M}^{\perp})$. Since $T|_{\mathscr{M}}$ is C_1-normal and T is C-hyponormal, we have

$$\begin{aligned}
&(CT)^{\#}(CT) - (CT)(CT)^{\#} = T^*T - CTT^*C \\
&= \begin{pmatrix} (T|_{\mathscr{M}})^* & 0 \\ X^* & Y^* \end{pmatrix} \begin{pmatrix} T|_{\mathscr{M}} & X \\ 0 & Y \end{pmatrix} - \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} T|_{\mathscr{M}} & X \\ 0 & Y \end{pmatrix} \begin{pmatrix} (T|_{\mathscr{M}})^* & 0 \\ X^* & Y^* \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \\
&= \begin{pmatrix} (T|_{\mathscr{M}})^*T|_{\mathscr{M}} - C_1T|_{\mathscr{M}}(T|_{\mathscr{M}})^*C_1 - C_1XX^*C_1 & * \\ * & * \end{pmatrix} \geqslant 0.
\end{aligned}$$

Therefore we get $(T|_{\mathscr{M}})^*T|_{\mathscr{M}} - C_1T|_{\mathscr{M}}(T|_{\mathscr{M}})^*C_1 - C_1XX^*C_1 \geqslant 0$ from [4]. Moreover, since $T|_{\mathscr{M}}$ is C_1-normal, $C_1XX^*C_1 \leqslant 0$ and so $X^* = 0$. Hence $T = \begin{pmatrix} T|_{\mathscr{M}} & 0 \\ 0 & Y \end{pmatrix}$. Thus $\mathscr{M}$ reduces T. $\square$

COROLLARY 2.4. *Let $T \in \mathscr{L}(\mathscr{H})$ be C-normal with a conjugation C. If $\mathscr{M} \in Lat(T)$ and $C = C_1 \oplus C_2$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$, then $T|_{\mathscr{M}}$ is C_1-normal $\Leftrightarrow$ $\mathscr{M}$ reduces T.*

Proof. Assume that $T|_{\mathscr{M}}$ is C_1-normal. If T is C-normal, then it is C-hyponormal. By Theorem 2.3, $\mathscr{M}$ reduces T. Conversely, if $\mathscr{M}$ reduces T, then $T = T|_{\mathscr{M}} \oplus T|_{\mathscr{M}^{\perp}}$. Since T is C-normal, $T|_{\mathscr{M}}$ is C_1-normal and $T|_{\mathscr{M}^{\perp}}$ is C_2-normal. $\square$

In the following lemma, we recapture the theorem of R. G. Douglas ([3]).

LEMMA 2.5. ([3]) *Let C be a conjugation on $\mathscr{H}$ and let $T \in \mathscr{L}(\mathscr{H})$. Then the following statements equivalent.*

(i) *T is C-hyponormal.*

(ii) *$ran(CT) \subset ran(T^*)$.*

(iii) *There exists a contraction antilinear operator D on $\mathscr{H}$ such that $T = CT^*D$.*

Proof. (i) $\Rightarrow$ (iii) Assume that T is C-hyponormal. Define an antilinear mapping D from $ran(T)$ to $ran(T^*C)$ such that $D(Tf) = T^*Cf$. Since $T^*T \geqslant CTT^*C$, it follows that

$$\begin{aligned}
\|D(Tf)\|^2 = \|T^*Cf\|^2 &= \langle CTT^*Cf, f\rangle \\
&\leqslant \langle T^*Tf, f\rangle = \|Tf\|^2
\end{aligned}$$

for $f \in \mathscr{H}$. Thus D is well-defined and it can be uniquely extended to $\overline{ran(T)}$. If we define D on $ran(T)^{\perp}$ to be 0, then $DT = T^*C$. Hence $CT = T^*D^*$ for a contraction operator D.

(iii) $\Rightarrow$ (i) If $T = CT^*D$ for a contraction antilnear operator D, then

$$\begin{aligned} TT^* &= (CT^*D)(CT^*D)^* \\ &= CT^*DD^*TC \\ &= \|D\|^2 CT^*(CT^*)^* - CT^*(\|D\|^2 I - DD^*)(CT^*)^* \\ &\leqslant \|D\|^2 CT^*(CT^*)^* \\ &\leqslant \|D\|^2 CT^*TC \\ &\leqslant CT^*TC. \end{aligned}$$

Therefore T is C-hyponormal.

(ii) $\Rightarrow$ (iii) Suppose that $ran(CT) \subset ran(T^*)$. Define an antilinear operator D on $\mathscr{H}$ as follows; for $f \in \mathscr{H}$, $CTf \in ran(CT) \subset ran(T^*)$, there exists $h \in \ker(T^*)^\perp$ such that $T^*h = CTf$. Set $Df = h$. Then $CT = T^*D$. Since D is defined on all of $\mathscr{H}$, we show that D has a closed graph. If $\{(f_n, h_n)\}_{n=1}^\infty$ is a sequence of elements in the graph of D such that $\lim_{n\to\infty}(f_n, h_n) = (f, h)$, then $\lim_{n\to\infty} CTf_n = CTf$ and $\lim_{n\to\infty} T^*h_n = T^*h$. Since $\ker(T^*)$ is closed, $CTf = T^*h$. Thus $h \in \ker(T^*)^\perp$ such that $Df = h$. Hence D is bounded.

