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# GENERALIZED INTEGRATION OPERATORS FROM THE BESOV SPACE INTO GENERAL FUNCTION SPACES

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*Abstract.* The boundedness of the inclusion mapping from the Besov space $B_p$ into a class of the tent type space $\mathscr{T}_s^{p,n}(\mu)$ is studied. As an application, the boundedness, compactness and essential norm of the generalized integral operators $T_g^{n,k}$ and $S_g^{n,0}$ from the Besov space $B_p$ to general function spaces are also investigated.

## 1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be its boundary, $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. Let $H^\infty = H^\infty(\mathbb{D})$ denote the space of all bounded analytic functions with the supremum norm $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$. The Bloch space $\mathscr{B}$ is the class of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathscr{B}_0$, consists of all $f \in H(\mathbb{D})$ such that $\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0$. For some information on these spaces see, e.g., [46]. For $0 < p < \infty$, a function $f \in H(\mathbb{D})$ belongs to the Besov space $B_p$ if

$$\|f\|_{B_p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty.$$

When $p = 2$, it is just the Dirichlet space. If $0 < p < q < \infty$, we have that $B_p \subset B_q$ and the inclusion is proper. The Bloch space plays a very important role in geometric function theory. The Bloch space and analytic Besov spaces are invariant under Möbius transformations, which makes them useful in various applications. Indeed, the Bloch space represents the largest Banach space of analytic functions that maintains this Möbius invariance and $B_2$ is the only one nontrivial Möbius invariant Hilbert space of analytic functions in the unit disk.

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Let $0 < p, s < \infty$, $-2 < q < \infty$. The general function space $F(p,q,s)$, which was introduced by Zhao in [44], consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) < \infty.$$

Here $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. $F(p,q,s)$ is a Banach space under the norm $\|\cdot\|_{F(p,q,s)}$ when $p \geqslant 1$. Also, it is known that $F(p,q,s)$ contains only constant functions if $s+q \leqslant -1$. $F(p,p,0)$ is just the Bergman space. When $p=2$ and $q=0$, it gives the $Q_s$ space. Especially, $Q_1$ is the *BMOA* space, the space of analytic functions in the Hardy space whose boundary functions have bounded mean oscillation.

For $I \subset \partial\mathbb{D}$, let $S(I)$ denote the Carleson box based on the arc $I$, i.e.,

$$S(I) = \left\{ z \in \mathbb{D} : 1-|I| \leqslant |z| < 1 \ \text{ and } \ \frac{z}{|z|} \in I \right\}.$$

If $I = \partial\mathbb{D}$, then $S(I) = \mathbb{D}$. Let $0 < s < \infty$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. The measure $\mu$ is called an $s$-Carleson measure if

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

When $s=1$, it gives the classical Carleson measure. Carleson measure is a significant concept in complex analysis and function theory, particularly in the study of Hardy spaces on the unit disk. Understanding Carleson measures helps in determining when certain operators (like composition operators) are bounded on some analytic function spaces.

Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and nonnegative integers, respectively. Let $0 < p, s < \infty$, $n \in \mathbb{N}_0$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. We say that an $f \in H(\mathbb{D})$ belongs to $\mathscr{T}_s^{p,n}(\mu)$ if (see [27])

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} \left| f^{(n)}(z)(1-|z|^2)^n \right|^p d\mu(z) < \infty.$$

For $g \in H(\mathbb{D})$, let

$$I_g f(z) = \int_0^z g(w) f'(w) dw \quad \text{and} \quad T_g f(z) = \int_0^z g'(w) f(w) dw, \quad f \in H(\mathbb{D}).$$

These operators and their $n$-dimensional counterparts have been investigated a lot (see, e.g., [1, 2, 3, 5, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 30, 31, 32, 33, 34, 45, 47] and the references therein). In 2008 several authors started studying products of these integral type operators with some other concrete operators (see, e.g., [13, 35, 36, 37, 38, 39, 40, 41, 43] and the related references therein). This motivated many authors to study other extensions of the operators.

