Journal of Mathematical Inequalities
Volume 19, Number 1 (2025), 223–245
doi:10.7153/jmi-2025-19-15

A NEW APPROACH TO SELECTING CONSTANTS FOR SOME ANALYTIC INEQUALITIES

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(*Communicated by M. Krnić*)

Abstract. The subject of the paper is a new approach to selecting real constants for which certain analytic inequalities hold. This approach is based on the introduction and analysis of the corresponding families of functions that are stratified and such that each function from the family has certain Taylor expansions. The approach is illustrated on some D'Aurizio-Sándor-type inequalities that were previously proved only for values of parameters that are natural numbers. In this paper, we analyse and prove those inequalities for all real values of the parameters for which they are defined. Our approach has enabled selecting the best real constants for which those inequalities hold.

1. Introduction

Many inequalities

$$f(x) > 0,\ x \in (a,b)$$

where $f:[a,b] \longrightarrow R$, are generalised by introducing a real parameter to inequalities of the form

$$\varphi_p(x) > 0,\ x \in (a,b) \tag{1}$$

where $\varphi_p(x)$ is a function from the real family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\emptyset \neq \mathbb{P} \subseteq \mathbb{R}$, defined on $[a,b]$ (Chapter 2.14.6 [2, 5, 7, 17, 20, 21, 25, 27–30, 32, 34, 35, 37–39, 44, 46, 47, 49]). The main problem is to determine, when possible, values of the parameter p for which functions from the family are positive on (a,b). The inequalities $f(x) < 0$ and their generalisations of the form $\varphi_p(x) < 0$ are considered analogously.

Our approach to solving these problems is based on the use of so-called stratification of the family of functions [5,25,27–30,32,35]. The family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is increasingly stratified on the interval (a,b) iff

$$(\forall x \in (a,b))(\forall p_1, p_2 \in \mathbb{P})\ p_1 < p_2 \Longleftrightarrow \varphi_{p_1}(x) < \varphi_{p_2}(x),$$

Mathematics subject classification (2020): 41A44, 26D05, 26D07.
Keywords and phrases: Taylor expansions, stratified families of functions, D'Aurizio-Sándor inequalities.

This work was financially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract numbers: 451-03-137/2025-03/200103 (for the first, third and fourth authors) and 451-03-136/2025-03/200103 (for the second author). This research was conducted at the Center for Applied Mathematics, the Palace of Science; therefore, the authors wish to express their gratitude to the Miodrag Kostić Endowment, Belgrade.

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and, conversely, the family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ is decreasingly stratified on the interval (a,b) iff

$$(\forall x\in(a,b))(\forall p_1,p_2\in\mathbb{P})\ p_1<p_2 \Longleftrightarrow \varphi_{p_1}(x)>\varphi_{p_2}(x).$$

Let $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ be a stratified family of functions on the interval (a,b). We say that *the constant* $p_1\in\mathbb{P}$ *is the best possible* for the inequality $\varphi_p(x)>0$ on (a,b) and for $p\in\mathbb{P}$ iff $\varphi_{p_1}(x)>0$ on (a,b) and there is no constant $p_2\in\mathbb{P}$ such that

$$(\forall x\in(a,b))\ \varphi_{p_1}(x)>\varphi_{p_2}(x)>0.$$

Of particular interest in this paper are families of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, defined on $[a,b]$, $a<b$, which are stratified and such that each function $\varphi_p(x)$ has Taylor expansions at the points a and b with at least one non-zero coefficient. For such functions $\varphi_p(x)$, we examine the sign on the interval (a,b).

Let us outline the further structure of the paper. In Section 2, a method for proving mixed trigonometric polynomial inequalities is briefly described. Section 3 contains the main theoretical results of the paper through which we select possible real values of the parameter for which the functions from the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ are positive, i.e. negative on the corresponding interval. In Section 4, based on the method for proving mixed trigonometric polynomial inequalities and the results from Section 3, we extend D'Aurizio-Sándor-type inequalities [17, 39] from the values of parameters that are natural numbers to real values and obtain the best values for those real parameters. In the conclusion, we propose an open problem.

