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THE C–N–STAR, S–STAR AND C–MINUS PARTIAL ORDERS

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Abstract. In this paper, we derive several characterizations of the s-star partial order in terms of the core-nilpotent decomposition, and establish the conditions under which the s-star partial order implies the C-N-star partial order. By applying the core-EP decomposition, we introduce a new partial order, the c-minus partial order, which generalizes the core-minus partial order. Additionally, we provide several characterizations and properties of the c-minus partial order.

1. Introduction

In this paper, we use the following symbols. Let $\mathbb{C}^{m\times n}$ be the set of $m\times n$ complex matrices, A^*, $\mathscr{R}(A)$ and $\mathrm{rk}(A)$ denote the respective conjugate transpose, range (column space) and rank of $A\in\mathbb{C}^{m\times n}$, and I_n be the identity matrix of order n. For $A\in\mathbb{C}^{n\times n}$, the index of A is the smallest positive integer k such that $\mathrm{rk}(A^{k+1})=\mathrm{rk}(A^k)$, and is denoted by $\mathrm{Ind}(A)=k$. For A is a rectangular $m\times n$ matrix, if there exists a $X\in\mathbb{C}^{n\times m}$ satisfying the following four equations:

$$(1)\ \ AXA=A,\ \ (2)\ \ XAX=X,\ \ (3)\ \ (AX)^*=AX,\ \ (4)\ \ (XA)^*=XA,$$

then X is called the Moore-Penrose inverse of A, and denoted as $X=A^\dagger$. Especially, if $m=n=rk(A)$, we have $A^\dagger=A^{-1}$. If $AA^\dagger=A^\dagger A$, then A is EP [28]. It is well known that A is EP if and only if $\mathscr{R}(A)=\mathscr{R}(A^*)$, see [20]. The set of all EP matrices on $\mathbb{C}^{n\times n}$ is denoted as $\mathbb{C}_n^{\mathrm{EP}}$:

$$\mathbb{C}_n^{\mathrm{EP}}=\left\{A\mid\mathscr{R}(A)=\mathscr{R}(A^*),A\in\mathbb{C}^{n\times n}\right\}.$$

The i-EP matrix is an extension of the EP matrix. If A^k is EP and k is the index of A, then A is said to be i-EP. The set of all i-EP matrices on $\mathbb{C}^{n\times n}$ is denoted as $\mathbb{C}_n^{\mathrm{iE}}$:

$$\mathbb{C}_n^{\mathrm{iE}}=\left\{A\mid A^k\in\mathbb{C}_n^{\mathrm{EP}},\mathrm{Ind}(A)=k,A\in\mathbb{C}^{n\times n}\right\}.\qquad(1.1)$$

For further conclusions on the properties and characterization of EP matrices and i-EP matrices, see [10, 14, 23, 27, 31].

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Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$. If there exists a $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations:

$$(1^k)\ \ XA^{k+1} = A^k,\ \ (2)\ \ XAX = X,\ \ (5)\ \ AX = XA,$$

then X is called the Drazin inverse of A, and denoted as $X = A^D$. In particular, when $k = 1$, X is called the group inverse of A, and denoted as $A^{\#}$, see [28]. Furthermore, we denote

$$\mathbb{C}_n^{\mathrm{CM}} = \left\{A \mid \mathrm{Ind}(A) = 1, A \in \mathbb{C}^{n\times n}\right\}.$$

Manjunatha Prasad and Mohana [15] introduced the core-EP inverse and gave some characterizations and properties of the core-EP inverse. Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A)$ $= k$. If there exists a $X \in \mathbb{C}^{n\times n}$ satisfying the following conditions:

$$(1^k)\ \ XA^{k+1} = A^k,\ \ (2)\ \ XAX = X,\ \ (3)\ \ (AX)^* = AX,\ \ (6)\ \ \mathscr{R}(X) \subseteq \mathscr{R}\left(A^k\right),$$

then X is called the core-EP inverse of A, and denoted as $X = A^{\textcircled{\dagger}}$. In particular, when $k = 1$, X is called the core inverse of A, and denoted as $A^{\textcircled{\#}}$, see [2].

Generalized inverses are one of the main tools for studying the partial order of matrices. Recently, the theory of partial order and its applications have received widespread attention, [1, 3, 4, 5, 7, 8, 9, 13, 17, 18, 22, 26, 32, 33, 34, 35]. A partial order is a binary relation that satisfies reflexivity, transitivity, and antisymmetry. It is well known that the classical partial orders are the minus order "$\overset{-}{\leqslant}$", the star order "$\overset{*}{\leqslant}$" and the sharp order "$\overset{\#}{\leqslant}$", see [6, 11, 19]. Let $A, B \in \mathbb{C}^{n\times n}$, then

(1) $A \overset{-}{\leqslant} B \Leftrightarrow A^{-}A = A^{-}B,\ AA^{=} = BA^{=}$, for some $A^{-}, A^{=} \in A\{1\}$;

(2) $A \overset{*}{\leqslant} B \Leftrightarrow A^*A = A^*B,\ AA^* = BA^*$;

(3) $A \overset{\#}{\leqslant} B \Leftrightarrow A^{\#}A = A^{\#}B,\ AA^{\#} = BA^{\#},\ \ \mathrm{Ind}(A) = \mathrm{Ind}(B) = 1$.

Another major tool for studying partial order is matrix decomposition. Matrix decomposition is also a primary tool for studying generalized inverses of matrices. Here, we present two matrix decompositions, one of which is the core-nilpotent decomposition.