One of them, denoted by $T_g^{n,k}$, was introduced in [6]. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $0 \leqslant k < n$ and $g \in H(\mathbb{D})$. Then the operator is defined by

$$T_g^{n,k} f(z) = I^n(f^{(k)}(z) g^{(n-k)}(z)), \quad f \in H(\mathbb{D}).$$

Here, $I^n$ is the $n$-th iteration of the integration operator $If(z) = \int_0^z f(t)dt$.

In [27], Qian and the first author of this paper introduced the following operator

$$S_g^{n,k} f(z) = I^n(f^{(n-k)}(z)g^{(k)}(z)) \quad f \in H(\mathbb{D}).$$

Note that $T_g^{1,0} f = T_g f$ and $S_g^{1,0} f = I_g f$.

Since $T_g^{n,k} = S_g^{n,n-k}$, $k \neq 0$, we will only consider the operator $S_g^{n,0}$. It is easy to see that

$$M_g f(z) = f(0)g(0) + I_g f(z) + T_g f(z).$$

Here $M_g$ is the multiplication operator, which is defined by $M_g f(z) = f(z)g(z)$.

In [6] was studied the boundedness of the operator $T_g^{n,k}$ on Hardy spaces $H^p$. For example, it was shown that $T_g^{n,k} : H^p \to H^p$ is bounded if and only if $g \in \mathscr{B}$ when $k \geqslant 1$. $T_g^{n,k} : H^p \to H^q$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} (1-|z|^2)^{\frac{1}{q}-\frac{1}{p}+n-k} |g^{(n-k)}(z)| < \infty$$

when $0 < p < q < \infty$. In [8], Du, Li and Qu studied the boundedness, compactness and essential norm of the operator $T_g^{n,k}$ on a class of weighted Bergman spaces induced by doubling weights. For more results on the operator $T_g^{n,k}$, see [6, 8, 27]. For some previous results on integral type operators from or to $F(p,q,s)$ spaces, see [16, 40, 41].

In this paper, we will prove that the inclusion mapping $Id : B_p \to \mathscr{T}_s^{p,n}(\mu)$ is bounded if and only if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\int_{S(I)} (1-|z|^2)^{pn} d\mu(z)}{|I|^{pn+s}} < \infty.$$

As an application, we study the boundedness of generalized integration operators $T_g^{n,k}$ and $S_g^{n,0}$ acting from $B_p$ to $F(p,p-2,s)$. Moreover, the essential norm and compactness of $T_g^{n,k}$ and $S_g^{n,0}$ acting from $B_p$ to $F(p,p-2,s)$ are also investigated.

Throughout this paper, we say that $f \lesssim g$ if there exists a constant $C$ such that $f \leqslant Cg$. The symbol $f \approx g$ means that $f \lesssim g \lesssim f$.

## 2. Boundedness of $Id : B_p \to \mathscr{T}_s^{p,n}(\mu)$

We will study the boundedness of $Id : B_p \to \mathscr{T}_s^{p,n}(\mu)$ in this section. For this purpose, some lemmas are given firstly.

LEMMA 1. [46, Theorem 5.4] *If $g \in H(\mathbb{D})$ and $m \geqslant 2$, then $g \in \mathscr{B}$ if and only if $f \in H^\infty$, where $f(w) = \left(1-|w|^2\right)^m g^{(m)}(w)$. Furthermore,*

$$|g(0)| + |g'(0)| + \cdots + |g^{(m-1)}(0)| + \sup_{w \in \mathbb{D}} (1-|w|^2)^m |g^{(m)}(w)| \approx \|g\|_{\mathscr{B}}.$$

LEMMA 2. [46, Theorem 4.28] *Suppose* $1 < p < \infty$ *and* $n \in \mathbb{N}$. *Then* $g \in B_p$ *if and only if*

$$\int_{\mathbb{D}} |g^{(n)}(w)|^p (1-|w|^2)^{pn-2} dA(w) < \infty.$$

LEMMA 3. *Let* $1 < p < \infty$ *and* $0 < s < \infty$. *Then*

$$\|h\|_{F(p,p-2,s)} \lesssim \|h\|_{B_p}.$$

*Proof.* The result follows from the following inequality.