2. Mixed trigonometric polynomial inequalities

By a mixed trigonometric polynomial (MTP) inequality, we refer to an inequality of the form

$$f(x)>0,\ x\in\mathbb{I} \tag{2}$$

where $\mathbb{I}$ is an open, semi-open, or closed interval, and

$$f(x)=\sum_{i=1}^{n}\alpha_i x^{p_i}\cos^{q_i}x\sin^{r_i}x,$$

for $\alpha_i\in\mathbb{R}\backslash\{0\}, p_i,q_i,r_i\in\mathbb{N}_0$ and $n\in\mathbb{N}$, is an MTP function, see [9, 11–13, 18, 19, 22–24, 26, 36, 45].

One method for proving MTP inequalities over $\mathbb{I}$ was proposed in [5, 26]. According to that method, we first transform the MTP function $f(x)$ in the form

$$f(x)=\sum_{i=1}^{m}\beta_i x^{s_i}\mathbf{trig}_i(kx), \tag{3}$$

where $\beta_i\in\mathbb{R}\backslash\{0\}, s_i\in\mathbb{N}_0$, $\mathbf{trig}_i=\cos$ or $\mathbf{trig}_i=\sin$, $k\in\mathbb{Z}$ and $m\in\mathbb{N}$. Then, for the function $f(x)$, we find a positive downward polynomial approximation $P(x)$ (if it exists) by determining the corresponding polynomial approximations of the functions $\mathbf{trig}_i(kx)$ for $i = 1,2,\ldots,m$. If the coefficients of the polynomial $P(x)$ are not rational numbers, according to [23], it is possible to determine a corresponding positive downward polynomial approximation of the polynomial $P(x)$ with the polynomial $Q(x)$ whose coefficients are rational numbers. Thus, proving the inequality (2) is reduced to proving the polynomial inequality $P(x) > 0$, i.e. $Q(x) > 0$, for $x \in \mathbb{I}$. To effectively prove the positivity of polynomials, we use Sturm's theorem ([41], Theorem 4.1 and 4.2 [14]).

Let us note that the results provided in Lemmas 1.1 and 1.2 from [26] are used when forming the polynomial $P(x)$. By using those Lemmas, it is also possible to determine upward/downward polynomial approximations for the functions $y = \cos(kx)$ and $y = \sin(kx)$, where $k \in \mathbb{Z}$, which appears in the expression (3).

In this paper, we also consider the determination of upward/downward polynomial approximations for the functions $y = \cos(\tau x)$ and $y = \sin(\tau x)$, where $\tau \in \mathbb{R}$. For a function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ and $\tau \in \mathbb{R}$, let us denote

$$\phi_\tau(x) = \phi(\tau x).$$

If $T_n^{\phi,a}$ is the Taylor expansion of order n of a function ϕ in a neighbourhood of the point a, then for any $\tau \in \mathbb{R}$, it holds:

$$T_n^{\phi_\tau,0}(x) = T_n^{\phi,0}(\tau x). \tag{4}$$

The equality (4) for $\phi = \cos$ or $\phi = \sin$, based on Lemmas 1.1 and 1.2 from [26], allows us to determine some upward/downward polynomial approximations for the functions $y = \cos(\tau x)$ and $y = \sin(\tau x)$ as used in Section 4.

Let us note that since the method for proving MTP inequalities from [26] is computer-implemented within the prover SimTheP [4, 5], MTP inequalities could also be proved automatically.

3. Main results

The following theorem holds.

THEOREM 1. *Let $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\emptyset \neq \mathbb{P} \subseteq \mathbb{R}$, be a family defined on $[a,b]$, $a < b$, such that each function $\varphi_p(x)$ has Taylor expansions at the points a and b with at least one non-zero coefficient.*

(i) Let $\mathfrak{c}_n^{\varphi_p,a}$ be the first non-zero coefficient in the Taylor expansion

$$\varphi_p(x) = \sum_{i=0}^{n+\nu} \mathfrak{c}_i^{\varphi_p,a} (x-a)^i + o\left((x-a)^{n+\nu}\right), \tag{5}$$

where $n \in \mathbb{N}_0$, $\nu \in \mathbb{N}$.