LEMMA 1.1. ([20]) *Let $A \in \mathbb{C}^{n\times n}$ and* $\mathrm{Ind}(A) = k$*, then A can be uniquely written as the sum of A_1 and A_2, i.e., $A = A_1 + A_2$, where*

(1) $\mathrm{Ind}(A_1) = 1$*;*

(2) $A_2^k = 0$*;*

(3) $A_1A_2 = A_2A_1 = 0$.

Furthermore, there exists an invertible matrix P *such that*

$$A_1 = P\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \quad A_2 = P\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}P^{-1}, \tag{1.2}$$

where T *is invertible,* N *is nilpotent and* $\mathrm{Ind}(N) = k$.

In the above decomposition, we say that A_1 is the core part of A, and A_2 is the nilpotent part of A. According to the decomposition, we can obtain

$$A^D = P\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}.$$

Especially, if $\mathrm{Ind}(A) = 1$, we have $N = 0$ and

$$A^{\#} = P\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}.$$

The other one is the core-EP decomposition.

LEMMA 1.2. ([29]) *Let* $A \in \mathbb{C}^{n\times n}$ *and* $\mathrm{Ind}(A) = k$*, then* A *can be uniquely written as the sum of* $\widehat{A}_1$ *and* $\widehat{A}_2$*, i.e.,* $A = \widehat{A}_1 + \widehat{A}_2$*, where*

(1) $\mathrm{Ind}\left(\widehat{A}_1\right) = 1$;

(2) $\widehat{A}_2^k = 0$;

(3) $\widehat{A}_1^*\widehat{A}_2 = \widehat{A}_2\widehat{A}_1 = 0$.

Furthermore, there exists a unitary matrix U *such that*

$$\widehat{A}_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \quad \widehat{A}_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{1.3}$$

where T *is invertible,* N *is nilpotent and* $\mathrm{Ind}(N) = k$.

According to the core-EP decomposition, we can obtain

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Especially, when $\mathrm{Ind}(A) = 1$, we have

$$A = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \tag{1.4}$$

and

$$A^{\textcircled{\#}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{1.5}$$

By applying the core inverse, Baksalary and Trenkler [2] introduced the core partial order on $\mathbb{C}_n^{\mathrm{CM}}$.

LEMMA 1.3. ([2]) *Let* $A,B\in\mathbb{C}_n^{\mathrm{CM}}$*, and let* A *be of the form as* (1.4)*. The following conditions are equivalent:*

(1) $A\overset{\textcircled{\#}}{\leqslant}B$;

(2) $B=U\begin{bmatrix}T & S\\ 0 & Z\end{bmatrix}U^*$, *where* $Z\in\mathbb{C}_{n-r}^{\mathrm{CM}}$;

(3) $A^\dagger A=A^\dagger B,\ A^2=BA$.

It has become common practice to construct partial orders using matrix decomposition. For example, Hauke and Markiewicz [12] introduced the GL partial order based on the polar decomposition.

Let $A,B\in\mathbb{C}^{n\times n}$, $A=\widehat{A}_1+\widehat{A}_2$ and $B=\widehat{B}_1+\widehat{B}_2$ be the core-EP decompositions of A and B, respectively. Wang [29] introduced the core-minus partial order:

$$A\overset{\mathrm{CM}}{\leqslant}B:\ \widehat{A}_1\overset{\textcircled{\#}}{\leqslant}\widehat{B}_1,\ \widehat{A}_2\overset{-}{\leqslant}\widehat{B}_2. \tag{1.6}$$

And let $A=A_1+A_2$ and $B=B_1+B_2$ be the core-nilpotent decompositions of A and B, respectively. Mitra and Hartwig [21] considered the C-N partial order:

$$A\overset{\#,-}{\leqslant}B:\ A_1\overset{\#}{\leqslant}B_1,\ A_2\overset{-}{\leqslant}B_2. \tag{1.7}$$

Mitra, Bhimasankaram and Malik [20] established the S-minus partial order:

$$A\overset{\ominus}{\leqslant}B:\ A\overset{-}{\leqslant}B,\ A_1\overset{\#}{\leqslant}B_1. \tag{1.8}$$

Based on (1.7) and (1.8), Mitra raised the open problem [20, Problem 16.3.1]: What are necessary and sufficient conditions under which the S-minus partial order implies the C-N partial order? Wang and Liu [30] studied the problem.

Furthermore, based on the core-nilpotent decomposition, Mitra, Bhimasankaram and Malik [20] introduced two new partial orders based on the star and sharp partial orders, which did not exist before. Let $A=A_1+A_2$ and $B=B_1+B_2$ be the core-nilpotent decompositions of A and B respectively. The forms of A_1 and B_1 are as shown in the first equation of (1.2). The first is the C-N-star partial order, and is denoted as "$\overset{\#,*}{\leqslant}$":

$$A\overset{\#,*}{\leqslant}B:\ A_1\overset{\#}{\leqslant}B_1,\ A_2\overset{*}{\leqslant}B_2,\ A_1,B_1\in\mathbb{C}_n^{\mathrm{EP}}. \tag{1.9}$$

The second is the s-star partial order, and is denoted as "$\overset{\circleddash}{\leqslant}$":

$$A\overset{\circleddash}{\leqslant}B:\ A\overset{*}{\leqslant}B,\ A_1\overset{\#}{\leqslant}B_1,\ A_1,B_1\in\mathbb{C}_n^{\mathrm{EP}}. \tag{1.10}$$