$$\begin{aligned} &\sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} (1-|\sigma_b(w)|^2)^s dA(w) \\ &\leqslant \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} dA(w). \quad \square \end{aligned}$$

LEMMA 4. [29, Theorem 3.2] *Let* $0 < s < \infty$, $1 < p < \infty$ *and* $n \in \mathbb{N}$. *Then the following statements are equivalent.*

*(i)*

$$\Theta_1 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^s dA(z) < \infty.$$

*(ii)*

$$\Theta_2 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{p(n-1)+p-2} (1-|\sigma_a(z)|^2)^s dA(z) < \infty.$$

*(iii)*

$$\Theta_3 := \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f^{(n)}(z)|^p (1-|z|^2)^{pn-2+s} dA(z) < \infty.$$

*Moreover, when* $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$,

$$\|f\|^p_{F(p,p-2,s)} = \Theta_1 \approx \Theta_2 \approx \Theta_3.$$

The proof of the following result is standard. See, for example [27, Lemma 2.4]. We omit the proof here.

LEMMA 5. *Let* $0 < s < \infty$, $1 \leqslant p < \infty$ *and* $n \in \mathbb{N}$. *If* $f \in F(p,p-2,s)$, *then*

$$|f^{(n)}(z)| \lesssim \frac{\|f\|_{F(p,p-2,s)}}{(1-|z|^2)^n}, \quad z \in \mathbb{D}.$$

Now we are in a position to state and prove our main result in this section.

THEOREM 1. *Suppose that* $\mu$ *is a positive Borel measure on* $\mathbb{D}$. *Let* $1 < p < \infty$, $0 < s \leqslant 2$, $m \in \mathbb{N}$. *Then the identity mapping* $Id : B_p \to \mathscr{T}_s^{p,m}(\mu)$ *is bounded if and only if*

$$\sup_{I\subset\partial\mathbb{D}} \frac{\int_{S(I)} (1-|w|^2)^{pm} d\mu(w)}{|I|^{pm+s}} < \infty. \tag{1}$$

*Proof.* First suppose that $Id : B_p \to \mathscr{T}_s^{p,m}(\mu)$ is bounded. Taking

$$f(w) = w^m \in B_p,$$

we obtain

$$\int_{\mathbb{D}} (1-|w|^2)^{pm} d\mu(w) < \infty.$$

Let $I \subset \partial\mathbb{D}$. Set $a = (1-|I|)\xi$. Here $\xi$ is the center point of $I$. Take

$$f_a(w) = \frac{(1-|a|^2)}{\overline{a}^m(1-\overline{a}w)}, \quad w \in \mathbb{D}.$$

By Lemma 3.10 of [46], we see that $f_a \in B_p$. Moreover, $\sup_{a\in\mathbb{D}} \|f_a\|_{B_p} < \infty$. From [10, p. 232], we see that

$$|1-\overline{a}w| \approx 1-|a|^2 \approx |I|, \quad w \in S(I). \tag{2}$$

It is easy to check that

$$|f_a^{(m)}(w)|^p \approx |I|^{-pm}, \quad w \in S(I).$$

From the assumption that $Id : B_p \to \mathscr{T}_s^{p,m}(\mu)$ is bounded, we get

$$\frac{1}{|I|^s} \int_{S(I)} |f_a^{(m)}(w)|^p (1-|w|^2)^{pm} d\mu(w) \lesssim \|f_a\|_{B_p}^p < \infty,$$

which implies

$$\sup_{I\subset\partial\mathbb{D}} \frac{\int_{S(I)} (1-|w|^2)^{pm} d\mu(w)}{|I|^{pm+s}} < \infty.$$

