Journal of Mathematical Inequalities
Volume 19, Number 1 (2025), 135–149 doi:10.7153/jmi-2025-19-09

MONOTONICITY PROPERTIES OF THE GAUSSIAN HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS

XIAO-YAN MA, XING-HUA HE AND GEN-HONG ZHONG*

(*Communicated by G. Nemes*)

Abstract. This paper shows the monotonicity properties of combined functions involving the Gaussian hypergeometric function with certain conditions satisfied by their parameters. These results have important applications in the theories of quasiconformal mappings and Ramanujan's modular equations. By these results, several inequalities for the complete elliptic integrals and the generalized elliptic integrals, the Grötzsch ring function and the generalized Grötzsch ring function, the solutions of Ramanujan's modular equation and Ramanujan's generalized modular equation are obtained.

1. Introduction

Throughout this paper, we let $\mathbb{N} = \{1,2,3\cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a,b,c \in \mathbb{R}$ with $c \neq 0,-1,-2,\cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \quad \text{for} \quad |x| < 1, \tag{1}$$

where (a,n) denotes the shifted factorial function

$$(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$$

for $n \in \mathbb{N}$, and $(a,0) = 1$ for $a \neq 0$. $F(a,b;c;x)$ is said to be zero-balanced if $c = a+b$.

It is well known that $F(a,b;c;x)$ has many important applications in various fields of the mathematical and natural sciences, and many other special functions in mathematical physics are particular cases of this function (cf. [5, 9, 12, 13, 17, 19, 20, 24]).

One of the important special cases of $F(a,b;c;x)$ is as follows

$$F_s(x) = F\left(\frac{1}{2}-s, \frac{1}{2}+s; 1; x\right) = \sum_{n=0}^{\infty} \frac{(1/2-s,n)(1/2+s,n)}{(n!)^2} x^n, \tag{2}$$

Mathematics subject classification (2020): 33C05, 33C75, 33E05.
Keywords and phrases: Gaussian hypergeometric function, complete elliptic integrals, generalized elliptic integrals, inequality.

* Corresponding author.

for $s \in (-1/2, 1/2)$ and $x \in (0,1)$. It is interesting that some properties of the function F_s can be directly applied to obtain properties of the compound means, and the elliptic series for $1/\pi$ (see [7]).

There have been many studies of the properties of F_s for the special cases when $s = 0, 1/6, 1/4, 1/3$ ([5, 6, 8, 15, 18, 21, 22, 23]). Naturally, it would be more significant for us to show the monotonicity properties of combined functions involving the Gaussian hypergeometric function with certain conditions satisfied by their parameters.

As particular cases of Gaussian hypergeometric functions, the generalized elliptic integrals of the first and second kinds are defined by

$$\mathscr{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1, r^2), \quad \mathscr{K}_a'(r) = \mathscr{K}_a(r') \tag{3}$$

and

$$\mathscr{E}_a(r) = \frac{\pi}{2} F(a-1, 1-a; 1, r^2), \quad \mathscr{E}_a'(r) = \mathscr{E}_a(r') \tag{4}$$

for $a, r \in (0,1)$, respectively. In particular, $\mathscr{K} = \mathscr{K}_{1/2}$ and $\mathscr{K}' = \mathscr{K}_{1/2}'$ ($\mathscr{E} = \mathscr{E}_{1/2}$ and $\mathscr{E}' = \mathscr{E}_{1/2}'$) are the well-known complete elliptic integrals of the first (second, respectively) kind. By the symmetry, we may assume that $a \in (0, 1/2]$, without loss of generality.

For $x, y \in (0, \infty)$, the classical gamma, beta and psi functions are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{5}$$

respectively (cf. [2, 5, 4, 17]). By [5, 6.1.15, 6.1.17, 6.1.30],

$$\Gamma(z+1) = z\Gamma(z), \tag{6}$$

$$\Gamma(z)\Gamma(1-z) = \pi / \sin(\pi z), \tag{7}$$

$$\Gamma(1/2+z)\Gamma(1/2-z) = \pi / \cos(\pi z). \tag{8}$$

Some important properties of $F(a,b;c;x)$ are related to the Ramanujan constant $R(a,b)$ and its special case $R(a)$, defined by

$$R(a,b) = -2\gamma - \psi(a) - \psi(b) \text{ for } a, b \in (0, \infty),$$
$$R(a) = R(a, 1-a) = -2\gamma - \psi(a) - \psi(1-a), \; R(1/2) = \log 16,$$

which are in fact functions of the parameters a and b in $F(a,b;c;x)$ (cf. [16]).

For convenience, let

$$B(a) = B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \pi / \sin(\pi a). \tag{9}$$

As we know, many of the known results study the properties of certain combinations defined mainly in terms of $\mathscr{K}_a(r)$ (or $\mathscr{K}(r)$) and $\mathscr{E}_a(r)$ (or $\mathscr{E}(r)$) such as $[\mathscr{E}_a(r) - r'^2 \mathscr{K}_a(r)]/[\mathscr{E}(r) - r'^2 \mathscr{K}(r)]$ and $[\mathscr{K}_a(r) - \mathscr{E}_a(r)]/[\mathscr{K}(r) - \mathscr{E}(r)]$ (cf. [14, Theorm 3.6]). As the applications of our main results, the other purpose of this paper is to reveal the monotonicity properties of certain combinations such as $[\mathscr{E}_a(r)-r'^2\mathscr{K}_a(r)]/[\mathscr{E}(r)-r'^2\mathscr{K}(r)]$ and $\mathscr{K}_a(r)/\mathscr{K}(r)$. Such kind of results can be applied to obtain some properties of the generalized Grötzsch ring function and some other related special functions (see [11, Corollary 2.12, Corollary 2.15]).

