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COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF NEGATIVELY DEPENDENT RANDOM VARIABLES UNDER SUB–LINEAR EXPECTATIONS

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Abstract. By Rosenthal's inequality for negatively dependent random variables under sub-linear expectations, we study complete convergence and complete moment convergence for weighted sums of negatively dependent random variables. The results complement that of Li and Shen [9] in some extent.

Peng [12, 13] gave the important concepts of the sub-linear expectations space to study the uncertainty in probability. The works of Peng [12, 13] encouraged many people to investigate the results under sub-linear expectations space, which extend the corresponding ones in probability space. Zhang [26–28] got Donsker's invariance principle, exponential inequalities and Rosenthal's inequality under sub-linear expectations. Under sub-linear expectations, Xu and Kong [22] investigated complete qth moment convergence of moving average processes for m-widely acceptable random variables. For more limit theorems under sub-linear expectations, the interested readers could refer to Xu and Zhang [24, 25], Zhang and Lin [30], Zhong and Wu [31], Chen [2], Zhang [29], Hu, Chen, and Zhang [6], Gao and Xu [4], Kuczmaszewska [8], Xu and Cheng [17–19], Xu et al. [20], [21], Xu and Kong [23], Chen and Wu [1], Xu [15], [14], [16], and the references therein.

In probability space, Li and Shen [9] investigated complete moment convergence for weighted sums of extended negatively dependent random variables. For references on complete moment convergence in linear expectation space, the interested reader could refer to Zhou [32], Ko [7], Hosseini and Nezakati [5], Meng et al. [11], and the refercences therein. Inspired by the works of Li and Shen [9], we try to investigate complete moment convergence for weighted sums of negatively dependent random variables under sub-linear expectations, which complements the relevant ones in Li and Shen [9].

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We organize the rest of this paper as follows. We present necessary basic notions, concepts and relevant properties, and give necessary lemma under sub-linear expectations in the next section. In Section 3, we give our main result, Theorem 2.1, the proof of which is postponed in Section 4.

1. Preliminaries

As in Xu and Cheng [17], we use similar notations as in the work by Peng [13], Zhang [27]. Suppose that $(\Omega,\mathscr{F})$ is a given measurable space. Assume that $\mathscr{H}$ is a subset of all random variables on $(\Omega,\mathscr{F})$ such that $X_1,\cdots,X_n\in\mathscr{H}$ implies $\varphi(X_1,\cdots,X_n)$ $\in\mathscr{H}$ for each $\varphi\in\mathscr{C}_{l,Lip}(\mathbb{R}^n)$, where $\mathscr{C}_{l,Lip}(\mathbb{R}^n)$ represents the linear space of (local lipschitz) function φ fulfilling

$$|\varphi(\mathbf{x})-\varphi(\mathbf{y})|\leqslant C(1+|\mathbf{x}|^m+|\mathbf{y}|^m)(|\mathbf{x}-\mathbf{y}|),\ \ \forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n$$

for some $C>0$, $m\in\mathbb{N}$ relying on φ.

DEFINITION 1.1. A sub-linear expectation $\mathbb{E}$ on $\mathscr{H}$ is a functional $\mathbb{E}:\mathscr{H}\mapsto\bar{\mathbb{R}}:=[-\infty,\infty]$ fulfilling the following properties: for all $X,Y\in\mathscr{H}$, we have

(a) If $X\geqslant Y$, then $\mathbb{E}[X]\geqslant\mathbb{E}[Y]$;

(b) $\mathbb{E}[c]=c$, $\forall c\in\mathbb{R}$;

(c) $\mathbb{E}[\lambda X]=\lambda\mathbb{E}[X]$, $\forall\lambda\geqslant 0$;

(d) $\mathbb{E}[X+Y]\leqslant\mathbb{E}[X]+\mathbb{E}[Y]$ whenever $\mathbb{E}[X]+\mathbb{E}[Y]$ is not of the form $\infty-\infty$ or $-\infty+\infty$.

DEFINITION 1.2. $\{X_n,n\geqslant 1\}$ is said to be stochastically dominated by a random variable X under $(\Omega,\mathscr{H},\mathbb{E})$, if there exists a contant C such that $\forall n\geqslant 1$, for all non-negative $h\in\mathscr{C}_{l,Lip}(\mathbb{R})$, $\mathbb{E}(h(X_n))\leqslant C\mathbb{E}(h(X))$.