It is easy to see that $A\overset{\#,*}{\leqslant}B$ implies $A\overset{\circleddash}{\leqslant}B$. Obviously, both of these partial orders are the C-N partial orders. It follows that

$$A\overset{*}{\leqslant}B\Rightarrow A\overset{\#,*}{\leqslant}B\Rightarrow A\overset{\circleddash}{\leqslant}B\Rightarrow A\overset{\#,-}{\leqslant}B\Rightarrow A\overset{-}{\leqslant}B.$$

It should be pointed out that Marovt [16, 17] further discussed the characterizations and properties of these two partial orders. It is well known that the core-nilpotent decomposition is applied to study the sharp partial order, and the singular value decomposition is applied to study the star partial order. The C-N-star partial order and the s-star partial order are both generated by the combination of the sharp partial order and the star partial order. An interesting fact about the C-N-star (s-star) partial order is that constraint A_1 exists in the set $\mathbb{C}_n^{\mathrm{EP}}$. It follows that the two partial orders are established on a special set of matrices. So, what is this set? Furthermore, how can we establish a generalized partial order in the set $\mathbb{C}^{n\times n}$? These factors result in the C-N (S-minus) partial order and the C-N-star (s-star) partial order, although structurally similar, having different levels of difficulty.

Although $A \overset{\#,*}{\leqslant} B$ implies $A \overset{\circleddash}{\leqslant} B$, the reverse is not true, that is, $A \overset{\circleddash}{\leqslant} B$ does not imply $A \overset{\#,*}{\leqslant} B$. Therefore, Mitra, Bhimasankaram and Malik raised the open problem [20, Problem 16.3.2]. Let $A = A_1 + A_2$ be the core-nilpotent decomposition of A. The form of A_1 is the first equation of (1.2). Furthermore, let us denote

$$\mathfrak{C}^{n\times n} = \left\{A \mid A_1 \in \mathbb{C}_n^{\mathrm{EP}},\ A = A_1 + A_2 \text{ is the core-nilpotent decomposition of } A \in \mathbb{C}^{n\times n}\right\}. \tag{1.11}$$

PROBLEM 1.1. ([20, Problem 16.3.2]) What are the necessary and sufficient conditions under which $A \overset{\circleddash}{\leqslant} B$ implies $A \overset{\#,*}{\leqslant} B$?

Marovt studied Problem 1.1 by providing some new characterizations of C-N-star partial order in [16].

In this paper, we apply the core-nilpotent decomposition to study the s-star partial order, derive several characterizations of the s-star partial order, consider the above Problem 1.1 and get some new conditions under which $A \overset{\circleddash}{\leqslant} B$ implies $A \overset{\#,*}{\leqslant} B$. Based on the core partial order and the minus partial order, we introduce a new partial order called the c-minus partial order, get some characterizations of the partial order, and the relationships between the c-minus and core-minus partial orders.

The structure of the rest of the paper is as follows. In Section 2, we provide characterizations of the s-star partial order. In Section 3, we study the relationships between the C-N-star and s-star partial order. In Section 4, we present properties of the c-minus partial order. Finally, we conclude in Section 5.

2. Characterizations of the s-star partial order on $\mathbb{C}_n^{\mathrm{iE}}$

The EP matrix is a special matrix. In [24], Pearl gave it a nice characterization.

LEMMA 2.1. ([24]) *Let $A \in \mathbb{C}^{n\times n}$. Then A is EP if and only if there is a unitary matrix U such that*

$$A = U \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{2.1}$$

where T is invertible.

REMARK 2.1. ([20]) Let $A, B \in \mathbb{C}_n^{\mathrm{EP}}$. It is obvious that $A^{\#} = A^{\dagger}$ and $B^{\#} = B^{\dagger}$. Therefore, $A \overset{\#}{\leqslant} B$ if and only if $A \overset{*}{\leqslant} B$.

Wang and Liu gave a characterization of the i-EP matrix in [31].

LEMMA 2.2. ([31]) *Let $A \in \mathbb{C}^{n\times n}$. Then A is i-EP if and only if there is a unitary U such that*

$$A = U \begin{bmatrix} T & 0 \\ 0 & N \end{bmatrix} U^*, \tag{2.2}$$

where T is invertible, and N is nilpotent.

THEOREM 2.3. *Let A and B be i-EP matrices of the same order. Then $A \overset{*}{\leqslant} B$ if and only if there exists a unitary matrix U such that*

$$A = U \begin{bmatrix} T & 0 \\ 0 & N \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T & 0 \\ 0 & B_{14} \end{bmatrix} U^*, \tag{2.3}$$

where T is invertible, B_{14} is i-EP, N is nilpotent and $N \overset{*}{\leqslant} B_{14}$.

Proof. Since A is i-EP, then it is of the form (2.2). Let B be partitioned as the following form according to the block form of A:

$$B = U \begin{bmatrix} B_{11} & B_{12} \\ B_{13} & B_{14} \end{bmatrix} U^*. \tag{2.4}$$

Since $A \overset{*}{\leqslant} B$, we have $AA^* = BA^*$ and $A^*A = A^*B$. By applying (2.2) and (2.4), it follows that $B_{11} = T$, $B_{12} = 0$, $B_{13} = 0$, $NN^* = B_{14}N^*$ and $N^*N = N^*B_{14}$. Therefore, B is the form as in (2.3) and $N \overset{*}{\leqslant} B_{14}$. Since B is i-EP, then B_{14} is i-EP. Therefore, we get (2.3).