Throughout this paper, by the symmetry of the parameters a and b in the function $F(a,b;a+b;x)$, we assume that $a<b$, without loss of generality. We denote $r'=\sqrt{1-r^2}$ for each $r\in[0,1]$. For $a,b\in(0,\infty)$ with $c=a+b$, and for $r\in[0,1]$, let $|s|<1/2$, and let

$$C=\frac{c+1}{2},\ \ \rho=\frac{4ab}{C(1-4s^2)},\ \ \tau=\frac{(1-4s^2)\sin(\pi a)}{4a(1-a)\cos(\pi s)},$$

$$\delta_1=\frac{\cos(\frac{a-b}{2}\pi)}{\cos(\frac{a+b}{2}\pi)},\ \ \delta_2=\frac{\sin(\pi a)}{\pi}\left[R(a)-R\left(\frac{1}{2}-s\right)\right],$$

$$F(r)=F(a,b;C;r),\ \ F_s(r)=F\left(\frac{1}{2}-s,\frac{1}{2}+s;1;r\right),$$

$$F_+(r)=F(a+1,b+1;C+1;r),\ \ F_{s+}(r)=F\left(\frac{3}{2}-s,\frac{3}{2}+s;2;r\right).$$

Observe that the conditions $a\leqslant b$ and $c=a+b$ imply that

$$a\leqslant c/2\leqslant b\quad\text{and}\quad ab=a(c-a)\leqslant c^2/4. \tag{10}$$

By (10), for $a,b\in(0,\infty)$ with $c=a+b$, we have the following simple relations

$$\begin{aligned}c\leqslant 1\Rightarrow \frac{11}{16}-\frac{7c}{16}-\frac{c+3}{4}s^2\leqslant \frac{7}{4}-\frac{3c}{2}-s^2,\\ c\geqslant 1\Rightarrow \frac{7}{4}-\frac{3c}{2}-s^2\leqslant \frac{11}{16}-\frac{7c}{16}-\frac{c+3}{4}s^2.\end{aligned}$$

The following formulas are well-known

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \ c>a+b, \tag{11}$$

$$\frac{d}{dx}F(a,b;c;x)=\frac{ab}{c}F(a+1,b+1;c+1;x), \tag{12}$$

$$F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x),\ \ c<a+b, \tag{13}$$

$$B(a,b)F(a,b;c;x)=R(a,b)-\log(1-x)+O((1-x)\log(1-x)),\ \ c=a+b \tag{14}$$

as $r\to 1$ (see [5, 15.1.20, 15.2.1, 15.3.3, 15.3.10]).

2. Preliminaries

In this section, for readers' convenience, we shall give two lemmas, which are needed in the proofs of our main results proved in Section 3.

LEMMA 1. ([12, Lemma 2.1] and [16, Lemma 2.4]) *For $n \in \mathbb{N}_0$, let r_n and s_n be real numbers, and let the power series $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and $S(x) = \sum_{n=0}^{\infty} s_n x^n$ be convergent for $|x| < 1$. If $s_n \geqslant 0$ and not all vanish for $n \in \mathbb{N}_0$, and if r_n/s_n is strictly increasing (decreasing) in $n \in \mathbb{N}_0$, then the function $x \mapsto R(x)/S(x)$ is strictly increasing (decreasing, respectively) on $(0,1)$.*

LEMMA 2. ([16, Lemma 2.4]) *Suppose that $r \in (0,\infty)$ is the common radius of convergence of the real power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ with $b_n > 0$, and $\{a_n/b_n\}$ is a non-constant sequence. Let $\varphi(x) = A(x)/B(x)$.*

(1) *If there is an $n_0 \in \mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \leqslant n \leqslant n_0$, and decreasing (increasing) for $n \geqslant n_0$, then φ is increasing (decreasing) on $(0,r)$ if and only if $\varphi'(r^-) \geqslant 0$ $(\varphi'(r^-) \leqslant 0)$, respectively.*

(2) *If there is an $n_0 \in \mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \leqslant n \leqslant n_0$, and decreasing (increasing) for $n \geqslant n_0$, and if $\varphi'(r^-) < 0$ $(\varphi'(r^-) > 0)$, then there exists a number $x_0 \in (0,r)$ such that φ is strictly increasing (decreasing) on $(0,x_0]$ and decreasing (increasing, respectively) on $[x_0,r)$.*

For the convenience of proving the following theorems, we have made the following preparations.

Since $(C+2)-[(C-a)+(C-b)] > C+1-[(C-a)+(C-b)] = C > 0$, it follows from (11) that

$$F(C-a,C-b;C+1;1) = \frac{C\Gamma(C)^2}{ab\Gamma(a)\Gamma(b)}, \tag{15}$$

$$F(C-a,C-b;C+2;1) = \frac{(1+C)C^2\Gamma(C)^2}{ab(1+a)(1+b)\Gamma(a)\Gamma(b)}. \tag{16}$$

By (8), (11) and $3-[(\frac{1}{2}-s)+(\frac{1}{2}+s)] > 2-[(\frac{1}{2}-s)+(\frac{1}{2}+s)] = 1 > 0$, we have

$$F\left(\frac{1}{2}-s,\frac{1}{2}+s;2;1\right) = \frac{4\cos(\pi s)}{\pi(1-4s^2)}, \tag{17}$$

$$F\left(\frac{1}{2}-s,\frac{1}{2}+s;3;1\right) = \frac{32\cos(\pi s)}{(1-4s^2)(9-4s^2)\pi}. \tag{18}$$

And if $c < 1$, then $C-(a+b) = (1-c)/2 = 1-C > 0$, by (11),

$$F(1) = F(a,b;C;1) = \frac{\Gamma(C)\Gamma(1-C)}{\Gamma(C-a)\Gamma(C-b)}. \tag{19}$$

If $c > 1$, then $C-(a+b) = (1-c)/2 = 1-C < 0$, by (13),

$$F(r) = F(a,b;C;r) = \frac{1}{(1-r)^{C-1}} F(C-a, C-b; C; r), \tag{20}$$

and $C - [(C-a)+(C-b)] = (c-1)/2 = C-1 > 0$, then by (11),

$$F(C-a, C-b; C; 1) = \frac{\Gamma(C)\Gamma(C-1)}{\Gamma(a)\Gamma(b)}. \tag{21}$$