(iii) $\Rightarrow$ (ii) Since $CT = T^*D$, it is trivial that $ran(CT) \subset ran(T^*)$. So we complete the proof. □

THEOREM 2.6. *Let $T \in \mathscr{L}(\mathscr{H})$ be C-hyponormal with a conjugation C. Assume that $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ on $\mathscr{M} \oplus \mathscr{M}^\perp = \mathscr{H}$ where C_j are antilinear and at least one C_j is zero for $j = 1, 2, 3, 4$. Then the following arguments hold.*

(i) *$C\ker(T) \subset \ker(T^*)$ and $C\overline{ran(T)} \subset \overline{ran(T^*)}$.*

(ii) *If $\mathscr{M}$ is a reducing subspace for a C-hyponormal operator T on $\mathscr{H}$, then $T|_{\mathscr{M}}$ is C_1-hyponormal where C_1 is a conjugation on $\mathscr{M}$ and $T|_{\mathscr{M}^\perp}$ is C_4-hyponormal where C_4 is a conjugation on $\mathscr{M}^\perp$.*

(iii) *If T is an idempotent, then T is a projection.*

Proof. (i) Since T is C-hyponormal, $\|Tx\| \geqslant \|T^*Cx\|$ for all $x \in \mathscr{H}$. Let $x \in \ker T$. Then $T^*Cx = 0$ and so $Cx \in \ker T^*$. Hence $C\ker T \subset \ker T^*$. Since T is C-hyponormal, $ran(CT) \subset ran(T^*)$ from Lemma 2.5. If $y \in \overline{ran(CT)}$, then there exists a sequence $\{y_n\}$ in $ran(CT)$ such that $y_n \to y$ as $n \to \infty$. Thus $y_n \in CT\mathscr{H}$ and $y_n \in ran(CT) \subset ran(T^*)$. Since $y_n \to y$ as $n \to \infty$, it follows that $y \in \overline{ran(T^*)}$. We claim that $\overline{ran(CT)} = C\overline{ran(T)}$. If $y \in \overline{ran(CT)}$, there exists a sequence $\{y_n\}$ in $ran(CT)$ such that $y_n \to y$ as $n \to \infty$. Thus $y_n = CTx_n$ and $Cy_n = Tx_n \in ran(T)$. Therefore $Cy \in \overline{ran(T)}$. Hence $\overline{ran(CT)} \subset C\overline{ran(T)}$. If $y \in C\overline{ran(T)}$, then $Cy \in \overline{ran(T)}$. Thus there exists a sequence $\{z_n\}$ in $ran(T)$ such that $z_n \to Cy$ as $n \to \infty$. Therefore $Cz_n \to y$ as $n \to \infty$ and $Cz_n = CTx_n \in ran(CT)$. So $y \in \overline{ran(CT)}$. Hence $\overline{ran(CT)} \subset \overline{ran(T^*)}$ by the above claim.

(ii) Let $\mathscr{M}$ be a reducing subspace for a C-hyponormal operator on $\mathscr{H}$ where $T = T|_{\mathscr{M}} \oplus T|_{\mathscr{M}^\perp}$. Set $A = T|_{\mathscr{M}}$ and $D = T|_{\mathscr{M}^\perp}$. Then

$$T := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ on } \mathscr{M} \oplus \mathscr{M}^\perp.$$

Since T is C-hyponormal,

$$[(CT)^{\#},CT] = \begin{pmatrix} A^*A - C_1AA^*C_1 - C_2DD^*C_3 & * \\ * & D^*D - C_4DD^*C_4 - C_3AA^*C_2 \end{pmatrix} \geqslant 0$$

where $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ is a conjugation on $\mathscr{M} \oplus \mathscr{M}^{\perp}$. Since C is a conjugation, it follows from [7] that $C_3 = {C_2}^{\#}$ and C_1, C_4 are conjugations. This gives from [4] that

$$\begin{cases} A^*A - C_1AA^*C_1 - C_2DD^*C_2^{\#} \geqslant 0 \quad \text{and} \\ D^*D - C_4DD^*C_4 - {C_2}^{\#}AA^*C_2 \geqslant 0 \end{cases}$$

where $C_3 = {C_2}^{\#}$ is the Hermitian adjoint of C_2. Thus

$$A^*A - C_1AA^*C_1 \geqslant C_2DD^*C_2^{\#} \geqslant 0,$$

which means that A is C_1-hyponormal. Also,

$$D^*D - C_4DD^*C_4 \geqslant C_2^{\#}AA^*C_2 \geqslant 0,$$

which means that D is C_4-hyponormal. Hence $T|_{\mathscr{M}}$ is C_1-hyponormal and $T|_{\mathscr{M}^{\perp}}$ is C_4-hyponormal.