If the equation

$$\mathfrak{c}_n^{\varphi_p,a} = 0 \tag{6}$$

has the solution $p = p_a \in \mathbb{P}$ *and for* p_a *there exists* $j \leqslant \nu$ *such that*

$$\mathfrak{c}_{n+1}^{\varphi_{p_a},a} = \mathfrak{c}_{n+2}^{\varphi_{p_a},a} = \ldots = \mathfrak{c}_{n+j-1}^{\varphi_{p_a},a} = 0 \wedge \mathfrak{c}_{n+j}^{\varphi_{p_a},a} < 0$$

$$\left(i.e.\ \mathfrak{c}_{n+1}^{\varphi_{p_a},a} = \mathfrak{c}_{n+2}^{\varphi_{p_a},a} = \ldots = \mathfrak{c}_{n+j-1}^{\varphi_{p_a},a} = 0 \wedge \mathfrak{c}_{n+j}^{\varphi_{p_a},a} > 0 \right),$$

then, there exists $b_1 \in (a,b]$ *such that for each* $x \in (a,b_1)$, *it holds:*

$$\varphi_{p_a}(x) < 0 \quad (i.e.\ \varphi_{p_a}(x) > 0).$$

(ii) *Let* $\mathfrak{c}_m^{\varphi_p,b}$ *be the first non-zero coefficient in the Taylor expansion*

$$\varphi_p(x) = \sum_{i=0}^{m+\mu} \mathfrak{c}_i^{\varphi_p,b} (x-b)^i + o\left((x-b)^{m+\mu}\right), \tag{7}$$

where $m \in \mathbb{N}_0$, $\mu \in \mathbb{N}$.

If the equation

$$\mathfrak{c}_m^{\varphi_p,b} = 0 \tag{8}$$

has the solution $p = p_b \in \mathbb{P}$ *and for* p_b *there exists* $k \leqslant \mu$ *such that*

$$\mathfrak{c}_{m+1}^{\varphi_{p_b},b} = \mathfrak{c}_{m+2}^{\varphi_{p_b},b} = \ldots = \mathfrak{c}_{m+k-1}^{\varphi_{p_b},b} = 0 \wedge (-1)^{m+k}\mathfrak{c}_{m+k}^{\varphi_{p_b},b} < 0$$

$$\left(i.e.\ \mathfrak{c}_{m+1}^{\varphi_{p_b},b} = \mathfrak{c}_{m+2}^{\varphi_{p_b},b} = \ldots = \mathfrak{c}_{m+k-1}^{\varphi_{p_b},b} = 0 \wedge (-1)^{m+k}\mathfrak{c}_{m+k}^{\varphi_{p_b},b} > 0 \right),$$

then, there exists $a_1 \in [a,b)$ *such that for each* $x \in (a_1,b)$, *it holds:*

$$\varphi_{p_b}(x) < 0 \quad \left(i.e.\ \varphi_{p_b}(x) > 0\right).$$

Proof. Based on the properties of the Taylor expansion of functions. □

REMARK 1. If in Theorem 1, the Taylor expansion (5) is such that the first non-zero coefficient $\mathfrak{c}_n^{\varphi_p,a}$ has a constant sign for all values of the parameter $p \in \mathbb{P}$, then all functions from the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ has the same sign in some right neighbourhood of the point a. For the Taylor expansion (7), the similar assertion holds.

The first non-zero coefficient $\mathfrak{c}_n^{\varphi_p,a}$ (i.e. $\mathfrak{c}_m^{\varphi_p,b}$) from Theorem 1, we call *the main coefficient*. If the main coefficient is a monotonic function with respect to $p \in \mathbb{P}$, then a solution to the equation (6) (i.e. the equation (8)) is unique if it exists.

For the stratified family of functions $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, based on the properties of the Taylor expansions of functions and properties of the stratified families of functions, the following assertion holds.

THEOREM 2. *Let* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\emptyset \neq \mathbb{P} \subseteq \mathbb{R}$, *be a family of functions defined on* $[a,b]$, $a<b$, *for which, it holds:*

(1) *the family is increasingly* (*i.e. decreasingly*) *stratified on the interval* (a,b),

(2) *the family satisfies conditions from* (i) *and* (ii) *in Theorem* 1,

(3) *the main coefficients from Theorem* 1 *are monotonic functions with respect to* $p\in\mathbb{P}$,

(4) *the set* $\mathbb{P}$ *contains the constants* p_a *and* p_b *from Theorem* 1.