Conversely, assume that (1) holds. Set $d\nu(w) = (1-|w|^2)^{pm} d\mu(w)$. Then

$$\sup_{I\subset\partial\mathbb{D}} \frac{\nu(S(I))}{|I|^{pm+s}} < \infty,$$

which combing with [46, Theorem 7.4] imply that $Id : A_{pm-2+s}^p \to L^p(d\nu)$ is bounded. Let $f \in B_p$. By Lemma 2,

$$f^{(m)} \in A_{pm-2}^p \subseteq A_{pm+s-2}^p.$$

Set $l > 2s$. By Lemma 3 and (2), for any $I \subset \partial\mathbb{D}$,

$$\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)} |f^{(m)}(w)|^p(1-|w|^2)^{pm}d\mu(w)\\
\approx&(1-|a|^2)^{l-s}\int_{S(I)}\left|\frac{f^{(m)}(w)}{(1-\overline{a}w)^{l/p}}\right|^p d\nu(w)\\
\lesssim&(1-|a|^2)^{l-s}\int_{\mathbb{D}}\frac{|f^{(m)}(w)|^p}{|1-\overline{a}w|^l}(1-|w|^2)^{mp+s-2}dA(w)\\
\leqslant&\int_{\mathbb{D}}|f^{(m)}(w)|^p(1-|w|^2)^{pm-2}\frac{(1-|w|^2)^s(1-|a|^2)^s}{|1-\overline{a}w|^{2s}}dA(w)\\
\lesssim&\|f\|^p_{F(p,p-2,s)}\\
\lesssim&\|f\|^p_{B_p},
\end{aligned}$$

which implies the desired result. □

## 3. Boundedness of $T_g^{n,k}$ and $S_g^{n,0}$

In the present section, we give some characterizations for the boundedness of $T_g^{n,k}$ and $S_g^{n,0}$ from $B_p$ to $F(p,p-2,s)$.

THEOREM 2. *Let $1<p<\infty$, $0<s<2$, and let $k,n\in\mathbb{N}$ such that $0\leqslant k<n$. Suppose $g\in H(\mathbb{D})$ with $g(0)=g'(0)=\cdots=g^{(n-k-1)}(0)=0$. Then $T_g^{n,k}: B_p \to F(p,p-2,s)$ is bounded if and only if $g\in\mathscr{B}$. Moreover,*

$$\|T_g^{n,k}\|_{B_p\to F(p,p-2,s)} \approx \|g\|_{\mathscr{B}}. \tag{3}$$

*Proof.* Suppose that $T_g^{n,k}: B_p \to F(p,p-2,s)$ is bounded. For $a\in\mathbb{D}$, set

$$G_a(w) = \frac{1-|a|^2}{1-\overline{a}w}, \quad w\in\mathbb{D}.$$

It is easy to check that $G_a\in B_p$ by Lemma 3.10 of [46] and $\sup_{a\in\mathbb{D}}\|G_a\|_{B_p}\lesssim 1$. Moreover,

$$G_a^{(k)}(w) = \frac{k!\overline{a}^k(1-|a|^2)}{(1-\overline{a}w)^{1+k}}, \quad G_a^{(k)}(a) = \frac{k!\overline{a}^k}{(1-|a|^2)^k}.$$

By Lemma 5 we have

$$\frac{\|T_g^{n,k}G_a\|_{F(p,p-2,s)}}{(1-|a|^2)^n} \gtrsim |(T_g^{n,k}G_a)^{(n)}(a)| \gtrsim \frac{|g^{(n-k)}(a)||a|^k}{(1-|a|^2)^k}, \tag{4}$$

which implies that

$$\sup_{|a|>1/2}(1-|a|^2)^{n-k}|g^{(n-k)}(a)| \lesssim \|T_g^{n,k}G_a\|_{F(p,p-2,s)} \lesssim \|T_g^{n,k}\|_{B_p\to F(p,p-2,s)} < \infty. \tag{5}$$