It follows from (12) that

$$F'(r) = \frac{d}{dr} F(a,b;C;r) = \frac{ab}{C} F(a+1, b+1; C+1; r) = \frac{ab}{C} F_{+}(r), \tag{22}$$

$$\begin{aligned} F_s'(r) &= \frac{d}{dr} F\left(\frac{1}{2}-s, \frac{1}{2}+s; 1; r\right) = \left(\frac{1}{4}-s^2\right) F\left(\frac{3}{2}-s, \frac{3}{2}+s; 2; r\right) \\ &= \left(\frac{1}{4}-s^2\right) F_{s+}(r). \end{aligned} \tag{23}$$

Similarly, by (12), (13), we obtain

$$F_{+}'(r) = \frac{d}{dr} F(a+1, b+1; C+1; r) = \frac{(a+1)(b+1)}{(C+1)(1-r)^{C+1}} F(C-a, C-b; C+2; r), \tag{24}$$

$$F_{s+}'(r) = \frac{d}{dr} F\left(\frac{3}{2}-s, \frac{3}{2}+s; 2; r\right) = \frac{9-4s^2}{8(1-r)^2} F\left(\frac{1}{2}-s, \frac{1}{2}+s; 3; r\right). \tag{25}$$

3. Main results

THEOREM 1. *For $a, b \in (0, \infty)$ with $c = a+b$, and $|s| < 1/2$, $C = (c+1)/2$, and for $r \in (0,1)$, let $f_1(r) = F_{+}(r)/F_{s+}(r)$. Then $f_1(0^+) = 1$ and*

$$f_1(1^-) = \begin{cases} 0, & c < 1, \\ \tau, & c = 1, \\ \infty, & c > 1. \end{cases}$$

And we have the following conclusions:

(1) If $c < 1$.

(i) $ab \leqslant \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4} s^2$, then f_1 is decreasing on $(0,1)$, with $f_1(0^+) = 1$ and $f_1(1^-) = 0$.

(ii) $\frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4} s^2 < ab < \frac{7}{4} - \frac{3c^2}{2} - s^2$, then there exists a number $r_1 = r_1(a,b,s) \in (0,1)$ such that f_1 is increasing on $(0, r_1]$ and decreasing on $[r_1, 1)$.

(iii) $ab \geqslant \frac{7}{4} - \frac{3c^2}{2} - s^2$, then there exists a number $r_2 = r_2(a,b,s) \in (0,1)$ such that f_1 is increasing on $(0, r_2]$ and decreasing on $[r_2, 1)$.

(2) If $c = 1$.

(i) $a(1-a) \leqslant 1/4 - s^2$, *then* f_1 *is decreasing on* $(0,1)$, *with* $f_1(0^+) = 1$ *and* $f_1(1^-) = \tau$.

(ii) $a(1-a) \geqslant 1/4 - s^2$, *then* f_1 *is increasing on* $(0,1)$, *with* $f_1(0^+) = 1$ *and* $f_1(1^-) = \tau$.

(3) If $c > 1$.

(i) $ab \leqslant \frac{7}{4} - \frac{3c^2}{2} - s^2$, *then there exists a number* $r_3 = r_3(a,b,s) \in (0,1)$ *such that* f_1 *is decreasing on* $(0, r_3)$ *and increasing on* $[r_3, 1)$.

(ii) $\frac{7}{4} - \frac{3c^2}{2} - s^2 < ab < \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, *then there exists a number* $r_4 = r_4(a,b,s) \in (0,1)$ *such that* f_1 *is decreasing on* $(0, r_4]$ *and increasing on* $[r_4, 1)$.

(iii) $ab \geqslant \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, *then* f_1 *is increasing on* $(0,1)$, *with* $f_1(0^+) = 1$ *and* $f_1(1^-) = \infty$.

Proof. Clearly, $f_1(0^+) = 1$. Since $C + 1 - [(a+1) + (b+1)] = -C < 0$ and $2 - [(\frac{3}{2} - s) + (\frac{3}{2} + s)] = -1 < 0$, it follows from (13) that

$$F_+(r) = F(a+1, b+1; C+1; r) = \frac{1}{(1-r)^C} F(C-a, C-b; C+1; r), \tag{26}$$

$$F_{s+}(r) = F\left(\frac{3}{2} - s, \frac{3}{2} + s; 2; r\right) = \frac{1}{1-r} F\left(\frac{1}{2} - s, \frac{1}{2} + s; 2; r\right), \tag{27}$$

and it follows from (7), (15) and (17) that we obtain the limiting value

$$\begin{aligned} f_1(1^-) &= \lim_{r \to 1^-} \frac{(1-r)^{-C} F(C-a, C-b; C+1; r)}{(1-r)^{-1} F(\frac{1}{2} - s, \frac{1}{2} + s; 2; r)} \\ &= \frac{(1-4s^2)\pi C \Gamma(C)^2}{4ab\cos(\pi s)\Gamma(a)\Gamma(b)} \cdot \lim_{r \to 1^-} (1-r)^{1-C} = \begin{cases} 0, & c < 1, \\ \tau, & c = 1, \\ \infty, & c > 1. \end{cases} \end{aligned} \tag{28}$$

By (24), (25), differentiation gives

$$f_1'(r) = \frac{\frac{(a+1)(b+1)}{(C+1)(1-r)^{C+1}} F(C-a, C-b; C+2; r) F_{s+}(r) - \frac{9-4s^2}{8(1-r)^2} F\left(\frac{1}{2} - s, \frac{1}{2} + s; 3; r\right) F_+(r)}{F_{s+}^2(r)}. \tag{29}$$

Clearly, $f_1'(1^-)|_{C=1} = 0$ by (15)–(18) and (29).