Conversely, let the forms of A and B be as in (2.3). It is easy to check that $A \overset{*}{\leqslant} B$. $\square$

LEMMA 2.4. ([20]) *Let $A, B \in \mathbb{C}^{m\times n}$ have the same block forms,*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_4 \end{bmatrix},$$

and A_4 and B_4 be of the same order. Then $A \overset{-}{\leqslant} B$ if and only if $A_1 = 0$, $A_2 = 0$, $A_3 = 0$ and $A_4 \overset{-}{\leqslant} B_4$.

LEMMA 2.5. *Let* $\mathbb{C}_n^{iE}$ *and* $\mathfrak{C}^{n\times n}$ *be as in* (1.1) *and* (1.11), *respectively. Then*

$$\mathbb{C}_n^{iE} = \mathfrak{C}^{n\times n}.$$

Proof. If $A \in \mathbb{C}_n^{\mathrm{iE}}$, applying (2.2) gives that the core part of A is EP, that is, $A \in \mathfrak{C}^{n\times n}$. Therefore, $\mathbb{C}_n^{\mathrm{iE}} \subseteq \mathfrak{C}^{n\times n}$.

If $A \in \mathfrak{C}^{n\times n}$, then A_1 is EP. Denote $\mathrm{rk}(A) = r$ and let $A = A_1 + A_2$ be the core-nilpotent decomposition of A. Then, applying Lemma 2.1 gives

$$A_1 = \widehat{U} \begin{bmatrix} \widehat{T} & 0 \\ 0 & 0 \end{bmatrix} \widehat{U}^*, \tag{2.5}$$

in which $\widehat{U}$ is unitary and $\widehat{T}$ is invertible. Furthermore, let A_2 be partitioned as

$$A_2 = \widehat{U} \begin{bmatrix} \widehat{X}_1 & \widehat{X}_2 \\ \widehat{X}_3 & \widehat{X}_4 \end{bmatrix} \widehat{U}^*, \tag{2.6}$$

in which $\widehat{X}_1 \in \mathbb{C}^{r\times r}$. Since $A = A_1 + A_2$ is the core-nilpotent decomposition of A, then $A_1A_2 = A_2A_1 = 0$. It follows from (2.5) and (2.6) that $\widehat{X}_1 = 0$, $\widehat{X}_2 = 0$, $\widehat{X}_3 = 0$ and $\widehat{X}_4$ is nilpotent. Therefore,

$$A = A_1 + A_2 = \widehat{U} \begin{bmatrix} \widehat{T} & 0 \\ 0 & \widehat{X}_4 \end{bmatrix} \widehat{U}^*. \tag{2.7}$$

Applying Lemma 2.2 gives that A is i-EP. Therefore, $\mathfrak{C}^{n\times n} \subseteq \mathbb{C}_n^{\mathrm{iE}}$.

In summary, we have $\mathbb{C}_n^{\mathrm{iE}}$ coincides with $\mathfrak{C}^{n\times n}$. $\square$

THEOREM 2.6. *Let* A *and* B *be i-EP matrices of the same order. Then* $A \overset{\ominus}{\leqslant} B$ *if and only if there exists a unitary matrix* U *such that*

$$A = U \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & N_2 \end{bmatrix} U^*, \tag{2.8}$$

where T *and* T_1 *are invertible,* N_{14} *and* N_2 *have the same order,* $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix}$ *and* N_2 *are nilpotent, and* $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \overset{*}{\leqslant} \begin{bmatrix} T_1 & 0 \\ 0 & N_2 \end{bmatrix}$.

Proof. Let $A \in \mathbb{C}_n^{\mathrm{iE}}$. Applying Lemma 2.2, we have the decomposition of A, $A = A_1 + A_2$, in which

$$A_1 = U_1 \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} U_1^*, \quad A_2 = U_1 \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U_1^*, \tag{2.9}$$

U_1 is unitary, T is invertible and N is nilpotent.

Since $A \overset{*}{\leqslant} B$, applying Theorem 2.3 gives that

$$B = U_1 \begin{bmatrix} T & 0 \\ 0 & B_{14} \end{bmatrix} U_1^*, \tag{2.10}$$

where $N \overset{*}{\leqslant} B_{14}$ and B_{14} is i-EP. Furthermore, applying Lemma 2.2, we have the core-EP decomposition of B_{14}

$$B_{14} = U_2 \begin{bmatrix} T_1 & 0 \\ 0 & N_2 \end{bmatrix} U_2^*, \tag{2.11}$$

in which U_2 is unitary, T_1 is invertible, and N_2 is nilpotent. Substituting (2.11) into (2.10), we have

$$B = U_1 \begin{bmatrix} T & 0 \\ 0 & U_2 \begin{bmatrix} T_1 & 0 \\ 0 & N_2 \end{bmatrix} U_2^* \end{bmatrix} U_1^*. \tag{2.12}$$

It follows from (2.12) that

$$B = U_1 \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & N_2 \end{bmatrix} \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 \\ 0 & U_2 \end{bmatrix}^* U_1^*. \tag{2.13}$$

Denote

$$U = U_1 \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 \\ 0 & U_2 \end{bmatrix}, \quad \begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} = U_2^* N U_2. \tag{2.14}$$

Applying (2.9), (2.13) and (2.14), we have (2.8).