Set $f(z) = z^k$. Then by the boundedness of $T_g^{n,k} : B_p \to F(p, p-2, s)$ and the fact that $f \in B_p$, we see that $T_g^{n,k} f \in F(p, p-2, s)$. By Lemma 5,

$$|f^{(k)}(z)g^{(n-k)}(z)| = |(T_g^{n,k} f)^{(n)}(z)| \lesssim \frac{\|T_g^{n,k} f\|_{F(p,p-2,s)}}{(1-|z|^2)^n} \lesssim \frac{\|T_g^{n,k}\|_{B_p \to F(p,p-2,s)}}{(1-|z|^2)^n}, \quad z \in \mathbb{D},$$

from which it easily follows that

$$\sup_{|a| \leqslant 1/2} (1-|a|^2)^{n-k} |g^{(n-k)}(a)| \lesssim \|T_g^{n,k}\|_{B_p \to F(p,p-2,s)} < \infty. \tag{6}$$

By (5) and (6), we see that $g \in \mathscr{B}$ by Lemma 1. Moreover, since $g(0) = g'(0) = \cdots = g^{(n-k-1)}(0) = 0$, we have

$$\|g\|_{\mathscr{B}} \approx \sup_{a \in \mathbb{D}} (1-|a|^2)^{n-k} |g^{(n-k)}(a)| \lesssim \|T_g^{n,k}\|_{B_p \to F(p,p-2,s)}. \tag{7}$$

Conversely, assume that $g \in \mathscr{B}$. Since

$$(T_g^{n,k} f)^{(m)}(0) = 0, \quad \text{when} \quad m = 0, 1, \ldots, n-1,$$

from Lemmas 3 and 4 we obtain

$$\begin{aligned}
\|T_g^{n,k} f\|_{F(p,p-2,s)}^p &\approx \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |(T_g^{n,k} f)^{(n)}(w)|^p (1-|w|^2)^{pn-2+s} dA(w) \\
&= \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f^{(k)}(w) g^{(n-k)}(w)|^p (1-|w|^2)^{pn-2+s} dA(w) \\
&\lesssim \|g\|_{\mathscr{B}}^p \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f^{(k)}(w)|^p (1-|w|^2)^{pk-2+s} dA(w) \\
&\lesssim \|f\|_{F(p,p-2,s)}^p \|g\|_{\mathscr{B}}^p \\
&\lesssim \|f\|_{B_p}^p \|g\|_{\mathscr{B}}^p,
\end{aligned} \tag{8}$$

which implies that $T_g^{n,k} : B_p \to F(p, p-2, s)$ is bounded and

$$\|T_g^{n,k}\|_{B_p \to F(p,p-2,s)} \lesssim \|g\|_{\mathscr{B}}. \tag{9}$$

From (7) and (9), we get (3). $\square$

THEOREM 3. *Let* $1 < p < \infty$, $0 < s < 2$, *and let* $n \in \mathbb{N}$. *Suppose* $g \in H(\mathbb{D})$. *Then* $S_g^{n,0} : B_p \to F(p, p-2, s)$ *is bounded if and only if*

$$g \in H^\infty.$$

*Moreover,*

$$\|S_g^{n,0}\|_{B_p \to F(p,p-2,s)} \approx \|g\|_{H^\infty}. \tag{10}$$

*Proof.* We first suppose that $g \in H^\infty$. Let $h \in B_p$. Since

$$(S_g^{n,0} f)^{(m)}(0) = 0, \quad \text{when} \quad m = 0, 1, \ldots, n-1,$$

by Lemmas 3 and 4 we obtain

$$\begin{aligned}
\|S_g^{n,0} h\|_{F(p,p-2,s)}^p &\approx \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} (1-|w|^2)^{pn-2+s} |(S_g^{n,0} h)^{(n)}(w)|^p dA(w) \\
&= \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} (1-|w|^2)^{pn-2+s} |h^{(n)}(w) g(w)|^p dA(w) \\
&\lesssim \|g\|_{H^\infty}^p \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} (1-|w|^2)^{pn-2+s} |h^{(n)}(w)|^p dA(w) \\
&\lesssim \|h\|_{F(p,p-2,s)}^p \|g\|_{H^\infty}^p \\
&\lesssim \|h\|_{B_p}^p \|g\|_{H^\infty}^p,
\end{aligned}$$