It follows from (26) and (27) that

$$f_1'(r) = \frac{\left[\begin{array}{l} \frac{(a+1)(b+1)}{C+1} F(C-a, C-b; C+2; r) F\left(\frac{1}{2} - s, \frac{1}{2} + s; 2; r\right) \\ \qquad - \frac{9-4s^2}{8} F(C-a, C-b; C+1; r) F\left(\frac{1}{2} - s, \frac{1}{2} + s; 3; r\right) \end{array}\right]}{(1-r)^C F\left(\frac{1}{2} - s, \frac{1}{2} + s; 2; r\right)^2}.$$

Hence by (15)–(18),

$$f_1'(1^-) = \frac{\pi C(C-1)\Gamma(C)^2(1-4s^2)}{4ab\Gamma(a)\Gamma(b)\cos(\pi s)} \cdot \lim_{r\to 1^-} \frac{1}{(1-r)^C} = \begin{cases} -\infty, & C<1, \\ \infty, & C>1. \end{cases} \tag{30}$$

Next, for $a,b \in (0,\infty)$ with $c = a+b$, and for $n \in N_0$, let

$$a_n = \frac{(a+1,n)(b+1,n)}{(C+1,n)n!}, \quad b_n = \frac{(\frac{3}{2}-s,n)(\frac{3}{2}+s,n)}{(2,n)n!}, \quad c_n = \frac{a_n}{b_n},$$

$$\begin{aligned} \Delta_1 &= \Delta_1(n,a,b,s) \\ &= 4(a+b-1)n^2 + [8ab+12(a+b)+8s^2-14]n \\ &\quad + [16ab+7a+7b+4(a+b+3)s^2-11] \\ &= 4(c-1)n^2 + 8\left[ab - \left(\frac{7}{4} - \frac{3c}{2} - s^2\right)\right] n + 16\left[ab - \left(\frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2\right)\right]. \end{aligned} \tag{31}$$

Then by (1),

$$\begin{aligned} f_1(r) &= \frac{\sum_{n=0}^{\infty} a_n r^n}{\sum_{n=0}^{\infty} b_n r^n}, \\ \frac{c_{n+1}}{c_n} &= 1 + \frac{\Delta_1(n,a,b,s)}{8n^3+4(c+9)n^2+(12c-8s^2+54)n+(c+3)(9-4s^2)}. \end{aligned} \tag{32}$$

Case 1. If $c < 1$.

Subcase (i). $ab \leqslant \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, then $\Delta_1(n,a,b,s) \leqslant 0$, and c_n is decreasing in $n \in \mathbb{N}_0$ by (31) and (32). Hence f_1 is decreasing on $(0,1)$ by Lemma 1.

Subcase (ii). $\frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2 < ab < \frac{7}{4} - \frac{3c}{2} - s^2$, then c_n is increasing and then decreasing in $n \in \mathbb{N}_0$ by (31) and (32), and $f_1'(1^-) < 0$ by (30). Hence there exists a number $r_1 = r_1(a,b,s) \in (0,1)$ such that f_1 is increasing on $(0,r_1]$ and decreasing on $[r_1,1)$ by Lemma 2(2).

Subcase (iii). $ab \geqslant \frac{7}{4} - \frac{3c}{2} - s^2$, then c_n is increasing and then decreasing in $n \in \mathbb{N}_0$ by (32), and $f_1'(1^-) < 0$ by (30). Hence the piecewise monotonicity of f_1 follows from Lemma 2(2).

Case 2. If $c = 1$.

By (31), we obtain

$$\Delta_1(n,a,b,s) = 8(n+2)[a(1-a) - (1/4 - s^2)]. \tag{33}$$

Subcase (i). $a(1-a) \leqslant 1/4 - s^2$, it follows from (33) that $\Delta_1(n,a,b,s) \leqslant 0$, and c_n is decreasing in $n \in \mathbb{N}_0$ by (32). Hence f_1 is decreasing on $(0,1)$ by Lemma 1.

Subcase (ii). $a(1-a) \geqslant 1/4 - s^2$, it follows from (33) that $\Delta_1(n,a,b,s) \geqslant 0$, and c_n is increasing in $n \in \mathbb{N}_0$ by (32). Hence f_1 is increasing on $(0,1)$ by Lemma 1.

Case 3. If $c > 1$.

Subcase (i). $ab \leqslant \frac{7}{4} - \frac{3c}{2} - s^2$, then c_n is decreasing and then increasing in $n \in \mathbb{N}_0$ by (32), and $f_1'(1^-) > 0$ by (30). Hence there exists a number $r_3 = r_3(a,b,s) \in (0,1)$ such that f_1 is decreasing on $(0, r_3]$ and increasing on $[r_3, 1)$ Lemma 2(2).

Subcase (ii). $\frac{7}{4} - \frac{3c}{2} - s^2 < ab < \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, then c_n is decreasing and then increasing in $n \in \mathbb{N}_0$ by (32), and $f_1'(1^-) > 0$ by (30). Hence the piecewise monotonicity of f_1 follows from Lemma 2(2).

Subcase (iii). $ab \geqslant \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, then $\Delta_1(n,a,b,s) \geqslant 0$, and c_n is increasing in $n \in \mathbb{N}_0$ by (32). Hence f_1 is increasing on $(0,1)$ by Lemma 1. □

THEOREM 2. *For $(a,b) \in (0,\infty)$ with $c = a+b$, and $|s| < 1/2$, $C = (c+1)/2$, and for $r \in (0,1)$, let $f_2(r) = F(r) - \dfrac{F'(r)}{F_s'(r)} \cdot F_s(r)$. Then $f_2(0^+) = 1 - \rho$, and*

$$f_2(1^-) = \begin{cases} \delta_1, & c < 1, \\ \delta_2, & c = 1, \\ -\infty, & c > 1. \end{cases}$$

And we have the following conclusions:

(1) If $c < 1$.

(i) $ab \leqslant \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, then f_2 is increasing on $(0,1)$, with $f_2(0^+) = 1 - \rho$ and $f_2(1^-) = \delta_1$.

(ii) $\frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2 < ab < \frac{7}{4} - \frac{3c^2}{2} - s^2$, then there exists a number $r_5 = r_5(a,b,s) \in (0,1)$ such that f_2 is decreasing on $(0, r_5]$ and increasing on $[r_5, 1)$.

(iii) $ab \geqslant \frac{7}{4} - \frac{3c^2}{2} - s^2$, then there exists a number $r_6 = r_6(a,b,s) \in (0,1)$ such that f_2 is decreasing on $(0, r_6]$ and increasing on $[r_6, 1)$.