(iii) If $T^2 = T$, then $ran(T) = \{x \in \mathscr{H} : Tx = x\}$. Hence $ran(T)$ is closed and $ran(T) \in Lat(T)$. Therefore, T has the following form with respect to $ran(T) \oplus ran(T)^{\perp}$;

$$T = \begin{pmatrix} I & S \\ 0 & 0 \end{pmatrix}.$$

Thus $T^*T = \begin{pmatrix} I & S \\ S^* & S^*S \end{pmatrix}$ and $TT^* = \begin{pmatrix} I+SS^* & 0 \\ 0 & 0 \end{pmatrix}$. Since $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$ is a conjugation on $ran(T) \oplus ran(T)^{\perp} = \mathscr{H}$, it follows from [7] that C_1 and C_4 are conjugations. Then

$$T^*T - CTT^*C = \begin{pmatrix} I - C_1(I+SS^*)C_1 & * \\ * & * \end{pmatrix} \geqslant 0.$$

Thus we get from [4] that $C_1SS^*C_1 \leqslant 0$, and hence $S = 0$. So $T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, which means that T is a projection. □

COROLLARY 2.7. *Let $T \in \mathscr{L}(\mathscr{H})$ be C-normal with a conjugation C. Then $C\ker(T) = \ker(T^*)$ and $C\overline{ran(T)} = \overline{ran(T^*)}$.*

Proof. If T is C-normal, then C-hyponormal and so $C\ker(T) \subset \ker(T^*)$ by Theorem 2.6. Since T is C-normal, $C\ker T^* \subset \ker T$ from [8, Corollary 4]. Then $C\ker T \subset \ker T^* \subset C\ker T$. Hence $\ker T^* = C\ker T$. Since T is C-hyponormal, by Theorem 2.6 $C\overline{ran(T)} \subset \overline{ran(T^*)}$. If $y \in [C\ker(T^*)]^{\perp}$, then $0 = \langle y, Cx\rangle = \langle x, Cy\rangle$ for all

$x \in \ker(T^*)$. Thus $Cy \in [\ker(T^*)]^\perp$ and so $y \in C[\ker(T^*)]^\perp$. Therefore, $[C\ker(T^*)]^\perp \subseteq C[\ker(T^*)]^\perp$. Since $\ker T^* = [ranT]^\perp$ for $T \in \mathscr{L}(\mathscr{H})$,

$$\overline{ran(T^*)} = [\ker T]^\perp = [C\ker(T^*)]^\perp \subseteq C[\ker(T^*)]^\perp = C\overline{ranT}.$$

Hence $C\overline{ran(T)} = \overline{ran(T^*)}$. □

COROLLARY 2.8. *Let $T \in \mathscr{L}(\mathscr{H})$ be C-hyponormal with a conjugation C. Then T^* is an isometry $\Leftrightarrow$ T is unitary.*

Proof. It suffices to show that $\Rightarrow$ holds. If T^* is an isometry, then it is bounded below. Since $\ker(T^*) = \{0\}$, $\ker(T) = \{0\}$ by Theorem 2.6. Hence $\overline{ran(T^*)} = (\ker T)^\perp = \mathscr{H}$. Since T^* has dense range and is bounded below, T^* is invertible. Hence T^* is unitary. Thus T is unitary. □

We next state several basic properties of C-hyponormal operators.

PROPOSITION 2.9. *Let $T \in \mathscr{L}(\mathscr{H})$ and let C be a conjugation on $\mathscr{H}$. Then the following properties hold.*

(i) *If T is C-hyponormal, then λT is C-hyponormal for all $\lambda \in \mathbb{C}$.*

(ii) *If T is invertible, then T is C-hyponormal if and only if so is T^{-1}.*

(iii) *The class $\mathscr{H}_C(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) \mid T \text{ is } C\text{-hyponormal}\}$ is closed in norm.*

Proof. (i) If T is C-hyponormal, then

$$(\lambda T)^*(\lambda T) - C(\lambda T)(\lambda T)^*C = |\lambda|^2(T^*T - CTT^*C) \geqslant 0.$$

Thus λT is C-hyponormal.

(ii) Note that if $T \in \mathscr{L}(\mathscr{H})$ is positive and invertible, then $T \geqslant I$ implies $T^{-1} \leqslant I$. Since $(CTC)^{-1} = CT^{-1}C$ for T, it follows that

$$I \geqslant (T^{-1})^*(CTC)(CT^*C)T^{-1}.$$

Equivalently,

$$T(CT^*C)^{-1}(CTC)^{-1}T^* \geqslant I.$$

Hence we get that

$$C(T^{-1})^*T^{-1}C - T^{-1}(T^{-1})^* \geqslant 0,$$

that is, T^{-1} is C-hyponormal. Conversely, if T^{-1} is C-hyponormal, then $T = (T^{-1})^{-1}$ is C-hyponormal by the previous proof.