which implies that $S_g^{n,0} : B_p \to F(p,p-2,s)$ is bounded and

$$\|S_g^{n,0}\|_{B_p \to F(p,p-2,s)} \lesssim \|g\|_{H^\infty}. \tag{11}$$

Conversely, assume that $S_g^{n,0} : B_p \to F(p,p-2,s)$ is bounded. For $b \in \mathbb{D}$ and $r > 0$, let $\mathbb{D}(b,r)$ denote the Bergman metric disk centered at $b$ with radius $r$, i.e., $\mathbb{D}(b,r) = \{z \in \mathbb{D} : \beta(b,z) < r\}$. For any $a \in \mathbb{D} \setminus \{0\}$, let

$$h_a(w) = \frac{(1-|a|^2)}{\overline{a}^n (1-\overline{a}w)}, \quad w \in \mathbb{D}.$$

We have that $h_a \in B_p$ and $\sup_{a \in \mathbb{D}} \|h_a\|_{B_p} \lesssim 1$ by Lemma 3.10 of [46]. Moreover,

$$|h_a^{(n)}(w)|^p \approx (1-|w|)^{-np}, \ \ w \in \mathbb{D}(a,r).$$

Therefore, using the fact that $(S_g^{n,0} h_a)^{(m)}(0) = 0$ when $m = 0, 1, \cdots, n-1$, we get

$$\begin{aligned}
\infty > &\|S_g^{n,0}\|_{B_p \to F(p,p-2,s)} \gtrsim \|S_g^{n,0} h_a\|_{F(p,p-2,s)}^p \\
\approx &\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |(S_g^{n,0} h_a)^{(n)}(w)|^p (1-|w|^2)^{pn-2} (1-|\sigma_b(w)|^2)^s dA(w) \\
= &\sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |h_a^{(n)}(w) g(w)|^p (1-|w|^2)^{pn-2} (1-|\sigma_b(w)|^2)^s dA(w) \\
\gtrsim &\int_{\mathbb{D}(a,r)} |h_a^{(n)}(w) g(w)|^p (1-|w|^2)^{pn-2} (1-|\sigma_a(w)|^2)^s dA(w) \\
\gtrsim &\int_{\mathbb{D}(a,r)} |g(w)|^p (1-|w|^2)^{-2} dA(w) \\
\gtrsim &|g(a)|^p,
\end{aligned} \tag{12}$$

which implies that $g \in H^\infty$ and

$$\|S_g^{n,0}\|_{B_p \to F(p,p-2,s)} \lesssim \|g\|_{H^\infty}. \tag{13}$$

From (11) and (13), we see that (10) holds. □

## 4. Essential norm

In this section, we investigate the essential norm of $T_g^{n,k}$ and $S_g^{n,0}$ from $B_p$ into $F(p,p-2,s)$. Recall that the essential norm of $T: X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf_K \{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

Here $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces and $T: X \to Y$ is a bounded linear operator. It is easy to see that $T: X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. For some results on the essential norm of concrete linear operators see, e.g., [7, 32, 38, 42] and the related references therein.

Given that $p \in (1,\infty)$, it is well known (see [46, Theorem 5.24]) that the Besov space is reflexive. By applying [7, Lemma 2.1], we arrive at the subsequent lemma, which will be utilized in the proofs of the main results in this section.

LEMMA 6. *Suppose $1<p<\infty$, $0<s<2$ and $T: B_p \to F(p,p-2,s)$ is bounded. Then $T$ is compact if and only if $\|Tf_n\|_{F(p,p-2,s)} \to 0$ as $n \to \infty$ whenever $\{f_n\}$ is bounded in $B_p$ and uniformly converges to 0 on any compact subset of $\mathbb{D}$ as $n \to \infty$.*

The proof of the following result can be proved similarly as [27, Lemma 5.1]. We omit the proof here.