A set function $V:\mathscr{F}\mapsto[0,1]$ is named to be a capacity if

(a) $V(\emptyset)=0$, $V(\Omega)=1$;

(b) $V(A)\leqslant V(B)$, $A\subset B$, $A,B\in\mathscr{F}$.

A capacity V is called sub-additive if $V(A\bigcup B)\leqslant V(A)+V(B)$, $A,B\in\mathscr{F}$.

In this sequel, given a sub-linear expectation space $(\Omega,\mathscr{H},\mathbb{E})$, set $\mathbb{V}(A):=\inf\{\mathbb{E}[\xi]:I_A\leqslant\xi,\xi\in\mathscr{H}\}$, $\forall A\in\mathscr{F}$ (see (2.3) and the definitions of $\mathbb{V}$ above (2.3) in Zhang [27]). $\mathbb{V}$ is a sub-additive capacity. Set

$$C_{\mathbb{V}}(X):=\int_0^\infty\mathbb{V}(X>x)\mathrm{d}x+\int_{-\infty}^0(\mathbb{V}(X>x)-1)\mathrm{d}x.$$

Suppose that $\mathbf{X}=(X_1,\cdots,X_m)$, $X_i\in\mathscr{H}$ and $\mathbf{Y}=(Y_1,\cdots,Y_n)$, $Y_i\in\mathscr{H}$ are two random vectors on $(\Omega,\mathscr{H},\mathbb{E})$. $\mathbf{Y}$ is said to be negatively dependent to $\mathbf{X}$, if for each $\psi_1 \in \mathscr{C}_{l,Lip}(\mathbb{R}^m)$, $\psi_2 \in \mathscr{C}_{l,Lip}(\mathbb{R}^n)$, we have $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] \leqslant \mathbb{E}[\psi_1(\mathbf{X})]\mathbb{E}[\psi_2(\mathbf{Y})]$ whenever $\psi_1(\mathbf{X}) \geqslant 0$, $\mathbb{E}[\psi_2(\mathbf{Y})] \geqslant 0$, $\mathbb{E}[|\psi_1(\mathbf{X})\psi_2(\mathbf{Y})|] < \infty$, $\mathbb{E}[|\psi_1(\mathbf{X})|] < \infty$, $\mathbb{E}[|\psi_2(\mathbf{Y})|] < \infty$, and either ψ_1 and ψ_2 are coordinatewise nondecreasing or ψ_1 and ψ_2 are coordinatewise nonincreasing (see Definition 2.3 of Zhang [27], Definition 1.5 of Zhang [28]). $\{X_n\}_{n=1}^{\infty}$ is named a sequence of negatively dependent random variables, if X_{n+1} is negatively dependent to $(X_1,\cdots,X_n)$ for each $n \geqslant 1$.

We cite results below.

LEMMA 1.1. (cf. Theorem 2.1 (a) and (b) and their proof of Zhang [28]) *Assume that $\{X_n, n \geqslant 1\}$ is a sequence of negatively dependent random variables with $\mathbb{E}(X_n) \leqslant 0$, $n \geqslant 1$, in the sub-linear expectation space $(\Omega,\mathscr{H},\mathbb{E})$. Then there exist a positive constant $C = C_p$ relying on p such that*

$$\mathbb{E}\left\{\left(\left(\sum_{j=1}^{n} X_j\right)^{+}\right)^{p}\right\} \leqslant C\left\{\sum_{j=1}^{n}\mathbb{E}(|X_j|^p) + \left(\sum_{j=1}^{n}\mathbb{E}(|X_j|^2)\right)^{p/2}\right\}, \quad \textit{for } p \geqslant 2, \tag{1.1}$$

$$\mathbb{E}\left\{\left(\left(\sum_{j=1}^{n} X_j\right)^{+}\right)^{p}\right\} \leqslant C\left\{\sum_{j=1}^{n}\mathbb{E}(|X_j|^p)\right\}, \quad \textit{for } 1 \leqslant p < 2. \tag{1.2}$$

LEMMA 1.2. (cf. Lemma 2.1 of Xu and Cheng [17]) *Suppose that Y is a random variable in the sub-linear expectation space $(\Omega,\mathscr{H},\mathbb{E})$. Then for any $\alpha > 0$, $\gamma > 0$, $\beta > -1$,*

(i) $\int_0^\infty u^\beta C_{\mathbb{V}}(|Y|^\alpha I(|Y| > u^\gamma))\,\mathrm{d}u \leqslant CC_{\mathbb{V}}\left(|Y|^{(\beta+1)/\gamma+\alpha}\right)$,

(ii) $\int_0^\infty u^\beta \ln(u) C_{\mathbb{V}}(|Y|^\alpha I(|Y| > u^\gamma))\,\mathrm{d}u \leqslant CC_{\mathbb{V}}\left(|Y|^{(\beta+1)/\gamma+\alpha}\ln(1+|Y|)\right)$.