(iii) Let $T \in \overline{\mathscr{H}_C(\mathscr{H})}$. Then there is a sequence $\{T_n\}$ in $\mathscr{H}_C(\mathscr{H})$ such that

$$\lim_{n\to\infty} \|T_n - T\| = 0.$$

Thus we obtain that

$$\begin{aligned}\|T^*T - CTT^*C\| \leqslant\ & \|T^*T - T_n^*T\| + \|T_n^*T - T_n^*T_n\| \\ & + \|T_n^*T_n - CT_nT_n^*C\| + \|CT_nT_n^*C - CTT_n^*C\| \\ & + \|CTT_n^*C - CTT^*C\| \\ \leqslant\ & \|T^* - T_n^*\|\|T\| + \|T_n^*\|\|T - T_n\| + 0 \\ & + \|C\|\|T_n - T\|\|T_n^*C\| + \|CT\|\|T_n^* - T^*\|\|C\| \to 0\end{aligned}$$

as $n \to \infty$. Hence $T \in \mathcal{N}_C(\mathcal{H}) \subset \mathcal{H}_C(\mathcal{H})$, which means that the class $\mathcal{H}_C(\mathcal{H})$ is norm closed in $\mathcal{L}(\mathcal{H})$ where $\mathcal{N}_C(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \mid T \text{ is } C\text{-normal}\}$. $\square$

PROPOSITION 2.10. *Let C be a conjugation on $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$ be C-hyponormal. Then the following statements hold.*

(i) $\|(CT)^n\| = \|T\|^n$ *for each* $n \geqslant 1$, *and hence* $\|T\| = \lim_{n\to\infty} \|(CT)^n\|^{\frac{1}{n}}$.

(ii) *If* $\mathcal{M} = \{Cx : \|Tx\| = \|T\|\|x\|\}$, *then* $TC\mathcal{M} \subset \mathcal{M}$, *and hence* $(CT)(C\mathcal{M}) \subset C\mathcal{M}$.

Proof. (i) Assume that T is C-hyponormal. Then $\|T^*Cx\| \leqslant \|Tx\|$ for all $x \in \mathcal{H}$. Therefore we get that for all $x \in \mathcal{H}$,

$$\begin{aligned}\|CTx\|^2 = \langle Tx, Tx\rangle &= \langle T^*Tx, x\rangle \\ &\leqslant \|T^*Tx\|\|x\| \\ &= \|T^*C(CTx)\|\|x\| \\ &\leqslant \|T(CTx)\|\|x\| \\ &= \|CT(CTx)\|\|x\| = \|(CT)^2x\|\|x\|.\end{aligned} \tag{2}$$

Replace x by CTx in (2). Then $\|(CT)^2x\|^2 \leqslant \|(CT)^3x\|\|CTx\|$ for all $x \in \mathcal{H}$. By a similar way, we have

$$\|(CT)^nx\|^2 \leqslant \|(CT)^{n+1}x\|\|(CT)^{n-1}x\| \leqslant \|(CT)^{n+1}\|\|(CT)^{n-1}\|\|x\|^2$$

for $x \in \mathcal{H}$. Therefore

$$\|(CT)^n\|^2 \leqslant \|(CT)^{n+1}\|\|(CT)^{n-1}\|^2. \tag{3}$$

We claim that $\|(CT)^n\| = \|CT\|^n$ for all $n \geqslant 1$. If $n = 1$, it is true. Assume that $\|(CT)^n\| = \|CT\|^n$ holds. Since $\|(CT)^n\|^2 \leqslant \|(CT)^{n+1}\|\|(CT)^{n-1}\|$ by (3), the induction hypothesis implies that

$$\|CT\|^{2n} = \|(CT)^n\|^2 \leqslant \|(CT)^{n+1}\|\|(CT)^{n-1}\|$$

and so $\|(CT)\|^{n+1} \leqslant \|(CT)^{n+1}\|$. By claim, $\|(CT)^n\| = \|CT\|^n = \|T\|^n$. Hence

$$\lim_{n\to\infty} \|(CT)^n\|^{\frac{1}{n}} = \|T\|.$$

So we complete the proof.

(ii) For all $y \in \mathscr{M}$, set $y = Cx$ for $x \in \mathscr{H}$. Then $\|Tx\| = \|T\|\|x\|$ holds. Thus

$$\|CT(CTx)\| \leqslant \|CT\|\|CTx\| \leqslant \|T\|\|Tx\| = \|T\|^2\|x\| = \|\|T\|^2x\|. \tag{4}$$

Note that

$$\begin{aligned}\|T^*Tx - \|T\|^2x\|^2 &= \|T^*Tx\|^2 - 2Re\langle T^*Tx, \|T\|^2x\rangle + \|T\|^4\|x\|^2\\ &= \|T^*Tx\|^2 - 2\|T\|^2\|Tx\|^2 + \|T\|^4\|x\|^2\\ &= \|T^*Tx\|^2 - 2\|T\|^4\|x\|^2 + \|T\|^4\|x\|^2\\ &= \|T^*Tx\|^2 - \|T\|^4\|x\|^2\\ &\leqslant \|T^*T\|^2\|x\|^2 - \|T\|^4\|x\|^2\\ &= \|T\|^4\|x\|^2 - \|T\|^4\|x\|^2 = 0.\end{aligned} \tag{5}$$