Conversely, let A and B have the forms as in (2.8) and $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \overset{*}{\leqslant} \begin{bmatrix} T_1 & 0 \\ 0 & N_2 \end{bmatrix}$, $A = A_1 + A_2$ and $B = B_1 + B_2$ be the core-nilpotent decompositions of A and B, respectively. Then

$$A_1 = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} U^*, \tag{2.15}$$

$$B_1 = U \begin{bmatrix} T & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad B_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_2 \end{bmatrix} U^*. \tag{2.16}$$

Applying (2.15) and (2.16), we have $A_1 \overset{\#}{\leqslant} B_1$ and $A \overset{*}{\leqslant} B$. Therefore, $A \overset{\circleddash}{\leqslant} B$. □

THEOREM 2.7. *Let A and B be i-EP matrices of the same order, then $A \overset{\circleddash}{\leqslant} B$ if and only if $A \overset{*}{\leqslant} B$.*

Proof. Let $A, B \in \mathbb{C}_n^{\text{iE}}$. If $A \overset{\circleddash}{\leqslant} B$, from (1.10), it is obvious that $A \overset{*}{\leqslant} B$.

Conversely, if $A \overset{*}{\leqslant} B$, then A and B have the forms as in (2.3). Since B_{14} is i-EP, then there exists a unitary matrix U_1 such that $U_1 B_{14} U_1^*$ can be partitioned as

$$U_1 B_{14} U_1^* = \begin{bmatrix} T_1 & 0 \\ 0 & N_2 \end{bmatrix}.$$

Obviously, $U_1 N U_1^*$ is nilpotent. We write $U_1 N U_1^* = \begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix}$. It follows from Theorem 2.6 that $A \overset{\circleddash}{\leqslant} B$. □

Marovt[16] gave a characterization of the C-N-star partial order in proper $*$-rings. In particular, the set $\mathbb{C}^{n\times n}$ is one special case of proper $*$-ring.

LEMMA 2.8. ([16]) *Let A and B be i-EP matrices of the same order. Then $A \overset{\#,*}{\leqslant} B$ if and only if there exists a unitary matrix U such that*

$$A = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & N_2 \end{bmatrix} U^*, \tag{2.17}$$

where T and T_1 are invertible, N_{14} and N_2 are nilpotent of the same order, and $N_{14} \overset{*}{\leqslant} N_2$.

3. Relationships between the C-N-star and s-star partial orders on $\mathbb{C}_n^{\text{iE}}$

In this section, we consider the relationships between the C-N-star and the s-star partial orders.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\text{Ind}(A) = 2$, $\text{Ind}(B) = 1$. Then we get the core-nilpotent decompositions of A and B, $A = A_1 + A_2$ and $B = B_1 + B_2$, in which

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$AA^* = BA^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^*A = A^*B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_1 A_1^\# = B_1 A_1^\# = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1^\# A_1 = A_1^\# B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 A_2^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq B_2 A_2^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2^* A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq A_2^* B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It can be seen from the above equation that $A \overset{\textcircled{=}}{\leqslant} B$, but not $A \overset{\#,*}{\leqslant} B$. It follows that $A \overset{\textcircled{=}}{\leqslant} B$ does not imply $A \overset{\#,*}{\leqslant} B$. So, in what condition(s) does $A \overset{\textcircled{=}}{\leqslant} B \Rightarrow A \overset{\#,*}{\leqslant} B$? This is also Problem 1.1. Marovt discussed this problem on the ring and gave some conclusions in [16]. Here we present some new results.

THEOREM 3.1. *Let $A, B \in \mathbb{C}_n^{\mathrm{iE}}$, $k = \max\{\mathrm{Ind}(A), \mathrm{Ind}(B)\}$, $A = A_1 + A_2$ and $B = B_1 + B_2$ be the core-nilpotent decompositions of A and B, respectively. If $A \overset{\textcircled{=}}{\leqslant} B$, then the following conditions are equivalent:*

(1) $A \overset{\#,*}{\leqslant} B$;

(2) $BB^DA = ABB^D$, $B^DA = A^DA$;

(3) $BB^{\textcircled{\dagger}}A = ABB^{\textcircled{\dagger}}$, $B^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$.

Proof. $(1) \Rightarrow (2)$–(3): Let $A, B \in \mathbb{C}_n^{\mathrm{iE}}$ and $A \overset{\#,*}{\leqslant} B$, then the forms of A and B are as in (2.17). It follows that

$$A^D = U \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad B^D = U \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & T_1^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$B^DA = U \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A^DA = U \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$BB^DA = U \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & I_{\mathrm{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$ABB^D = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & I_{\mathrm{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad B^{\textcircled{\dagger}} = U \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & T_1^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$B^{\textcircled{\dagger}}A = U \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A^{\textcircled{\dagger}}A = U \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$BB^{\textcircled{\dagger}}A = U \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & I_{\mathrm{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$ABB^{\textcircled{\dagger}} = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} \begin{bmatrix} I_{\mathrm{rk}(T)} & 0 & 0 \\ 0 & I_{\mathrm{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*.$$

Applying the above results and (2.17) gives (2)–(3).

Next, let $A \overset{\circleddash}{\leqslant} B$ and $A, B \in \mathbb{C}_n^{\text{iE}}$. Then the forms of A and B are as in (2.8). It is easy to check that

$$B^k = U \begin{bmatrix} T^k & 0 & 0 \\ 0 & T_1^k & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*. \tag{3.1}$$

$(2) \Rightarrow (1)$: Applying (2.8), we have

$$BB^DA = U \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & I_{\text{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$ABB^D = U \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & I_{\text{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & 0 \\ 0 & N_{13} & 0 \end{bmatrix} U^*,$$

$$B^DA = U \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & T_1^{-1}N_{11} & T_1^{-1}N_{12} \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A^DA = U \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*.$$

Since $BB^DA = ABB^D$, then $N_{12} = 0$ and $N_{13} = 0$. Since $B^DA = A^DA$, then $N_{11} = 0$ and $N_{12} = 0$. Therefore,

$$A = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} U^*, \quad A_1 = U \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} U^*. \tag{3.2}$$

From Theorem 2.6, we have $\begin{bmatrix} 0 & 0 \\ 0 & N_{14} \end{bmatrix} \overset{*}{\leqslant} \begin{bmatrix} T_1 & 0 \\ 0 & N_2 \end{bmatrix}$, so $N_{14} \overset{*}{\leqslant} N_2$. Therefore, applying Theorem 2.8 gives $A \overset{\#,*}{\leqslant} B$.