Then, it holds:

(i) *For each* $d\in(a,b]$ *such that* $(\forall x\in(a,d))$ $\varphi_{p_a}(x)<0$ *and for each* $p'<p_a$ (*i.e. for* $p'>p_a$), *it holds:*

$$x\in(a,d) \Longrightarrow \varphi_{p'}(x) < \varphi_{p_a}(x) < 0. \tag{9}$$

Particularly, if $d=b$, *then* p_a *is the best possible constant for the inequality* $\varphi_p(x)<0$ *on the interval* (a,b), *for* $p\in\mathbb{P}$.

(ii) *For each* $c\in[a,b)$ *such that* $(\forall x\in(c,b))$ $\varphi_{p_b}(x)>0$ *and for each* $p''>p_b$ (*i.e. for* $p''<p_b$), *it holds:*

$$x\in(c,b) \Longrightarrow 0 < \varphi_{p_b}(x) < \varphi_{p''}(x). \tag{10}$$

Particularly, if $c=a$, *then* p_b *is the best possible constant for the inequality* $\varphi_p(x)>0$ *on the interval* (a,b), *for* $p\in\mathbb{P}$.

(iii) *For each* $d\in(a,b]$ *such that* $(\forall x\in(a,d))$ $\varphi_{p_a}(x)>0$ *and for each* $p''>p_a$ (*i.e. for* $p''<p_a$), *it holds:*

$$x\in(a,d) \Longrightarrow 0 < \varphi_{p_a}(x) < \varphi_{p''}(x). \tag{11}$$

Particularly, if $d=b$, *then* p_a *is the best possible constant for the inequality* $\varphi_p(x)>0$ *on the interval* (a,b), *for* $p\in\mathbb{P}$.

(iv) *For each* $c\in[a,b)$ *such that* $(\forall x\in(c,b))$ $\varphi_{p_b}(x)<0$ *and for each* $p'<p_b$ (*i.e. for* $p'>p_b$), *it holds:*

$$x\in(c,b) \Longrightarrow \varphi_{p'}(x) < \varphi_{p_b}(x) < 0. \tag{12}$$

Particularly, if $c=a$, *then* p_b *is the best possible constant for the inequality* $\varphi_p(x)<0$ *on the interval* (a,b), *for* $p\in\mathbb{P}$.

Proof. Based on the conditions (2), (3) and (4), the constants p_a and p_b are unique. The implications (9), (10), (11) and (12) follow from (1). Based on (1) and the fact that the main coefficients change the sign at the points p_a and p_b (the condition (3)), if $c=a$ and $d=b$, there are no better constants than p_a and p_b on the interval (a,b) and for $p\in\mathbb{P}$. □

Based on the previous results, a brief outline of our approach is given bellow.

First of all, we consider families of functions $\{\varphi_p(x)\}_{p\in\mathscr{P}}$, $\emptyset\neq\mathscr{P}\subseteq\mathbb{R}$ and $x\in[a,b]$, $a<b$, such that the functions $\varphi_p(x)$ are differentiable at least once with respect to the parameter p and that for each $p\in\mathscr{P}$, the function $\varphi_p(x)$ has Taylor expansions at the points a and b with at least one non-zero coefficient.

Next, we select, if it exists, a set $\mathbb{P}\subseteq\mathscr{P}$ such that:

(1) the family is stratified on the interval (a,b),

(2) the main coefficients are monotonic functions on the set $\mathbb{P}$ and have the zeros $p_a, p_b\in\mathbb{P}$.

If the functions $\varphi_{p_a}(x)$ and $\varphi_{p_b}(x)$ have a constant sign on the entire interval (a,b), then, based on Theorem 2, the constants p_a and p_b are the best possible for the corresponding inequalities on (a,b) and for $p\in\mathbb{P}$.

In the next section, we will illustrate the previously described method with specific examples.

4. Applications to some D'Aurizio-Sándor-type inequalities

In this section, we apply the described method to generalise some D'Aurizio-Sándor-type inequalities. Instead of a parameter that is a natural number, we introduce a real parameter and determine the best real constants for the observed inequalities.