Since $CT^*TC \geqslant TT^*$, we have

$$\|T^*x\| \leqslant \|TCx\| \text{ for } x \in \mathscr{H}. \tag{6}$$

Hence we get from (4), (5), and (6) that

$$\|CT(CTx)\| \leqslant \|\|T\|^2x\| = \|T^*(Tx)\| \leqslant \|TC(Tx)\| = \|(CT)^2x\| \tag{7}$$

for $x \in \mathscr{H}$. From (4) and (7), $\|(CT)^2x\| = \|CT\|\|CTx\|$. Therefore, $C(CTx) \in \mathscr{M}$ and so $Tx = TCy \in \mathscr{M}$. Hence $TC\mathscr{M} \subset \mathscr{M}$ and $(TC)(C\mathscr{M}) \subset C\mathscr{M}$. □

PROPOSITION 2.11. *If $T = u \otimes v$ is C-hyponormal with a conjugation C, then* $|\langle Cu, v\rangle| = \|u\|\|v\|$.

Proof. Let $T = u \otimes v$ be C-hyponormal. Then $\|Tx\| \geqslant \|T^*Cx\|$ for all $x \in \mathscr{H}$. Since $T^* = v \otimes u$, we have

$$\|\langle x, v\rangle u\| \geqslant \|\langle Cx, u\rangle v\| \tag{8}$$

for all $x \in \mathscr{H}$. Take $x = v$ in (8). Then

$$\|u\|\|v\| \geqslant |\langle Cu, v\rangle|. \tag{9}$$

Take $x = Cu$ in (8). Then

$$\|u\|\|v\| \leqslant |\langle Cu, v\rangle|. \tag{10}$$

From (9) and (10), we have $|\langle Cu, v\rangle| = \|u\|\|v\|$. □

The C-hyponormality and hyponormality are equivalent for weighted shifts with respect to the given conjugation C as in Proposition 2.12.

PROPOSITION 2.12. *Let $\{e_n\}$ be an orthonormal basis on ℓ^2 and let $C : \ell^2 \to \ell^2$ be a conjugation given by $Ce_n = e_n$ for each $n \in \mathbb{N}$. If $W \in \mathscr{L}(\mathscr{H})$ is the weighted shift given by $We_n = \alpha_n e_{n+1}$ for all $n \geqslant 1$, then W is C-hyponormal if and only if W is hyponormal.*

Proof. If W is C-hyponormal, then for all $x \in \ell^2$, we get that

$$\begin{aligned}
&\langle W^*Wx,x\rangle - \langle CWW^*Cx,x\rangle \\
&= \langle W^*W\sum_{n=1}^{\infty} x_n e_n - CWW^*C\sum_{n=1}^{\infty} x_n e_n, \sum_{n=1}^{\infty} x_n e_n\rangle \\
&= \langle W\sum_{n=1}^{\infty} x_n e_n, W\sum_{n=1}^{\infty} x_n e_n\rangle - \langle C\sum_{n=1}^{\infty} x_n e_n, WW^*C\sum_{n=1}^{\infty} x_n e_n\rangle \\
&= \langle \sum_{n=1}^{\infty} x_n \alpha_n e_{n+1}, \sum_{n=1}^{\infty} x_n \alpha_n e_{n+1}\rangle - \langle \sum_{n=1}^{\infty} \overline{x_n} e_n, W\sum_{n=2}^{\infty} \overline{x_n}\, \overline{\alpha_{n-1}} e_{n-1}\rangle \\
&= \sum_{n=1}^{\infty} |x_n|^2|\alpha_n|^2 - \langle \sum_{n=1}^{\infty} \overline{x_n} e_n, \sum_{n=2}^{\infty} \overline{x_n} |\alpha_{n-1}|^2 e_n\rangle \\
&= \sum_{n=1}^{\infty} |x_n|^2|\alpha_n|^2 - \sum_{n=2}^{\infty} |x_n|^2|\alpha_{n-1}|^2 \\
&= |x_1|^2|\alpha_1|^2 + \sum_{n=2}^{\infty} |x_n|^2(|\alpha_n|^2 - |\alpha_{n-1}|^2) \geqslant 0 \qquad (11)
\end{aligned}$$

for all $n \in \mathbb{N}$. Take $x = e_j$. Then $|\alpha_j| \geqslant |\alpha_{j-1}|$ for $j = 2,3,\ldots$. Thus W is hyponormal. Conversely, if W is hyponormal, then $|a_{n+1}| \geqslant |a_n|$ for all $n \in \mathbb{N}$. Hence W is C-hyponormal from (11). □

Using Proposition 2.12, we can show that there is a C-hyponormal operator which is not C-normal.

EXAMPLE 2.13. If $S \in \mathscr{L}(\mathscr{H})$ is the unilateral shift given by $Se_n = e_{n+1}$ where $\{e_n\}$ is an orthonormal basis for $\mathscr{H}$, then S is hyponormal. If C is a conjugation given by $Ce_n = e_n$ for $n \in \mathbb{N}$, then S is C-hyponormal by Proposition 2.12. But S is not C-normal form Corollary 3 in [8].