(2) If $c = 1$.

(i) $a(1-a) \leqslant 1/4 - s^2$, then f_2 is increasing on $(0,1)$, with $f_2(0^+) = 1 - \rho$ and $f_2(1^-) = \delta_2$.

(ii) $a(1-a) \geqslant 1/4 - s^2$, then f_2 is decreasing on $(0,1)$, with $f_2(0^+) = 1 - \rho$ and $f_2(1^-) = \delta_2$.

(3) If $c > 1$.

(i) $ab \leqslant \frac{7}{4} - \frac{3c^2}{2} - s^2$, then there exists a number $r_7 = r_7(a,b,s) \in (0,1)$ such that f_2 is increasing on $(0, r_7)$ and decreasing on $[r_7, 1)$.

(ii) $\frac{7}{4} - \frac{3c^2}{2} - s^2 < ab < \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, then there exists a number $r_8 = r_8(a,b,s) \in (0,1)$ such that f_2 is increasing on $(0, r_8]$ and decreasing on $[r_8, 1)$.

(iii) $ab \geqslant \frac{11}{16} - \frac{7c}{16} - \frac{c+3}{4}s^2$, then f_2 is decreasing on $(0,1)$, with $f_2(0^+) = 1 - \rho$ and $f_2(1^-) = -\infty$.

Proof. Let f_1 be as in Theorem 1. Then by (22), (23),

$$\frac{F'(r)}{F_s'(r)} = \frac{4ab}{(1-4s^2)C}\cdot\frac{F_+(r)}{F_{s+}(r)} = \frac{4ab}{(1-4s^2)C}\cdot f_1(r), \tag{34}$$

so that

$$f_2(r) = F(r) - \frac{4ab}{(1-4s^2)C}F_s(r)f_1(r), \tag{35}$$

$$f_2'(r) = -\frac{4ab}{(1-4s^2)C}F_s(r)f_1'(r). \tag{36}$$

Hence the monotonicity properties of f_2 follow from (36) and Theorem 1. By (35), we see that $f_2(0^+) = 1-\rho$.

Next we show the limiting value $f_2(1^-)$.

- $c<1$, then $C<1$. By (26), (27) and (34), $f_2(r)$ can be written as

$$f_2(r) = F(r) - (1-r)^{1-C}\frac{4ab}{C(1-4s^2)}\cdot\frac{F(C-a,C-b;C+1;r)}{F(1/2-s,1/2+s;2;r)}\cdot F_s(r). \tag{37}$$

Hence by (15), (17), (19) and (14), we obtain

$$\begin{aligned}f_2(1^-) =&\frac{\Gamma(C)\Gamma(1-C)}{\Gamma(C-a)\Gamma(C-b)} - \frac{\pi\Gamma(C)^2}{\Gamma(a)\Gamma(b)\cos(\pi s)B(1/2-s)}\cdot\\ &\lim_{r\to1}(1-r)^{1-C}\{R(1/2-s)-\log(1-r)+O[(1-r)\log(1-r)]\}.\end{aligned}$$

It follows from (6), (7) and (9) that

$$\begin{aligned}f_2(1^-) =&\frac{\Gamma(C)\Gamma(1-C)}{\Gamma(C-a)\Gamma(C-b)} - \frac{\Gamma(C)^2}{\Gamma(a)\Gamma(b)}\lim_{r\to1}(1-r)^{1-C}\cdot\\ &\{R(1/2-s)-\log(1-r)+O[(1-r)\log(1-r)]\}\\ =&\frac{\Gamma(C)\Gamma(1-C)}{\Gamma(C-a)\Gamma(C-b)} = \frac{\Gamma(C)\Gamma(1-C)}{\Gamma(C-a)\Gamma[1-(C-a)]} = \frac{\pi}{\sin(C\pi)}\cdot\frac{\sin[(C-a)\pi]}{\pi}\\ =&\frac{\sin[(C-a)\pi]}{\sin(C\pi)} = \frac{\sin(\frac{b-a+1}{2}\pi)}{\sin(\frac{a+b+1}{2})\pi} = \frac{\cos(\frac{a-b}{2}\pi)}{\cos(\frac{a+b}{2}\pi)} = \delta_1.\end{aligned}$$

- $c=1$. It follows from (9), (14) and (35) that

$$f_2(1^-) = \frac{\sin(\pi a)}{\pi}[R(a)-R(1/2-s)] = \delta_2.$$

- $c>1$. By (20) and (37), $f_2(r)$ can be written as

$$f_2(r) = \frac{1}{(1-r)^{C-1}}\left[F(C-a,C-b;C;r) - \frac{4ab}{C(1-4s^2)}\cdot\frac{F(C-a,C-b;C+1;r)}{F(1/2-s,1/2+s;2;r)}\cdot F_s(r)\right].$$

It follows from (9), (14), (15), (17) and (21) that

$$\begin{aligned}f_2(1^-) =&\frac{\Gamma(C)\Gamma(C-1)}{\Gamma(a)\Gamma(b)}\cdot\lim_{r\to1}\frac{1}{(1-r)^{C-1}}\cdot\\ &\{1-(C-1)[R(1/2-s)-\log(1-r)+O[(1-r)\log(1-r)]]\} = -\infty. \qquad \square\end{aligned}$$

4. Applications

Our main results in section 3 not only present properties of Gaussian hypergeometric functions, but also have applications in some of mathematics such as the properties of complete and generalized elliptic integrals and in the theory of Ramanujan's modular equations. In this section, utilizing Theorem 1 and Theorem 2, we shall also derive some results involving the complete elliptic integrals, the generalized elliptic integrals, the Grötzsch ring function, the generalized Grötzsch ring function, the solutions of Ramanujan's modular equation and Ramanujan's generalized modular equation.