LEMMA 1.3. (cf. Lemma 4.5 (iii) of Zhang [27] or Lemma 2.3 of Xu and Cheng [17]) *If $\mathbb{E}$ is countably sub-additive and $C_{\mathbb{V}}(|X|) < \infty$, then*

$$\mathbb{E}(|X|) \leqslant C_{\mathbb{V}}(|X|).$$

In the paper we assume that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leqslant \sum_{n=1}^{\infty}\mathbb{E}(X_n)$, whenever $X \leqslant \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathscr{H}$, and $X \geqslant 0$, $X_n \geqslant 0$, $n = 1,2,\ldots$. Let C stand for a positive constant which may change from place to place. $I(A)$ or I_A represent the indicator function of A.

2. Main results

Our main result is the following.

THEOREM 2.1. *Suppose that $\beta > -1$, $r > 1$, $1 \leqslant q < r \bigwedge 2$. Assume that $\{X_n, n \geqslant 1\}$ is a sequence of negatively dependent random variables with $\mathbb{E}(X_n) = \mathbb{E}(-X_n) = 0$,*

$n \geqslant 1$, *in the sub-linear expectation space* $(\Omega, \mathscr{H}, \mathbb{E})$ *and* $\{X_n, n \geqslant 1\}$ *is also stochastically dominated by* X. *Suppose that* $\{a_{ni} \approx \left(\frac{i}{n}\right)^{\beta}(1/n), 1 \leqslant i \leqslant n, n \geqslant 1\}$. *Assume that*

$$\begin{cases} C_{\mathbb{V}}\left\{|X|^{(r-1)/(1+\beta)}\right\} < \infty, & -1 < \beta < -1/r; \\ C_{\mathbb{V}}\left\{|X|^{r}\log(1+|X|)\right\} < \infty, & \beta = -1/r; \\ C_{\mathbb{V}}\left\{|X|^{r}\right\} < \infty, & \beta > -1/r. \end{cases} \tag{2.1}$$

Then $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{E}\left\{\left(\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon\right)^{+}\right)^{q}\right\} < \infty, \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{V}\left\{\left|\sum_{i=1}^{n} a_{ni}X_i\right| > \varepsilon\right\} < \infty. \tag{2.3}$$

3. The proof of the main result

Proof of Theorem 2.1. We first prove (2.2). By C_r inequality,

$$\begin{aligned} &\mathbb{E}\left\{\left(\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon\right)^{+}\right)^{q}\right\} \\ &\leqslant \mathbb{E}\left\{\left(\left(\sum_{i=1}^{n} a_{ni}X_i - \varepsilon\right)^{+} + \left(\sum_{i=1}^{n}(-a_{ni}X_i) - \varepsilon\right)^{+}\right)^{q}\right\} \\ &\leqslant C\mathbb{E}\left\{\left(\left(\sum_{i=1}^{n} a_{ni}X_i - \varepsilon\right)^{+}\right)^{q}\right\} + \mathbb{E}\left\{\left(\left(\sum_{i=1}^{n}(-a_{ni}X_i) - \varepsilon\right)^{+}\right)^{q}\right\}, \end{aligned}$$

we only need to prove

$$\sum_{n=1}^{\infty} n^{r-2}\mathbb{E}\left\{\left(\left(\sum_{i=1}^{n} a_{ni}X_i - \varepsilon\right)^{+}\right)^{q}\right\} < \infty,\ \forall \varepsilon > 0.$$

For $1 \leqslant i \leqslant n$, $n \geqslant 1$, denote

$$a_{ni}Y_{ni} = -I(a_{ni}X_i < -1) + a_{ni}X_iI(|a_{ni}X_i| \leqslant 1) + I(a_{ni}X_i > 1),$$

$$a_{ni}Z_{ni} = (a_{ni}X_i + 1)I(a_{ni}X_i < -1) + (a_{ni}X_i - 1)I(a_{ni}X_i > 1),$$

$$a_{ni}Y_n = -I(a_{ni}X < -1) + a_{ni}XI(|a_{ni}X| \leqslant 1) + I(a_{ni}X > 1),$$