Let C be a conjugation on $\mathscr{H}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *C-quasinormal* with a conjugation C if CT and $(CT)^{\#}(CT)$ commute. As in the proof of the case for a bounded linear operator, it is obvious that every C-normal operator is C-quasinormal and every C-quasinormal operator is C-hyponormal. We also know that every isometry is C-quasinormal. For example, the unilateral shift S in Example 2.2 is C-quasinormal.

PROPOSITION 2.14. *Let C be a conjugation on $\mathscr{H}$ and let $T \in \mathscr{L}(\mathscr{H})$. Then the following statements hold.*

(i) *If T is C-hyponormal and CT is an isometry on $(\ker(CT))^{\perp}$, then T is C-quasinormal.*

(ii) *If nonzero T is a C-quaisnormal operator defined on $\mathscr{H}$ with $\dim\mathscr{H} > 1$, then either T is C-normal or $\ker[(CT)^{\#}, CT]$ and $\overline{ran(CT)}$ are nontrivial.*

Proof. (i) Since T is C-hyponormal, $\ker(CT) \subset \ker(CT)^{\#}$. Thus $(CT)\ker(CT) = \{0\} \subset \ker(CT)$ and $(CT)^{\#}\ker(CT) = \{0\} \subset \ker(CT)$. Since $CT : \mathscr{H} \to \mathscr{H}$ is bounded and antilinear, it follows that

$$CT :\to \ker(CT) \oplus (\ker(CT))^{\perp} \to \ker(CT) \oplus (\ker(CT))^{\perp}.$$

Hence as in $\mathcal{L}(\mathcal{H})$, CT has the following matrix form;

$$CT = \begin{pmatrix} (CT)|_{\ker(CT)} & 0 \\ 0 & (CT)|_{\ker(CT)}^{\perp} \end{pmatrix}.$$

Since $(CT)|_{\ker(CT)}^{\perp}$ is an isometry, $T^*T = (CT)^{\#}(CT) = 0 \oplus I$. Hence $(CT)(T^*T) = (T^*T)(CT)$. Thus T is C-quasinormal.

(ii) Assume that T is C-quasinormal. Then $[(CT)^{\#}, CT]CT = 0$. Hence either $[(CT)^{\#}, CT] = 0$ or $[(CT)^{\#}, CT] \neq 0$. If $[(CT)^{\#}, CT] = 0$, then T is C-normal. If $[(CT)^{\#}, CT] \neq 0$, then $ran(CT) \subset \ker[(CT)^{\#}, CT]$ and

$$(CT)ran(CT) \subset (CT)\ker[(CT)^{\#}, CT] \subset (CT)\mathcal{H} = ran(CT) \subset \ker[(CT)^{\#}, CT].$$

Since CT is nonzero, we have $ran(CT) \neq \{0\}$ and $\ker[(CT)^{\#}, CT] \neq \{0\}$. Since $[(CT)^{\#}, CT] \neq 0$, $\ker[(CT)^{\#}, CT] \neq \mathcal{H}$ and $\overline{ran(CT)} \neq \mathcal{H}$. Therefore $\{0\} = \ker[(CT)^{\#}, CT] \neq \mathcal{H}$ and $\{0\} \neq \overline{ran(CT)} \neq \mathcal{H}$. $\square$

THEOREM 2.15. *Let $T \in \mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Then the following arguments are equivalent.*

(i) *$T - \lambda I$ is C-hyponormal for all $\lambda \in \mathbb{C}$.*

(ii) *T is a complex symmetric operator with a conjugation C.*

(iii) *$T - \lambda I$ is C-normal for all $\lambda \in \mathbb{C}$.*

(iv) *$T - \lambda I$ is C-quasinormal for all $\lambda \in \mathbb{C}$.*

Proof. Since (ii) $\Rightarrow$ (iii) holds from [10] and (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are true, it suffices to show that (i) $\Rightarrow$ (ii) holds. If $T - \lambda I$ is C-hyponormal for all $\lambda \in \mathbb{C}$, then it follows that

$$\begin{aligned} 0 &\leqslant C(T-\lambda I)^*(T-\lambda I)C - (T-\lambda I)(T-\lambda I)^* \\ &= C(T^*T - \overline{\lambda}T - \lambda T^* + |\lambda|^2 I)C - (TT^* - \overline{\lambda}T - \lambda T^* + |\lambda|^2 I) \\ &= CT^*TC - \lambda CTC - \overline{\lambda}CT^*C - TT^* + \lambda T^* + \overline{\lambda}T. \end{aligned} \tag{12}$$

Set $\lambda = re^{i\theta}$ for any $\theta \in \mathbb{R}$. Thus (12) becomes

$$CT^*TC - re^{i\theta}CTC - re^{-i\theta}CT^*C \geqslant TT^* - re^{i\theta}T^* - re^{-i\theta}T.$$