Indeed, by (12), (13) and [3, Theorem 4.1(1)],

$$\frac{dF(a,1-a;1;r^2)}{dr} = \frac{2a(1-a)r}{r'^2}F(a,1-a;2;r^2)$$

and

$$\frac{2}{\pi}\frac{d\mathscr{K}_a(r)}{dr} = \frac{4(1-a)}{\pi r r'^2}\left[\mathscr{E}_a(r) - r'^2\mathscr{K}_a(r)\right],$$

yielding the following equality

$$F(a,1-a;2;r^2) = \frac{2}{\pi a}\frac{\mathscr{E}_a(r) - r'^2\mathscr{K}_a(r)}{r^2}. \tag{38}$$

In particular, for $a = 1/2$ and $r \in (0,1)$,

$$F\left(\frac{1}{2},\frac{1}{2};2;r^2\right) = \frac{4}{\pi}\frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{r^2}. \tag{39}$$

COROLLARY 1. *For each $a \in (0,1/2]$, define the functions g_1, g_2 and g_3 on $(0,1)$ by*

$$g_1(r) = \frac{\mathscr{E}_a(r) - r'^2\mathscr{K}_a(r)}{\mathscr{E}(r) - r'^2\mathscr{K}(r)},$$

$$g_2(r) = \mathscr{K}_a(r) - 2(1-a)\frac{\mathscr{E}_a(r) - r'^2\mathscr{K}_a(r)}{\mathscr{E}(r) - r'^2\mathscr{K}(r)}\mathscr{K}(r),$$

$$g_3(r) = \frac{\mathscr{K}_a(r)}{\mathscr{K}(r)},$$

respectively. Then we have the following conclusions:

(1) g_1 is decreasing from $(0,1)$ onto $(\sin(\pi a)/[2(1-a)], 2a)$. In particular, for $a \in (0,1/2]$ and $r \in (0,1)$,

$$\frac{\sin(\pi a)}{2(1-a)}\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right] \leqslant \mathscr{E}_a(r) - r'^2\mathscr{K}_a(r) \leqslant 2a\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]. \tag{40}$$

(2) g_2 is increasing from $(0,1)$ onto $(\pi(1-2a)^2/2, \sin(\pi a)[R(a) - \log 16]/2)$. In particular, for $a \in (0,1/2]$ and $r \in (0,1)$,

$$\frac{\pi(1-2a)^2}{2} \leqslant \mathscr{K}_a(r) - 2(1-a)\frac{\mathscr{E}_a(r) - r'^2\mathscr{K}_a(r)}{\mathscr{E}(r) - r'^2\mathscr{K}(r)}\mathscr{K}(r) \leqslant \frac{\sin(\pi a)}{2}\left[R(a) - \log 16\right]. \tag{41}$$

(3) g_3 is decreasing from $(0,1)$ onto $(\sin(\pi a),1)$. In particular, for $a \in (0,1/2]$ and $r \in (0,1)$,

$$\sin(\pi a)\mathscr{K}(r) \leqslant \mathscr{K}_a(r) \leqslant \mathscr{K}(r). \tag{42}$$

Proof. (1) Let f_1 be as in Theorem 1. Then by (26), (27), (38) and (39), $g_1(r)$ can be written as

$$g_1(r) = 2a\frac{F(a,1-a;2;r^2)}{F(\frac{1}{2},\frac{1}{2};2;r^2)} = 2af_1(r^2)\Big|_{\substack{c=1\\ s=0}},$$

which is decreasing on $(0,1)$ by Theorem 1(2)(i), with

$$g_1(0^+) = 2af_1(0^+)\Big|_{\substack{c=1\\ s=0}} = 2a, \text{ and } g_1(1^-) = 2af_1(1^-)\Big|_{\substack{c=1\\ s=0}} = \frac{\sin(\pi a)}{2(1-a)}.$$

(2) Let f_2 be as in Theorem 2. Then by (3), (38) and (39), $g_2(r)$ can be written as

$$g_2(r) = \frac{\pi}{2}f_2(r^2)\Big|_{\substack{c=1\\ s=0}},$$

which is increasing on $(0,1)$ by Theorem 3.2(2)(i),with

$$g_2(0^+) = \frac{\pi}{2}f_2(0^+)\Big|_{\substack{c=1\\ s=0}} = \frac{\pi(1-2a)^2}{2}, \text{ and}$$
$$g_2(1^-) = \frac{\pi}{2}f_2(1^-)\Big|_{\substack{c=1\\ s=0}} = \frac{\sin(\pi a)[R(a)-\log 16]}{2}.$$

(3) The limiting value $g_3(0^+) = 1$ is clear. By [3, Theorem 4.1(1)] and [4, 3.6], we can obtain

$$\frac{rr'^2\mathscr{K}^2}{\mathscr{E}(r)-r'^2\mathscr{K}(r)}\frac{d(\mathscr{K}_a/\mathscr{K})}{dr} = -g_2(r), \tag{43}$$

which is negative by the part (2). Hence the monotoneity of g_3 is obtained. By l'Hôpital's Rule (cf. [4, Theorem 1.25]) and [3, Theorem 4.1(1)], [4, 3.6], we have

$$\lim_{r\to 1^-} g_3(r) = 2(1-a)g_1(1^-) = \sin(\pi a).$$

The inequalities (40), (41) and (42) hold by the monotonicities of g_1, g_2 and g_3, the equality cases of these inequalities are obvious. □

COROLLARY 2. *For each $r \in (0,1), |s| < 1/2$, the function*

$$g_4(r) = \frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{r^2F\left(\frac{1}{2}-s,\frac{1}{2}+s;2;r^2\right)}$$

is increasing from $(0,1)$ *onto* $(\pi/4,[(1-4s^2)\pi]/[4\cos(\pi s)])$. *In particular, for* $|s| < 1/2$ *and* $r \in (0,1)$,

$$\frac{\pi r^2}{4}F\left(\frac{1}{2}-s,\frac{1}{2}+s;2;r^2\right) \leqslant \mathscr{E}(r)-r'^2\mathscr{K}(r) \leqslant \frac{(1-4s^2)\pi r^2}{4\cos(\pi s)}F\left(\frac{1}{2}-s,\frac{1}{2}+s;2;r^2\right). \tag{44}$$