$$a_{ni}Z_n = (a_{ni}X+1)I(a_{ni}X<-1)+(a_{ni}X-1)I(a_{ni}X>1).$$

Note $\mathbb{E}(X_n)=\mathbb{E}(-X_n)=0$, $a_{ni}X_i=a_{ni}(Y_{ni}+Z_{ni})$. By Proposition 1.3.7 of Peng [13], we have $\mathbb{E}(a_{ni}Y_{ni})=\mathbb{E}(-a_{ni}Z_{ni})=a_{ni}\mathbb{E}(-Z_{ni})$. We see that

$$\sum_{i=1}^{n}a_{ni}X_i=\sum_{i=1}^{n}a_{ni}\left(Y_{ni}-\mathbb{E}(Y_{ni})\right)+\sum_{i=1}^{n}a_{ni}\left(Z_{ni}-\mathbb{E}(Z_{ni})\right)+\sum_{i=1}^{n}a_{ni}\left(\mathbb{E}(Z_{ni})+\mathbb{E}(-Z_{ni})\right). \tag{3.1}$$

By C_r inequality, we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{r-2}\mathbb{E}\left\{\left(\left(\sum_{i=1}^{n}a_{ni}X_i-\varepsilon\right)^{+}\right)^{q}\right\}\\
&\quad\leqslant C\sum_{n=1}^{\infty}n^{r-2}\mathbb{E}\left\{\left(\left(\sum_{i=1}^{n}a_{ni}\left(Y_{ni}-\mathbb{E}(Y_{ni})\right)-\varepsilon\right)^{+}\right)^{q}\right\}\\
&\qquad+C\sum_{n=1}^{\infty}n^{r-2}\mathbb{E}\left\{\left(\left(\sum_{i=1}^{n}a_{ni}\left(Z_{ni}-\mathbb{E}(Z_{ni})\right)\right)^{+}\right)^{q}\right\}\\
&\qquad+\sum_{n=1}^{\infty}n^{r-2}\mathbb{E}\left\{\left(\sum_{i=1}^{n}a_{ni}\left(\mathbb{E}(Z_{ni})+\mathbb{E}(-Z_{ni})\right)\right)^{q}\right\}=:L_1+L_2+L_3.
\end{aligned}$$

Hence, in order to establish (2.2), it is sufficient to establish $L_1<\infty$, $L_2<\infty$, $L_3<\infty$.

Firstly, we prove $L_1<\infty$. For $n\geqslant 1$, we see that $\{Y_{ni}-\mathbb{E}(Y_{ni}),1\leqslant i\leqslant n\}$ are still negatively dependent random variables under sub-linear expectations. Hence, for $p>2$, by Lemma 1.3, Lemma 1.1, Markov's inequality under sub-linear expectations, Jensen's inequality under sub-linear expectations (cf. Lin [10]), and $q<2<p$, we see that

$$\begin{aligned}
L_1&\leqslant C\sum_{n=1}^{\infty}n^{r-2}C_{\mathbb{V}}\left\{\left(\left(\sum_{i=1}^{n}a_{ni}\left(Y_{ni}-\mathbb{E}(Y_{ni})\right)-\varepsilon\right)^{+}\right)^{q}\right\}\\
&\leqslant C\sum_{n=1}^{\infty}n^{r-2}\int_0^{\infty}\mathbb{V}\left\{\sum_{i=1}^{n}a_{ni}\left(Y_{ni}-\mathbb{E}(Y_{ni})\right)>t^{1/q}+\varepsilon\right\}\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty}n^{r-2}\int_0^{\infty}\frac{1}{\left(t^{1/q}+\varepsilon\right)^{p}}\mathbb{E}\left\{\left(\left(\sum_{i=1}^{n}a_{ni}\left(Y_{ni}-\mathbb{E}(Y_{ni})\right)\right)^{+}\right)^{p}\right\}\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty}n^{r-2}\sum_{i=1}^{n}\mathbb{E}|a_{ni}Y_{ni}|^{p}+C\sum_{n=1}^{\infty}n^{r-2}\left[\sum_{i=1}^{n}\mathbb{E}|a_{ni}Y_{ni}|^{2}\right]^{p/2}=:K_1+K_2.
\end{aligned}$$