Therefore we have

$$\frac{CT^*TC}{r} - e^{i\theta}CTC - e^{-i\theta}CT^*C \geqslant \frac{TT^*}{r} - e^{i\theta}T^* - e^{-i\theta}T.$$

Letting $r \to \infty$, we get that

$$-e^{i\theta}CTC - e^{-i\theta}CT^*C + e^{i\theta}T^* + e^{-i\theta}T \geqslant 0. \tag{13}$$

Taking $\theta = 0$, we get that $-CTC + T^* \geqslant -T + CT^*C$ and taking $\theta = \pi$, we have $CTC + CT^*C \geqslant T + T^*$. Hence we have $C(T+T^*)C = T + T^*$. So, $C(Re(T))C = Re(T)$. Taking $\theta = \frac{\pi}{2}$. Then (13) implies

$$-2C(Im(T))C + 2Im(T) \geqslant 0.$$

Taking $\theta = -\frac{\pi}{2}$ we get that (13) implies

$$2C(Im(T))C - 2Im(T) \geqslant 0.$$

Therefore $C(Im(T))C = Im(T)$. So,

$$\begin{aligned} CTC &= C[Re(T) + iIm(T)]C \\ &= CRe(T)C - iCIm(T)C \\ &= Re(T) - iIm(T) \\ &= [Re(T) + iIm(T)]^* = T^*. \end{aligned}$$

Hence T is a complex symmetric operator with the conjugation C. □

EXAMPLE 2.16. Let C be a conjugation operator on $\mathbb{C}^4$ defined by

$$C(x_1, x_2, x_3, x_4) = (\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_1})$$

and let $\{e_n\}_{n=1}^4$ be an orthonormal basis of $\mathbb{C}^4$. Suppose that T has the form

$$T = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{e_n\}_{n=1}^4$. Since T is complex symmetric if and only if T is unitarily equivalent to a complex symmetric matrix, it follows from [6, Theorem 1] that the trace of the following matrices must vanish:

$$(1)\quad T(TT^{*2} - TT^{*2}T)TT^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 49 & -2401 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(2)\quad T^2(TT^{*2} - T^{*2}T)T^2T^* = \begin{pmatrix} 0 & 0 & -2352 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(3)\quad T[(T^2T^*)^2 - (T^*T^2)^2]TT^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(4)\ \ T^2(TT^{*3}-T^{3*}T)T^2T^{*2}=\begin{pmatrix}49 & 0 & 0 & 0\\ 0 & -2401 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$(5)\ \ T(T^2T^{*2}-T^{*2}T^2)TT^*=\begin{pmatrix}0 & -49 & 0 & 0\\ 0 & 0 & -49 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$(6)\ \ T(T^2T^{*3}-T^{*3}T^2)TT^*=\begin{pmatrix}49 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2401 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad \text{and}$$

$$(7)\ \ T^2T^*(T^*T-TT^*)T^*T^2T^*=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & -115248 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

However, these traces do not vanish. Hence T is not unitarily equivalent to a complex symmetric matrix and hence T is not a complex symmetric operator. By Theorem 2.15, $T-\lambda I$ is not C-hyponormal for some $\lambda\in\mathbb{C}$.

Recall that two operators T_1 and T_2 are *doubly commuting* if $T_1T_2=T_2T_1$ and $T_1^*T_2=T_2T_1^*$ hold.

PROPOSITION 2.17. *Let C be a conjugation on $\mathscr{H}$. Assume that T_1 and T_2 are C-hyponormal in $\mathscr{L}(\mathscr{H})$ and T_1,T_2 are doubly commuting. If $T_1^*T_2$ is complex symmetric with a conjugation C, then T_1+T_2 is C-hyponormal.*

Proof. Since T_1 and T_2 are C-hyponormal, it follows that

$$\begin{aligned}
&(T_1+T_2)^*(T_1+T_2)-C(T_1+T_2)(T_1+T_2)^*C\\
&=T_1^*T_1+T_1^*T_2+T_2^*T_1+T_2^*T_2-C(T_1T_1^*+T_1T_2^*+T_2T_1^*+T_2T_2^*)C\\
&=(T_1^*T_1-CT_1T_1^*C)+(T_1^*T_2-C(T_1T_2^*)C)\\
&\quad+(T_2^*T_1-C(T_2T_1^*)C)+(T_2^*T_2-CT_2T_2^*C)\\
&\geqslant(T_1^*T_2-C(T_1T_2^*)C)+(T_2^*T_1-C(T_2T_1^*)C).
\end{aligned}$$

Moreover, since $T_1^*T_2$ is complex symmetric with $T_1^*T_2=T_2T_1^*$, we have

$$\begin{aligned}
&(T_1+T_2)^*(T_1+T_2)-C(T_1+T_2)(T_1+T_2)^*C\\
&\geqslant(T_1^*T_2-C(T_1T_2^*)C)+(T_2^*T_1-C(T_2T_1^*)C)=0.
\end{aligned}$$

Hence T_1+T_2 is C-hyponormal. $\square$

EXAMPLE 2.18. Let $\{e_n\}$ be an orthonormal basis and let C be the conjugation such that $Ce_n=e_n$ for n. Assume that D_1 and D_2 are diagonal operators in $\mathscr{L}(\mathscr{H})$, D_1 and D_2 are doubly commuting, and $D_1^*D_2$ is complex symmetric with a conjugation C. Since D_1 and D_2 are normal, D_1 and D_2 are C-hyponormal. Hence D_1+D_2 is C-hyponormal from Proposition 2.17.