Proof. Let f_1 be as in Theorem 1. Then by (39), $g_4(r)$ can be written as

$$g_4(r) = \frac{\pi}{4}f_1(r^2)\Big|_{\substack{c=1\\a=b=1/2}},$$

which is increasing on $(0,1)$ by Theorem 1(2)(ii), with

$$g_4(0^+) = \frac{\pi}{4}f_1(0^+)\Big|_{\substack{c=1\\a=b=1/2}} = \frac{\pi}{4}, \text{ and } g_4(1^-) = \frac{\pi}{4}f_1(1^-)\Big|_{\substack{c=1\\a=b=1/2}} = \frac{(1-4s^2)\pi}{4\cos(\pi s)}.$$

The double inequaltiy (44) and its equality case are clear. □

Second, our main results have applications in the theory of Ramanujan's modular equations. Recall that for $r \in (0,1)$, the Grötzsch ring function μ is defined as

$$\mu(r) = \frac{\pi}{2}\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}, \tag{45}$$

which is the conformal modulus of the Grötzsch ring $\mathbb{B}^2 \setminus (0,r)$ and plays an essential role in the theory of quasiconformal mappings (cf. [1, 4, 10]), and the generalized Grötzsch ring function is defined by

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)}\frac{\mathscr{K}_a'(r)}{\mathscr{K}_a(r)} \tag{46}$$

for $a \in (0,1/2]$ and $r \in (0,1)$, and it is decreasing from $(0,1)$ onto $(0,\infty)$ (cf. [3]). Clearly, $\mu_{1/2}(r) = \mu(r)$. Ramanujan's generalized modular equation with signature $1/a$ and order (or degree) p is as follows

$$\frac{F(a,1-a;1;1-v^2)}{F(a,1-a;1;v^2)} = p\frac{F(a,1-a;1;1-r^2)}{F(a,1-a;1;r^2)}. \tag{47}$$

By (46), (47) and its solution v can be rewritten as

$$\mu_a(v) = p\mu_a(r) \text{ and } \varphi_K(a,r) = \mu_a^{-1}(\mu_a(r)/K)(K=1/p), \tag{48}$$

respectively. Clearly, as a function of r, $\varphi_K(a,r)$ is an increasing homeomorphism from $[0,1]$ onto itself. In the case when $a = 1/2$, (47) is called Ramanujan's classical modular equation, and $\varphi_K(r) = \varphi_K(1/2,r)$ is exactly the well-known Hersch-Pfluger distortion function in the theory of quasiconformal mappings (cf. [3, 4, 8, 17]).

With the help of the above preliminary results, the following corollary gives relations between the generalized Grötzsch ring function $\mu_a(r)$, the solution of Ramanujan's generalized modular equation (47) $\varphi_K(a,r)$ and their corresponding classical forms $\mu(r)$, $\varphi_K(r)$ for $a = 1/2$.

COROLLARY 3. *For each $a\in(0,1/2]$, $r\in(0,1)$ and $K\in(1,\infty)$, define the functions g_5, g_6 and g_7 on $(0,1)$ by*

$$g_5(r)=\frac{\mu_a(r)}{\mu(r)},\quad g_6(r)=\frac{\varphi_K(a,r)}{\varphi_K(r)},\quad g_7(r)=\frac{\varphi_{1/K}(a,r)}{\varphi_{1/K}(r)},$$

respectively. Then $g_5(r)$, $g_7(r)$ is increasing and $g_6(r)$ is decreasing on $(0,1)$, with

$$g_5(0^+)=g_6(1^-)=g_7(1^-)=1,\quad g_5(1^-)=1/\sin^2(\pi a),$$
$$g_6(0^+)=(e^{R(a)/2}/4)^{1-1/K},\quad g_7(0^+)=(e^{R(a)/2}/4)^{1-K}.$$

In particular, for $a\in(0,1/2]$, $r\in(0,1)$ and $K\in(1,\infty)$,

$$\mu(r)\leqslant\mu_a(r)\leqslant\mu(r)/\sin^2(\pi a), \tag{49}$$

$$\varphi_K(r)\leqslant\varphi_K(a,r)\leqslant(e^{R(a)/2}/4)^{1-1/K}\varphi_K(r), \tag{50}$$

$$(e^{R(a)/2}/4)^{1-K}\varphi_{1/K}(r)\leqslant\varphi_{1/K}(a,r)\leqslant\varphi_{1/K}(r). \tag{51}$$

Proof. By [3, Theorem4.1(5)], [4, (5.9)], we have

$$\frac{d\mu_a(r)/dr}{d\mu(r)/dr}=\left(\frac{\mathscr{K}(r)}{\mathscr{K}_a(r)}\right)^2=\left(\frac{1}{g_3(r)}\right)^2. \tag{52}$$

So g_5 is increasing by l'Hôpital's Rule (cf. [4, Theorem 1.25]) and Corollary 1 (3).

For $K>1$, let $u=\varphi_K(r)$, $v=\varphi_K(a,r)$, by (48),

$$u=\varphi_K(r)\Rightarrow\mu(u)=\mu(r)/K<\mu(r)\Rightarrow u>r.$$

It follows from the monotonicity of g_5 that

$$g_5(u)\geqslant g_5(r)\Rightarrow\mu_a(u)\geqslant\mu_a(r)\mu(u)/\mu(r)=\mu_a(r)/K\Rightarrow u\leqslant\mu_a^{-1}(\mu_a(r)/K)=v.$$

By [3, Theorem 4.1(7)], differentiation gives

$$\frac{Kurr'^2\mathscr{K}_a(r)^2}{vu'^2\mathscr{K}_a(u)^2}g_6'(r)=\left[\frac{v'^2\mathscr{K}_a(v)^2}{u'^2\mathscr{K}_a(u)^2}-1\right]+\left[1-\frac{g_3^2(r)}{g_3^2(u)}\right],$$

which shows that $g_6'(r)<0$ by [3, Lemma 5.4(1)] and Corollary 1(3).

Similarly, the monotonicity of function g_7 can be obtained.