LEMMA 7. *Let $1<p<\infty$, $0<s<2$, $k,n \in \mathbb{N}$ such that $0 \leqslant k < n$. If $0<r<1$ and $g \in \mathscr{B}$, then $T_{g_r}^{n,k} : B_p \to F(p,p-2,s)$ is compact.*

For $0<r<1, z \in \mathbb{D}$ and $f \in \mathscr{B}$, set $f_r(z) = f(rz)$. Let $\operatorname{dist}_{\mathscr{B}}(f,\mathscr{B}_0)$ denote the distance from the Bloch function to the little Bloch space, that is

$$\operatorname{dist}_{\mathscr{B}}(f,\mathscr{B}_0) = \inf_{g\in\mathscr{B}_0} \|f-g\|_{\mathscr{B}}.$$

The following result can be found in [4].

LEMMA 8. *If $g \in \mathscr{B}$, then*

$$\limsup_{|z|\to 1^-} (1-|z|^2)|g'(z)| \approx \operatorname{dist}_{\mathscr{B}}(g,\mathscr{B}_0) \approx \limsup_{r\to 1^-} \|g-g_r\|_{\mathscr{B}}.$$

THEOREM 4. *Let $1<p<\infty$, $0<s<2$, and let $k,n \in \mathbb{N}$ such that $0 \leqslant k < n$. Suppose $g \in H(\mathbb{D})$ with $g(0) = g'(0) = \cdots = g^{(n-k-1)}(0) = 0$ such that $T_g^{n,k} : B_p \to F(p,p-2,s)$ is bounded. Then*

$$\|T_g^{n,k}\|_{e,B_p\to F(p,p-2,s)} \approx \limsup_{|w|\to 1^-} (1-|w|^2)|g'(w)| \approx \operatorname{dist}_{\mathscr{B}}(g,\mathscr{B}_0),$$

*Proof.* Assume that $\{z_t\}$ is a sequence in $\mathbb{D}$ such that $\lim_{t\to\infty}|z_t|=1$. For every $t$, set

$$G_{z_t}(w)=\frac{1-|z_t|^2}{1-\overline{z_t}w},\quad w\in\mathbb{D}.$$

Then $\{G_{z_t}\}$ is a bounded sequence in $B_p$ and $G_{z_t}\to 0$ uniformly on compact subset of $\mathbb{D}$ as $t\to\infty$. Suppose that $S:B_p\to F(p,p-2,s)$ is a compact operator. Then by Lemma 6 we get

$$\lim_{t\to\infty}\|SG_{z_t}\|_{F(p,p-2,s)}=0.$$

By (4),

$$\begin{aligned}
&\limsup_{t\to\infty}\|(T_g^{n,k}-S)G_{z_t}\|_{F(p,p-2,s)}\\
\gtrsim&\limsup_{t\to\infty}\|T_g^{n,k}G_{z_t}\|_{F(p,p-2,s)}-\limsup_{t\to\infty}\|SG_{z_t}\|_{F(p,p-2,s)}\\
\approx&\limsup_{t\to\infty}\|T_g^{n,k}G_{z_t}\|_{F(p,p-2,s)}\\
\gtrsim&\limsup_{t\to\infty}|g^{(n-k)}(z_t)|(1-|z_t|^2)^{n-k},
\end{aligned}$$

which implies that

$$\|T_g^{n,k}\|_{e,B_p\to F(p,p-2,s)}\gtrsim\limsup_{t\to\infty}|g^{(n-k)}(z_t)|(1-|z_t|^2)^{n-k}.$$

It follows from the arbitrariness of $\{z_t\}$ that

$$\begin{aligned}
\|T_g^{n,k}\|_{e,B_p\to F(p,p-2,s)}&\gtrsim\limsup_{|a|\to1^-}|g^{(n-k)}(a)|(1-|a|^2)^{n-k}\\
&\approx\limsup_{|a|\to1^-}(1-|a|^2)|g'(a)|\\
&\approx\operatorname{dist}_{\mathscr{B}}(g,\mathscr{B}_0).
\end{aligned}$$