THEOREM 2.19. *Let T^* be p-hyponormal in $\mathcal{L}(\mathcal{H})$ for $0 < p \leqslant 1$ and let C be a conjugation on $\mathcal{H}$. Then the following statements equivalent.*

(i) *T is C-hyponormal.*

(ii) *T is normal.*

(iii) *T is a complex symmetric operator.*

(iv) *T is C-normal.*

Proof. Since every normal operator is a complex symmetric operator by [5] and any complex symmetric operator is C-normal by [10], it suffices to show that (i) $\Rightarrow$ (ii). Assume that T is C-hyponormal and T^* is p-hyponormal. It suffices to consider when $p = \frac{1}{2^n}$ for some $n \in \mathbb{N}$. Since $C|T^*|^2 C \leqslant |T|^2$, it follows from Löwner's lemma that

$$(C|T^*|^2 C)^{\frac{1}{2}} \leqslant |T|.$$

Since $(C|T^*|C)^2 = C|T^*|^2 C$, it follows that $C|T^*|C = (C|T^*|^2 C)^{\frac{1}{2}} \leqslant |T|$. By induction, we can prove that $C|T^*|^{\frac{1}{2^n}} C \leqslant |T|^{\frac{1}{2^n}}$. Thus

$$C|T^*|^p C \leqslant |T|^p \tag{14}$$

for $0 < p \leqslant 1$. Since T^* is p-hyponormal, $|T|^p \leqslant |T^*|^p$ holds and so $C|T|^p C \leqslant C|T^*|^p C$. From (14), $C|T|^p C \leqslant C|T^*|^p C \leqslant |T|^p$. Then $C|T|^p C \leqslant |T|^p$ and so $|T|^p \leqslant C|T|^p C$. Hence $C|T|^p C = |T|^p$. So, $C|T|^p = |T|^p C$ for $0 < p \leqslant 1$. Since T is C-hyponormal, $C|T|^2 C \geqslant |T^*|^2$. By Löwner's lemma, $C|T|^{2p} C \geqslant |T^*|^{2p}$ for $0 < p \leqslant 1$. Since $C|T|^p = |T|^p C$, $|T|^{2p} \geqslant |T^*|^{2p}$. Hence T is p-hyponormal. Since T^* is p-hyponomal, $|T|^{2p} = |T^*|^{2p}$ for $0 < p \leqslant 1$. Now we consider the case when $p = \frac{1}{2^n}$ again. Since

$$(|T|^{2\cdot\frac{1}{2^n}})^2 - (|T^*|^{2\cdot\frac{1}{2^n}})^2 = (|T|^{2\cdot\frac{1}{2^n}} + |T^*|^{2\cdot\frac{1}{2^n}})(|T|^{2\cdot\frac{1}{2^n}} - |T^*|^{2\cdot\frac{1}{2^n}}) = 0,$$

$(|T|^{2\cdot\frac{1}{2^n}})^2 = (|T^*|^{2\cdot\frac{1}{2^n}})^2$. Assume that $(|T|^{2\cdot\frac{1}{2^n}})^{2^k} - (|T^*|^{2\cdot\frac{1}{2^n}})^{2^k}$ holds. Then

$$(|T|^{2\cdot\frac{1}{2^n}})^{2^{k+1}} - (|T^*|^{2\cdot\frac{1}{2^n}})^{2^{k+1}} = ((|T|^{2\cdot\frac{1}{2^n}})^{2^k} + (|T^*|^{2\cdot\frac{1}{2^n}})^{2^k})((|T|^{2\cdot\frac{1}{2^n}})^{2^k} - (|T^*|^{2\cdot\frac{1}{2^n}})^{2^k}) = 0.$$

By induction, $|T|^2 = |T^*|^2$. Hence T is normal. $\square$

COROLLARY 2.20. *Let T be C-hyponormal with a conjugation C. If T^* be p-hyponormal for $0 < p \leqslant 1$, then $|T|^p C = C|T|^p$.*

Proof. The proof follows from the proof of Theorem 2.19. $\square$

COROLLARY 2.21. *Let C be a conjugation on $\mathcal{H}$. Assume that T_1 and T_2 are commuting C-hyponormal operators in $\mathcal{L}(\mathcal{H})$. If T_1^* is p-hyponormal and $T_1^* T_2$ is a complex symmetric operator with a conjugation C, then $T_1 + T_2$ is C-hyponormal.*

Proof. By Theorem 2.19, T_1 is normal. Since $T_1T_2 = T_2T_1$, by Fuglede-Putnam Theorem, T_1 and T_2 are doubly commuting. By Proposition 2.17, $T_1 + T_2$ is C-hyponormal. □

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