By l'Hôpital's Rule (cf. [4, Theorem 1.25]), the limits $g_5(0^+)$ and $g_5(1^-)$ follow from (52) and Corollary 1(3). The limits $g_6(1^-)$ and $g_7(1^-)$ are trivial. The limits $g_6(0^+)$ and $g_7(0^+)$ follow from [3, Theorem 6.7], [4, Theorem 10.9(1)].

The inequalities (49), (50) and (51) hold by the monotonicities of g_5, g_6 and g_7, the equality cases of these inequalities are obvious. □

REMARK 1. Corollary 1(1), (3) and Corollary 3 have been proved in [11, Theorem 2.11, Corollary 2.12].

Acknowledgements. This research is supported by Shaoxing Basic Public Welfare Plan Project (2024A11021).

Conflict of interest. The authors declare that they have no conflict of interest.

REFERENCES

[1] L. V. AHLFORS, *Lectures on Quasiconformal Mappings*, (with additional chapters by C. J. Earle, I. Kra, et al., University Lecture Series, vol. 38, Amer. Math. Soc., Providence, R. I., 2006.

[2] G. D. ANDERSON, R. ASKEY, R. ROY, *Special functions*, Encyclopedia of Mathmatics and Its Applications **71** ,Cambridge Univ. Press, 1999.

[3] G. D. ANDERSON, S. L. QIU, M. K. VAMANAMURTHY, *Generalized elliptic integrals and modular equations*, Pacific J. Math. **192** (2000), 1–37.

[4] G. D. ANDERSON, M. K. VAMANAMURTHY, M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[5] M. ABRAMOWITZ, I. A. STEGUN (Eds), *Handbook of mathematical functions with formulas, Graphs, and Mathematical Tables*, New York, Dover, 1965.

[6] Á. BARICZ, *Landen Inequalities for special functions*, Proc. Amer. Math. Soc. **142** (2014), 3059–3066.

[7] J. M. BORWEIN, P. B. BORWEIN, *Pi and the AGM: A study in analytic number theory and computational complexity*, John Wiley & Sons, New York, 1987.

[8] B. C. BERNDT, S. BHARGAVA, F. G. GARVAN, *Ramanujan's theories of elliptic functions to alternative bases*, Trans. Am. Math. Soc. **347**, 11 (1995), 4163–4244.

[9] P. HASTO, S. PONNUSAMY, M. VUORINEN, *Starlikeness of the Gaussian hypergeometric functions*, Complex Var. Elliptic Equ. **55**, 1–3 (2010), 173–184.

[10] O. LEHTO AND K. I. VIRTANEN, *Quasiconformal Mappings in the Plane*, 2nd ed., Die Grundlehren der Math. Wiss. **126**, Springer-Verlag, New York-Berlin, 1973.

[11] X.-Y. MA, *Some properties of the generalized elliptic integrals and solutions of modular equation* (Excellent Master's Thesis in Zhejiang Province), Hangzhou Dianzi University for the Degree of Master, 2005.

[12] S. PONNUSAMY, M. VUORINEN, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44**, 2 (1997), 278–301.

[13] S. PONNUSAMY, M. VUORINEN, *Univalence and convexity properties for Gaussian hypergeometric functions*, Rocky Mountain J. Math. **31**, 1 (2001), 327–353.

[14] S.-L. QIU, X.-Y. MA, Q. BAO, *Monotonicity properties of generalized elliptic integrals with respect to the parameter*, J. Math. Anal. Appl. **492** (2020), Paper No. 124469, 1–31.

[15] S.-L. QIU, X.-Y. MA, Y.-M. CHU, *Extensions of quadratic transformation identities for hypergeometric functions*, Math. Inequal. Appl. **23**, 4 (2020), 1391–1423.

[16] S.-L. QIU, X.-Y. MA, Y.-M. CHU, *Sharp Landen transformation inequalities for hypergeometric functions, with applications*, J. Math. Anal. Appl. **474**, 2 (2019), 1306–1337.

[17] S.-L. QIU, M. VUORINEN, *Special functions in geometric function theory*, In book: Handbook of Complex Analysis: Special Function in Geometric Function Theory, Elsevier B.V., Amsterdam, **2** (2005), 621–659.

[18] L.-C. SHEN, *A note on Ramanujan's identities involving the hypergeometric function* ${}_2F_1(1/6,5/6;1;z)$, Ramanujan J. **30**, 2 (2013), 211–222.

[19] S. TAKEUCHI, *A new form of the generalized complete elliptic integrals*, Kodai Math. J. **39**, 1 (2016), 202–226.

[20] S. TAKEUCHI, *The complete p-elliptic integrals and a computation formula of* π_p *for* $p=4$, Ramanujan J. **46**, (2018), 309–321.

[21] M.-K. WANG, Y.-M. CHU, Y.-Q. SONG, *Ramanujan's cubic transformation and generalized modular equation*, Sci. China Math. **58**, 11 (2015), 2387–2404.

[22] M.-K. WANG, Y.-M. CHU, Y.-P. JIANG, *Ramanujan's cubic transformation and inequalities for zero-balanced hypergeometric functions*, Rocky Mountain J. Math. **46**, 2 (2016), 679–691.

[23] M.-K. WANG, Y.-M. CHU, Y.-P. JIANG, D. D. YAN, *A class of quadtatic transformation inequations for zero-balanced hypergeometric functions*, Acta Math. Sci. Ser. A (Chin. Ed.), **34 A**, 4 (2014), 999–1007.

[24] W. WANG, J.-J. PAN, S.-G. YANG, *Inequalities for Gaussian hypergeometric functions*, J. Math. Inequal. **11**, 1 (2017), 27–39.

(Received April 25, 2024)

Xiao-Yan Ma
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou 310018, Zhejiang, P.R. China
e-mail: mxy@zstu.edu.cn

Xing-Hua He
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou 310018, Zhejiang, P.R. China
e-mail: 1163897221@qq.com

Gen-Hong Zhong
Keyi College of Zhejiang Sci-Tech University
Shaoxing 312300, Zhejiang, P.R. China
e-mail: genhongz@126.com