$(3) \Rightarrow (1)$: Applying (2.8), we have

$$BB^{\textcircled{\dagger}}A = U \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & I_{\text{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & 0 & 0 \end{bmatrix} U^*,$$

$$ABB^{\textcircled{\dagger}} = U \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & I_{\text{rk}(T_1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T & 0 & 0 \\ 0 & N_{11} & 0 \\ 0 & N_{13} & 0 \end{bmatrix} U^*,$$

$$B^{\textcircled{\dagger}}A = U \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & T_1^{-1}N_{11} & T_1^{-1}N_{12} \\ 0 & 0 & 0 \end{bmatrix} U^*, \; A^{\textcircled{\dagger}}A = U \begin{bmatrix} I_{\text{rk}(T)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*.$$

Since $BB^{\textcircled{\dagger}}A = ABB^{\textcircled{\dagger}}$, then $N_{12} = 0$ and $N_{13} = 0$. Since $B^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}A$, then $N_{11} = 0$ and $N_{12} = 0$. From Theorem 2.6 and Theorem 2.8, it follows that $A \overset{\#,*}{\leqslant} B$. $\square$

4. Characterizations and properties of the c-minus partial order

By Lemma 1.3 and (1.6), we see that the core-minus partial order and the core partial order coincide in $\mathbb{C}_n^{\mathrm{CM}}$. Wang [29] used the core-EP decomposition to give the characterization of the core-minus partial order, as follows:

LEMMA 4.1. ([29]) *Let $A, B \in \mathbb{C}^{n\times n}$, then $A \overset{\mathrm{CM}}{\leqslant} B$ if and only if there exists a unitary matrix U such that*

$$A = U\begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & 0 & 0 \\ 0 & 0 & N_1 \end{bmatrix}U^*, \quad B = U\begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & N_2 \end{bmatrix}U^*, \tag{4.1}$$

where T_1 and T_2 are non-singular, N_1 and N_2 are nilpotent, satisfying $N_1 \overset{-}{\leqslant} N_2$.

In this section we introduce the c-minus partial order, and consider the relationships between the c-minus partial order and the core-minus partial order.

DEFINITION 4.1. Let A, $B \in \mathbb{C}^{n\times n}$, $A = \widehat{A}_1 + \widehat{A}_2$ and $B = \widehat{B}_1 + \widehat{B}_2$ be the core-EP decompositions of A and B, respectively, where $\widehat{A}_1$ and $\widehat{B}_1$ are core-invertible, and, $\widehat{A}_2$ and $\widehat{B}_2$ are nilpotent. Then A is below B under the c-minus order if

$$A \overset{-}{\leqslant} B, \quad \widehat{A}_1 \overset{\#}{\leqslant} \widehat{B}_1.$$

Whenever this happens, we write $A \overset{©}{\leqslant} B$. Since the core-EP decomposition of a given matrix is unique, and the core order and the minus order are both partial orders, it is easy to get the following theorem:

THEOREM 4.2. *The c-minus order $A \overset{©}{\leqslant} B$ is a partial order.*

THEOREM 4.3. *Let A, $B \in \mathbb{C}^{n\times n}$. Then $A \overset{©}{\leqslant} B$ if and only if there exists a unitary matrix U such that*

$$A = U\begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix}U^*, \quad B = U\begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & N_2 \end{bmatrix}U^*, \tag{4.2}$$

where T_1 and T_2 are non-singular, $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix}$ and N_2 are nilpotent, and $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \overset{-}{\leqslant} \begin{bmatrix} T_2 & S_3 \\ 0 & N_2 \end{bmatrix}$.

Proof. Let $A \in \mathbb{C}^{n\times n}$, and the core-EP decompositions of A be as in (1.3). And let $B = \widehat{B}_1 + \widehat{B}_2$ be the core-EP decomposition of B.

Let $A \overset{\copyright}{\leqslant} B$. Then $\widehat{A}_1 \overset{\#}{\leqslant} \widehat{B}_1$. It follows from Lemma 1.3 that

$$\widehat{A}_1 = U \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad \widehat{B}_1 = U \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & 0 \end{bmatrix} U^*. \tag{4.3}$$

Therefore,

$$\widehat{B}_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_2 \end{bmatrix} U^*,$$

in which N_2 is nilpotent. Furthermore, write $\widehat{A}_2 = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{13} & N_{14} \end{bmatrix} U^*$, in which $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix}$ is nilpotent. Then A is the form as in (4.2).