Obviously, by the definition of Y_{ni}, and Lemma 1.3, we obtain

$$\begin{aligned}
K_1 &\leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{E}|a_{ni}Y_n|^p \leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n} C_{\mathbb{V}}\{|a_{ni}Y_n|^p\}\\
&\leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\int_0^1 \mathbb{V}\{1\cdot I\{|a_{ni}X|>1\}>x\}\,\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\int_0^1 \mathbb{V}\{|a_{ni}X|^p I\{|a_{ni}X|\leqslant 1\}>x\}\,\mathrm{d}x\\
&= C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{V}\{|a_{ni}X|>1\}+C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\int_0^1 \mathbb{V}\{|a_{ni}X|^p I\{|a_{ni}X|\leqslant 1\}>x\}\,\mathrm{d}x\\
&=: K_{11}+K_{12}.
\end{aligned}$$

Next, we first prove $K_{11}<\infty$. By the proof of Lemma 2.2 of Zhong and Wu [31], and (2.1), we see that

$$\begin{aligned}
K_{11} &\leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{V}\left\{|X|>Cn^{1+\beta}i^{-\beta}\right\}\\
&\approx C\int_1^{\infty} x^{r-2}\mathrm{d}x\int_1^{x}\mathbb{V}\left\{|X|>Cx^{1+\beta}y^{-\beta}\right\}\mathrm{d}y\\
&\quad\left(\text{letting } u=x^{1+\beta}y^{-\beta},\ v=y\right)\\
&\leqslant C\int_1^{\infty}\mathrm{d}u\int_1^{u} u^{(r-2-\beta)/(1+\beta)}v^{\beta(r-1)/(1+\beta)}\mathbb{V}\{|X|>Cu\}\,\mathrm{d}v\\
&\approx \begin{cases} C\int_1^{\infty} u^{\frac{r-1}{1+\beta}-1}\mathbb{V}\{|X|>Cu\}\,\mathrm{d}u, & -1<\beta<-1/r;\\ C\int_1^{\infty} u^{r-1}\log(u)\mathbb{V}\{|X|>Cu\}\,\mathrm{d}u, & \beta=-1/r;\\ C\int_1^{\infty} u^{r-1}\mathbb{V}\{|X|>Cu\}\,\mathrm{d}u, & \beta>-1/r\end{cases}\\
&\leqslant \begin{cases} C_{\mathbb{V}}\left\{|X|^{\frac{r-1}{1+\beta}}\right\}<\infty, & -1<\beta<-1/r;\\ C_{\mathbb{V}}\{|X|^r\log(1+|X|)\}<\infty, & \beta=-1/r;\\ C_{\mathbb{V}}\{|X|^r\}<\infty, & \beta>-1/r.\end{cases}
\end{aligned} \tag{3.2}$$

And we choose p large enough such that $(r-1)/(1+\beta)-1-p<-1$, $r-1-p<-1$. By the proof of Lemma 2.2 of Zhong and Wu [31], and (2.1), we see that

$$\begin{aligned}
K_{12} &\approx C\int_1^{x} x^{r-2}\mathrm{d}x\int_1^{x}\mathrm{d}y\int_0^1\mathbb{V}\left\{|X|^p I\left\{|X|\leqslant Cy^{-\beta}x^{1+\beta}\right\}>C(x^{1+\beta}y^{-\beta})^p z\right\}\mathrm{d}z\\
&\approx C\int_1^{\infty}\mathrm{d}u\int_1^{u} u^{\frac{r-2-\beta}{1+\beta}}v^{\frac{(r-1)\beta}{1+\beta}}\mathrm{d}v\int_0^1\mathbb{V}\{|X|^p I\{|X|\leqslant Cu\}>Cu^p z\}\,\mathrm{d}z\\
&\qquad\left(\text{setting } u=x^{1+\beta}y^{-\beta},\ v=y\right)\\
&\approx \begin{cases} C\int_1^{\infty} u^{\frac{r-1}{1+\beta}-1}\mathrm{d}u\int_0^1\mathbb{V}\{|X|^p I\{|X|\leqslant Cu\}>Cu^p z\}\,\mathrm{d}z, & -1<\beta<-1/r;\\ C\int_1^{\infty} u^{r-1}\log(u)\mathrm{d}u\int_0^1\mathbb{V}\{|X|^p I\{|X|\leqslant Cu\}>Cu^p z\}\,\mathrm{d}z, & \beta=-1/r;\\ C\int_1^{\infty} u^{r-1}\mathrm{d}u\int_0^1\mathbb{V}\{|X|^p I\{|X|\leqslant Cu\}>Cu^p z\}\,\mathrm{d}z, & \beta>-1/r\end{cases}
\end{aligned}$$