On the other hand, for $0<t<1$, by Lemma 7 we see that $T_{g_t}^{n,k}:B_p\to F(p,p-2,s)$ is compact. From Theorem 2, we get

$$\|T_g^{n,k}\|_{e,B_p\to F(p,p-2,s)}\leqslant\|T_g^{n,k}-T_{g_t}^{n,k}\|=\|T_{g-g_t}^{n,k}\|\approx\|g-g_t\|_{\mathscr{B}}.$$

Then, we have

$$\|T_g^{n,k}\|_{e,B_p\to F(p,p-2,s)}\lesssim\limsup_{t\to1^-}\|g-g_t\|_{\mathscr{B}}\approx\operatorname{dist}_{\mathscr{B}}(g,\mathscr{B}_0).\quad\square$$

From Theorem 4, we get

COROLLARY 1. *Let $1<p<\infty$, $0<s<2$, and let $k,n\in\mathbb{N}$ such that $0\leqslant k<n$. Suppose $g\in H(\mathbb{D})$ with $g(0)=g'(0)=\cdots=g^{(n-k-1)}(0)=0$. Then $T_g^{n,k}:B_p\to F(p,p-2,s)$ is compact if and only if $g\in\mathscr{B}_0$.*

THEOREM 5. *Let* $1 < p < \infty$, $0 < s < 2$, *and let* $n \in \mathbb{N}$. *Suppose* $g \in H(\mathbb{D})$ *such that* $S_g^{n,0} : B_p \to F(p, p-2, s)$ *is bounded. Then*

$$\|S_g^{n,0}\|_{e,B_p \to F(p,p-2,s)} \approx \|g\|_{H^\infty}.$$

*Proof.* Assume that $\{z_t\}$ is a sequence in $\mathbb{D}$ such that $\lim_{t\to\infty} |z_t| = 1$. For every $t$, set

$$f_{z_t}(w) = \frac{(1-|z_t|^2)}{\overline{z_t}^n (1-\overline{z_t}w)}, \quad w \in \mathbb{D}.$$

Then $\{f_{z_t}\}$ is a bounded sequence in $B_p$ and $f_{z_t} \to 0$ uniformly on compact subset of $\mathbb{D}$ as $t \to \infty$. Since $S : B_p \to F(p, p-2, s)$ is compact, we obtain

$$\lim_{t\to\infty} \|Sf_{z_t}\|_{F(p,p-2,s)} = 0.$$

Hence,

$$\begin{aligned}
\|S_g^{n,0} - K\| &\gtrsim \limsup_{t\to\infty} \|(S_g^{n,0} - S)f_{z_t}\|_{F(p,p-2,s)} \\
&\gtrsim \limsup_{t\to\infty} \|S_g^{n,0} f_{z_t}\|_{F(p,p-2,s)} - \limsup_{t\to\infty} \|Sf_{z_t}\|_{F(p,p-2,s)} \\
&= \limsup_{t\to\infty} \|S_g^{n,0} f_{z_t}\|_{F(p,p-2,s)}.
\end{aligned}$$

Therefore, by (12),

$$\|S_g^{n,0}\|_{e,B_p\to F(p,p-2,s)} \gtrsim \limsup_{t\to\infty} \|S_g^{n,0} f_{z_t}\|_{F(p,p-2,s)} \gtrsim \limsup_{t\to\infty} |g(z_t)|,$$

which implies that

$$\|S_g^{n,0}\|_{e,B_p\to F(p,p-2,s)} \gtrsim \|g\|_{H^\infty}.$$

By Theorem 3 we have

$$\|S_g^{n,0}\|_{e,B_p\to F(p,p-2,s)} = \inf_K \|S_g^{n,0} - K\| \leqslant \|S_g^{n,0}\|_{B_p\to F(p,p-2,s)} \lesssim \|g\|_{H^\infty}.$$

Then the desired result follows. □

COROLLARY 2. *Let* $1 < p < \infty$, $0 < s < 2$, *and* $n \in \mathbb{N}$. *Suppose* $g \in H(\mathbb{D})$. *Then* $S_g^{n,0} : B_p \to F(p, p-2, s)$ *is compact if and only if* $g = 0$.

*Proof.* The result follows immediately from Theorem 5. □

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