Since $A \overset{\copyright}{\leqslant} B$, then $A \overset{-}{\leqslant} B$, that is, $\mathrm{rk}(B-A) = \mathrm{rk}(B) - \mathrm{rk}(A)$. It follows that

$$\mathrm{rk}(T_2) + \mathrm{rk}(N_2) - \mathrm{rk} \begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} = \mathrm{rk}\left(\begin{bmatrix} T_2 & T_3 \\ 0 & N_2 \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \right).$$

Therefore,

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \overset{-}{\leqslant} \begin{bmatrix} T_1 & T_3 \\ 0 & N_2 \end{bmatrix}.$$

Conversely, let the forms of A and B be as in (4.2). Obviously, $\widehat{A}_1 \overset{\#}{\leqslant} \widehat{B}_1$, since $\begin{bmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{bmatrix} \overset{-}{\leqslant} \begin{bmatrix} T_1 & T_3 \\ 0 & N_2 \end{bmatrix}$, then $\mathrm{rk}(B-A) = \mathrm{rk}(B) - \mathrm{rk}(A)$, that is $A \overset{-}{\leqslant} B$. Therefore, $A \overset{\copyright}{\leqslant} B$. □

From (2.8) and (4.2), we can see the relationship between the c-minus partial order and the s-star partial order.

REMARK 4.1. The s-star partial order coincides with the c-minus partial order on $\mathbb{C}_n^{\mathrm{iE}}$.

From Lemma 4.1 and Theorem 4.3, it is easy to check that the core-minus partial order implies the c-minus partial order, and the c-minus partial order implies the minus partial order, that is,

$$A \overset{\mathrm{CM}}{\leqslant} B \Rightarrow A \overset{\copyright}{\leqslant} B \Rightarrow A \overset{-}{\leqslant} B.$$

But the c-minus partial order does not imply the core-minus partial order. This can be verified by the following example.

EXAMPLE 4.1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We can write the core-EP decompositions of A and B as $A = \widehat{A}_1 + \widehat{A}_2$ and $B = \widehat{B}_1 + \widehat{B}_2$, respectively, where

$$\widehat{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \widehat{B}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Obviously, we can know that $A \overset{-}{\leqslant} B$, $\widehat{A}_1 \overset{\#}{\leqslant} \widehat{B}_1$, and $\widehat{A}_2$ is not below $\widehat{B}_2$ under the minus order. That is $A \overset{©}{\leqslant} B$, but not $A \overset{\text{CM}}{\leqslant} B$.

Under what condition(s) is the c-minus order equivalent to the core-minus order? We aim to answer this question with the following results.

THEOREM 4.4. *Let A, $B \in \mathbb{C}^{n\times n}$, and A and B be the forms as in* (4.2)*. If $N_{11} = 0$, $N_{12} = 0$ and $N_{13} = 0$, then $A \overset{©}{\leqslant} B$ is equivalent to $A \overset{\text{CM}}{\leqslant} B$.*

Proof. Let $A \overset{\text{CM}}{\leqslant} B$, A and B be the forms as in (4.1). Since $N_1 \overset{-}{\leqslant} N_2$, then $\mathrm{rk}(N_2 - N_1) = \mathrm{rk}(N_2) - \mathrm{rk}(N_1)$. It follows that $\mathrm{rk}(B - A) = \mathrm{rk}(B) - \mathrm{rk}(A)$. Then $A \overset{-}{\leqslant} B$. Therefore, $A \overset{©}{\leqslant} B$.

Conversely, let $A \overset{©}{\leqslant} B$, $N_{11} = 0$, $N_{12} = 0$ and $N_{13} = 0$. Then from Theorem 4.3, there exists a unitary matrix U such that

$$A = U \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & 0 & 0 \\ 0 & 0 & N_{14} \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & T_2 & S_3 \\ 0 & 0 & N_2 \end{bmatrix} U^*,$$

where T_1 and T_2 are non-singular, N_{14} and N_2 are nilpotent, and $\begin{bmatrix} 0 & 0 \\ 0 & N_{14} \end{bmatrix} \overset{-}{\leqslant} \begin{bmatrix} T_1 & T_3 \\ 0 & N_2 \end{bmatrix}$. Since $A \overset{-}{\leqslant} B$, then $\mathrm{rk}(N_2 - N_{14}) = \mathrm{rk}(N_2) - \mathrm{rk}(N_{14})$, that is, $N_{14} \overset{-}{\leqslant} N_2$. Therefore, $A \overset{\text{CM}}{\leqslant} B$. $\square$

THEOREM 4.5. *Let A, $B \in \mathbb{C}^{n\times n}$ and $k = \max\{\mathrm{Ind}(A), \mathrm{Ind}(B)\}$. If $A \overset{©}{\leqslant} B$, then the following conditions are equivalent:*

(1) $A \overset{\text{CM}}{\leqslant} B$;

(2) $BB^{\textcircled{\dagger}}AB^k = AB^k$, $BB^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A$;

(3) $AA^{\textcircled{\dagger}} = AB^{\textcircled{\dagger}}$, $A^{\textcircled{\dagger}}A = B^{\textcircled{\dagger}}A$;

(4) $B^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$, $AA^{\textcircled{\dagger}} = AB^{\textcircled{\dagger}}$;

(5) $AA^{\textcircled{\dagger}} = AB^{\textcircled{\dagger}}$, $BB^{\textcircled{\dagger}}A = AA^{\textcircled{\dagger}}A$.