$$\approx \begin{cases} C\int_1^\infty u^{\frac{r-1}{1+\beta}-1}\mathrm{d}u\int_0^{Cu^p}\mathbb{V}\{|X|^pI\{|X|\leqslant Cu\}>z\}u^{-p}\mathrm{d}z, & -1<\beta<-1/r;\\ C\int_1^\infty u^{r-1}\log(u)\mathrm{d}u\int_0^{Cu^p}\mathbb{V}\{|X|^pI\{|X|\leqslant Cu\}>z\}u^{-p}\mathrm{d}z, & \beta=-1/r;\\ C\int_1^\infty u^{r-1}\mathrm{d}u\int_0^{Cu^p}\mathbb{V}\{|X|^pI\{|X|\leqslant Cu\}>z\}u^{-p}\mathrm{d}z, & \beta>-1/r \end{cases}$$

$$\leqslant \begin{cases} \int_0^\infty\mathbb{V}\{|X|^p>z\}\mathrm{d}z\int_{1\bigvee(C/z^{1/p})}^\infty u^{\frac{r-1}{1+\beta}-1-p}\mathrm{d}u, & -1<\beta<-1/r;\\ \int_0^\infty\mathbb{V}\{|X|^p>z\}\mathrm{d}z\int_{1\bigvee(C/z^{1/p})}^\infty u^{r-1-p}\log(u)\mathrm{d}u, & \beta=-1/r;\\ \int_0^\infty\mathbb{V}\{|X|^p>z\}\mathrm{d}z\int_{1\bigvee(C/z^{1/p})}^\infty u^{r-1-p}\mathrm{d}u, & \beta>-1/r \end{cases}$$

$$\leqslant \begin{cases} C\int_0^\infty\mathbb{V}\{|X|^p>z\}z^{\left(\frac{r-1}{1+\beta}-p\right)/p}\mathrm{d}z, & -1<\beta<-1/r;\\ \int_0^\infty\mathbb{V}\{|X|^p>z\}z^{r/p-1}\log(z)\mathrm{d}z, & \beta=-1/r;\\ \int_0^\infty\mathbb{V}\{|X|^p>z\}z^{r/p-1}\mathrm{d}z, & \beta>-1/r \end{cases}$$

$$\leqslant \begin{cases} CC_{\mathbb{V}}\left\{|X|^{\frac{r-1}{1+\beta}}\right\}<\infty, & -1<\beta<-1/r;\\ CC_{\mathbb{V}}\{|X|^r\log(1+|X|)\}<\infty, & \beta=-1/r;\\ CC_{\mathbb{V}}\{|X|^r\}<\infty, & \beta>-1/r. \end{cases}$$

Next, we will prove $K_2<\infty$. By the definition of Y_{ni}, C_r inequality, we see that

$$\begin{aligned} K_2 &\leqslant C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n\mathbb{V}(|a_{ni}X|>1)+\sum_{i=1}^n C_{\mathbb{V}}\left\{|a_{ni}X|^2I\{|a_{ni}X|\leqslant 1\}\right\}\right)^{p/2}\\ &\leqslant C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n\mathbb{V}(|a_{ni}X|>1)\right)^{p/2}\\ &\quad +C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n C_{\mathbb{V}}\left\{|a_{ni}X|^2I\{|a_{ni}X|\leqslant 1\}\right\}\right)^{p/2}\\ &=:K_{21}+K_{22}. \end{aligned} \tag{3.3}$$

We choose p sufficiently large such that $r-2-pr(1+\beta)/2<-1$ and $r-2-(r-1)p/2<-1$. By Markov's inequality under sub-linear expectations, Lemma 1.3, Lemma 1.2, and (2.1), we have

$$\begin{aligned} K_{21} &\leqslant C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n\mathbb{V}\left(|X|>Cn^{1+\beta}i^{-\beta}\right)\right)^{p/2}\leqslant C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n\frac{\mathbb{E}|X|^r}{(n^{1+\beta}i^{-\beta})^r}\right)^{p/2}\\ &\leqslant C\sum_{n=1}^\infty n^{r-2}\left(C_{\mathbb{V}}\{|X|^r\}n^{-r(1+\beta)}\sum_{i=1}^n i^{r\beta}\right)^{p/2}\\ &\leqslant \begin{cases} C\sum_{n=1}^\infty n^{r-2-pr(1+\beta)/2}<\infty, & -1<\beta<-1/r;\\ C\sum_{n=1}^\infty n^{r-2-p(r-1)/2}(\log n)^{p/2}<\infty, & \beta=-1/r;\\ C\sum_{n=1}^\infty n^{r-2-p(r-1)/2}<\infty, & \beta>-1/r. \end{cases} \end{aligned}$$

In order to get $K_{22}<\infty$, we investigate the following two cases.

(1) When $1<r<2$, choose p large enough such that $r-2-pr(1+\beta)/2<-1$, $r-2-(r-1)p/2<-1$. By $C_{\mathbb{V}}\{|X|^r\}<\infty$, we obtain

$$K_{22} \leqslant C\sum_{n=1}^{\infty} n^{r-2}\left(\sum_{i=1}^{n} C_{\mathbb{V}}\{|a_{ni}X|^r I\{|a_{ni}X|\leqslant 1\}\}\right)^{p/2}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{r-2}\left(\sum_{i=1}^{n}|a_{ni}|^r\right)^{p/2} \approx C\sum_{n=1}^{\infty} n^{r-2}\left(\sum_{i=1}^{n} n^{-r(1+\beta)} i^{r\beta}\right)^{p/2}$$
$$\leqslant \begin{cases} C\sum_{n=1}^{\infty} n^{r-2-pr(1+\beta)/2}<\infty, & -1<\beta<-1/r; \\ C\sum_{n=1}^{\infty} n^{r-2-p(r-1)/2}(\log n)^{p/2}<\infty, & \beta=-1/r; \\ C\sum_{n=1}^{\infty} n^{r-2-p(r-1)/2}<\infty, & \beta>-1/r. \end{cases}$$

(2) When $r\geqslant 2$, note that (2.1) implies $C_{\mathbb{V}}\left\{|X|^2\right\}<\infty$. Choose p large enough such that $r-2-p(1+\beta)<-1$, $r-2-p/2<-1$. We conclude that

$$K_{22} \leqslant C\sum_{n=1}^{\infty} n^{r-2}\left(\sum_{i=1}^{n}|a_{ni}|^2\right)^{p/2} \approx C\sum_{n=1}^{\infty} n^{r-2}\left(\sum_{i=1}^{n} n^{-2(1+\beta)} i^{2\beta}\right)^{p/2}$$
$$\leqslant \begin{cases} C\sum_{n=1}^{\infty} n^{r-2-p(1+\beta)}<\infty, & -1<\beta<-1/2; \\ C\sum_{n=1}^{\infty} n^{r-2-p/2}(\log n)^{p/2}<\infty, & \beta=-1/2; \\ C\sum_{n=1}^{\infty} n^{r-2-p/2}<\infty, & \beta>-1/2. \end{cases}$$

Next, we establish $L_2<\infty$. We note that $\{a_{ni}Z_{ni}-a_{ni}\mathbb{E}(Z_{ni})\}$ are still negatively dependent random variables under sub-linear expectations. By Lemma 1.1, C_r inequality, Jensen-inequality under sub-linear expectations (cf. Proposition 2.1 of Chen et al. [3]), Lemma 1.3, we see that

$$L_2 \leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{E}|a_{ni}Z_n-a_{ni}\mathbb{E}(Z_{ni})|^q \leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}(\mathbb{E}|a_{ni}Z_n|^q+|\mathbb{E}(a_{ni}Z_n)|^q)$$
$$\leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{E}|a_{ni}Z_n|^q \leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n} C_{\mathbb{V}}\{|a_{ni}Z_n|^q\}$$
$$\leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n}\mathbb{V}\{|a_{ni}X|>1\}+C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n} C_{\mathbb{V}}\{|a_{ni}X|^q I\{|a_{ni}X|>1\}\}$$
$$=: H_1+H_2.$$

By $K_{11}<\infty$, we get $H_1<\infty$. Next we establish $H_2<\infty$. By Lemma 1.2, $1\leqslant q<r\bigwedge 2$, and (2.1), we obtain

$$H_2 \leqslant C\sum_{n=1}^{\infty} n^{r-2}\sum_{i=1}^{n} n^{-q(1+\beta)} i^{q\beta} C_{\mathbb{V}}\left\{|X|^q I\left\{|X|>Cn^{1+\beta}i^{-\beta}\right\}\right\}$$
$$\approx C\int_1^{\infty} x^{r-2}\mathrm{d}x\int_1^x x^{-q(1+\beta)} y^{q\beta} C_{\mathbb{V}}\left\{|X|^q I\left\{|X|>Cx^{1+\beta}y^{-\beta}\right\}\right\}\mathrm{d}y$$

$$\leqslant C\int_1^\infty \mathrm{d}u\int_1^u u^{\frac{r-2-\beta}{1+\beta}-q}v^{\frac{(r-1)\beta}{1+\beta}}C_{\mathbb{V}}\{|X|^qI\{|X|>Cu\}\}\mathrm{d}v$$