Proof. $(1) \Rightarrow (2)-(5)$: Let $A \overset{\text{CM}}{\leqslant} B$, A and B be the forms as in (4.1). Then

$$B^k = U \begin{bmatrix} T_1^k & \widehat{T} & \widetilde{T} \\ 0 & T_2^k & T' \\ 0 & 0 & 0 \end{bmatrix} U^*, \tag{4.4}$$

$$A^{\textcircled{\dagger}} = U \begin{bmatrix} T_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \tag{4.5}$$

$$B^{\textcircled{\dagger}} = U \begin{bmatrix} T_1^{-1} & -T_1^{-1} S_1 T_2^{-1} & 0 \\ 0 & T_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \tag{4.6}$$

where $\overrightarrow{T}$, $\widehat{T}$, $\widetilde{T}$, T' are some suitable matrices. It follows that

$$A^{\textcircled{\dagger}} A = U \begin{bmatrix} I_{\text{rk}(T_1)} & T_1^{-1} S_1 & T_1^{-1} S_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = B^{\textcircled{\dagger}} A = A^{\textcircled{\dagger}} B,$$

$$A A^{\textcircled{\dagger}} = U \begin{bmatrix} I_{\text{rk}(T_1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = A B^{\textcircled{\dagger}},$$

$$B B^{\textcircled{\dagger}} A = U \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = A A^{\textcircled{\dagger}} A,$$

$$B B^{\textcircled{\dagger}} A B^k = U \begin{bmatrix} T_1^{k+1} & T_1 \widehat{T} + S_1 T_2^k & T_1 \widetilde{T} + S_1 T' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* = A B^k,$$

So $(2)-(5)$ are obtained.

Let $A \overset{\copyright}{\leqslant} B$ and A, B be the forms as in (4.2), then B^k, $A^{\textcircled{\dagger}}$ and $B^{\textcircled{\dagger}}$ are the forms as in (4.4), (4.5) and (4.6). It is easy to check that

$$B B^{\textcircled{\dagger}} A B^k = U \begin{bmatrix} T_1^{k+1} & T_1 \widehat{T} + S_1 T_2^k & T_1 \widetilde{T} + S_1 T' \\ 0 & N_{11} T_2^k & N_{11} T' \\ 0 & 0 & 0 \end{bmatrix} U^*, \tag{4.7}$$

$$A B^k = U \begin{bmatrix} T_1^{k+1} & T_1 \widehat{T} + S_1 T_2^k & T_1 \widetilde{T} + S_1 T' \\ 0 & N_{11} T_2^k & N_{11} T' \\ 0 & N_{13} T_2^k & N_{13} T' \end{bmatrix} U^*, \tag{4.8}$$

$$B B^{\textcircled{\dagger}} A = U \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & N_{11} & N_{12} \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A A^{\textcircled{\dagger}} A = U \begin{bmatrix} T_1 & S_1 & S_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \tag{4.9}$$

$$A A^{\textcircled{\dagger}} = U \begin{bmatrix} I_{\text{rk}(T_1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \quad A B^{\textcircled{\dagger}} = U \begin{bmatrix} I_{\text{rk}(T_1)} & 0 & 0 \\ 0 & N_{11} T_2^{-1} & 0 \\ 0 & N_{13} T_2^{-1} & 0 \end{bmatrix} U^*, \tag{4.10}$$

$$A^{\oplus}A = U\begin{bmatrix} I_{\mathrm{rk}(T_1)} & T_1^{-1}S_1 & T_1^{-1}S_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*, \tag{4.11}$$

$$B^{\oplus}A = U\begin{bmatrix} I_{\mathrm{rk}(T_1)} & T_1^{-1}S_1 - T_1^{-1}S_1T_2^{-1}N_{11} & T_1^{-1}S_2 - T_1^{-1}S_1T_2^{-1}N_{12} \\ 0 & T_2^{-1}N_{11} & T_2^{-1}N_{12} \\ 0 & 0 & 0 \end{bmatrix} U^*, \tag{4.12}$$

$$A^{\oplus}B = U\begin{bmatrix} I_{\mathrm{rk}(T_1)} & T_1^{-1}S_1 & T_1^{-1}S_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^*. \tag{4.13}$$

$(2) \Rightarrow (1)$: Since $BB^{\oplus}AB^k = AB^k$, $BB^{\oplus}A = AA^{\oplus}A$ and T_2 is non-singular, from (4.7), (4.8) and (4.9), we have $N_{11} = 0$, $N_{12} = 0$ and $N_{13} = 0$. Applying Theorem 4.4 gives $A \overset{\mathrm{CM}}{\leqslant} B$.

$(3) \Rightarrow (1)$: Since $AA^{\oplus} = AB^{\oplus}$ and $A^{\oplus}A = B^{\oplus}A$, from (4.10), (4.11) and (4.12), we have $N_{11} = 0$, $N_{12} = 0$ and $N_{13} = 0$. Therefore, $A \overset{\mathrm{CM}}{\leqslant} B$.

$(4) \Rightarrow (1)$: Since $B^{\oplus}A = A^{\oplus}B$ and $AA^{\oplus} = AB^{\oplus}$, from (4.10), (4.12) and (4.13), we have $N_{11} = 0$, $N_{12} = 0$ and $N_{13} = 0$. Therefore, $A \overset{\mathrm{CM}}{\leqslant} B$.

$(5) \Rightarrow (1)$: Since $AA^{\oplus} = AB^{\oplus}$ and $BB^{\oplus}A = AA^{\oplus}A$, from (4.9) and (4.10), we have $N_{11} = 0$, $N_{12} = 0$ and $N_{13} = 0$. Therefore, $A \overset{\mathrm{CM}}{\leqslant} B$. $\square$

5. Conclution

This paper provides several characterizations of the s-star partial order, explores the relationships between the C-N-star partial order and the s-star partial order, and provided further characterizations of problem 16.3.2 in [20]. Furthermore, this paper introduces a new partial order, the c-minus partial order, which generalizes the core-minus partial order. The s-star partial order implies the c-minus partial order on $\mathbb{C}_n^{\mathrm{iE}}$.

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