$$\left(\text{setting } u=x^{1+\beta}y^{-\beta},\ v=y\right)$$

$$\approx\begin{cases}C\int_1^\infty u^{\frac{r-1}{1+\beta}-1-q}C_{\mathbb{V}}\{|X|^qI\{|X|>Cu\}\}\mathrm{d}u, & -1<\beta<-\frac{1}{r};\\ C\int_1^\infty u^{r-1-q}\log(u)C_{\mathbb{V}}\{|X|^qI\{|X|>Cu\}\}\mathrm{d}u, & \beta=-\frac{1}{r};\\ C\int_1^\infty u^{r-1-q}C_{\mathbb{V}}\{|X|^qI\{|X|>Cu\}\}\mathrm{d}u, & \beta>-\frac{1}{r}\end{cases}$$

$$\leqslant\begin{cases}CC_{\mathbb{V}}\left\{|X|^{\frac{r-1}{1+\beta}}\right\}<\infty, & -1<\beta<-\frac{1}{r};\\ CC_{\mathbb{V}}\{|X|^r\log(1+|X|)\}<\infty, & \beta=-\frac{1}{r};\\ CC_{\mathbb{V}}\{|X|^r\}<\infty, & \beta>-\frac{1}{r}.\end{cases}$$

Hence $L_2<\infty$.

Finally, we prove L_3 in two cases. When $1<q<r\wedge 2$, by $\mathbb{E}(X_n)=\mathbb{E}(-X_n)=0$, Lemma 1.3, we see that

$$L_3=C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n[\mathbb{E}(a_{ni}Z_{ni})+\mathbb{E}(-a_{ni}Z_{ni})]\right)^q\leqslant C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n\mathbb{E}|a_{ni}Z_{ni}|\right)^q$$

$$=C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n\mathbb{E}|a_{ni}Z_n|\right)^q\leqslant C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n C_{\mathbb{V}}\{|a_{ni}Z_n|\}\right)^q$$

$$\leqslant\begin{cases}C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n\left[|a_{ni}|^{\frac{r-1}{1+\beta}}C_{\mathbb{V}}\left\{|X|^{\frac{r-1}{1+\beta}}\right\}\right]\right)^q, & -1<\beta<-\frac{1}{r};\\ C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n[|a_{ni}|^rC_{\mathbb{V}}\{|X|^r\}]\right)^q, & \beta=-\frac{1}{r};\\ C\sum_{n=1}^\infty n^{r-2}\left(\sum_{i=1}^n[|a_{ni}|^rC_{\mathbb{V}}\{|X|^r\}]\right)^q, & \beta>-\frac{1}{r}\end{cases}$$

$$\leqslant\begin{cases}C\sum_{n=1}^\infty n^{r-2-q(r-1)}<\infty, & -1<\beta<-\frac{1}{r};\\ C\sum_{n=1}^\infty n^{r-2-q(r-1)}(\log(n))^q<\infty, & \beta=-\frac{1}{r};\\ C\sum_{n=1}^\infty n^{r-2-q(r-1)}<\infty, & \beta>-\frac{1}{r}.\end{cases}$$

When $q=1$, by Lemma 1.3, the similar proof of III on page 11 of Xu and Cheng [17], and (2.1), we see that

$$L_3=C\sum_{n=1}^\infty n^{r-2}\sum_{i=1}^n\mathbb{E}|a_{ni}Z_n|\leqslant C\sum_{n=1}^\infty n^{r-2}\sum_{i=1}^n C_{\mathbb{V}}\{|a_{ni}Z_n|\}$$

$$\leqslant C\sum_{n=1}^\infty n^{r-2}\sum_{i=1}^n n^{-(1+\beta)}i^\beta C_{\mathbb{V}}\left\{|X|I\left\{|X|>Cn^{1+\beta}i^{-\beta}\right\}\right\}<\infty.$$

Next, we establish (2.3). Obviously, we see that $\forall\varepsilon>0$,

$$\infty>\sum_{n=1}^\infty n^{r-2}\mathbb{E}\left\{\left(\left(\left|\sum_{i=1}^n a_{ni}X_i\right|-\varepsilon\right)^+\right)^q\right\}$$

$$\geqslant\sum_{n=1}^\infty n^{r-2}\varepsilon^q\mathbb{V}\left\{\left|\sum_{i=1}^n a_{ni}X_i\right|>2\varepsilon\right\}.$$

By (2.2), we prove (2.3). This completes the proof. □

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