The D'Aurizio-Sándor inequalities [15, 40] are given by:

THEOREM 3. *Let $x\in\left(0,\frac{\pi}{2}\right)$. Then, it holds that*

$$\frac{3}{8}<\frac{1-\dfrac{\cos x}{\cos\frac{x}{2}}}{x^2}<\frac{4}{\pi^2} \tag{13}$$

and

$$\frac{4}{\pi^2}\left(2-\sqrt{2}\right)<\frac{2-\dfrac{\sin x}{\sin\frac{x}{2}}}{x^2}<\frac{1}{4}.$$

The inequality (13) was generalised in [39] by introducing a parameter which is a natural number, as follows:

THEOREM 4. *For any $x\in\left(0,\frac{\pi}{2}\right)$ and all positive integers $n\geqslant 3$, one has*

$$\frac{4}{\pi^2}<\frac{1-\dfrac{\cos x}{\cos\frac{x}{n}}}{x^2}<\frac{n^2-1}{2n^2}.$$

In [17], the authors gave an additional generalisation of the D'Aurizio-Sándor inequalities through the following assertion.

THEOREM 5. *Let* $x \in \left(0, \dfrac{\pi}{2}\right)$. *Then the two double inequalities*

$$\frac{4}{\pi^2} < \frac{1 - \dfrac{\cos x}{\cos \frac{x}{p}}}{x^2} < \frac{p^2-1}{2p^2} \tag{14}$$

and

$$\frac{4}{\pi^2}\left(p - \csc\left(\frac{\pi}{2p}\right)\right) < \frac{p - \dfrac{\sin x}{\sin \frac{x}{p}}}{x^2} < \frac{p^2-1}{6p} \tag{15}$$

hold for $p = 3,4,5,\ldots$. *In particular,* (15) *remains true when* $p = 2$, *while* (14) *is reversed when* $p = 2$.

In the following two subsections, we analyse the double inequality (14) for real values of the parameter p and $x \in (0, \pi/2)$.

4.1. Extension of the left side of the inequality (14)

Based on the left side of the inequality (14), we introduce the family of functions $\{\varphi_p(x)\}_{p\in\mathscr{P}}$, where

$$\varphi_p(x) = \begin{cases} \dfrac{1 - \dfrac{\cos x}{\cos \frac{x}{p}}}{x^2} - \dfrac{4}{\pi^2}, & x \in (0, \pi/2] \\[2ex] \dfrac{1}{2} - \dfrac{1}{2p^2} - \dfrac{4}{\pi^2}, & x = 0, \end{cases} \tag{16}$$

which is defined for each $x \in [0, \pi/2]$ and the parameter $p \in \mathscr{P} = \mathbb{R} \setminus [-1, 1]$.

Due to the evenness of all functions $\varphi_p(x)$, it is sufficient to analyse the aforementioned family for $p \in (1, +\infty)$. Additionally, for this family, the following assertion holds.

LEMMA 1. *The family of functions* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\mathbb{P} = (1, +\infty)$, *is increasingly stratified on the interval* $(0, \pi/2)$.

Proof. It holds that

$$\frac{\partial \varphi_p(x)}{\partial p} = \frac{\cos x \sin \frac{x}{p}}{x\left(p\cos \frac{x}{p}\right)^2} > 0$$

for $p \in (1, +\infty)$ and $x \in (0, \pi/2)$. □

Further, we consider the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ for $\mathbb{P} = (1, +\infty)$ and $x \in [0, \pi/2]$.

The Taylor expansion of $\varphi_p(x)$ at the point 0 is

$$\varphi_p(x) = \left(\frac{1}{2} - \frac{1}{2p^2} - \frac{4}{\pi^2}\right) + \left(-\frac{1}{24} + \frac{1}{4p^2} - \frac{5}{24p^4}\right)x^2 + o\left(x^2\right) \tag{17}$$

and the Taylor expansion of $\varphi_p(x)$ at the point $\pi/2$ is

$$\begin{aligned}\varphi_p(x) = &\left(\frac{4}{\pi^2\cos\frac{\pi}{2p}} - \frac{16}{\pi^3}\right)\left(x - \frac{\pi}{2}\right)\\ &+ \left(\frac{4\sin\frac{\pi}{2p}}{p\pi^2\left(\cos\frac{\pi}{2p}\right)^2} - \frac{16}{\pi^3\cos\frac{\pi}{2p}} + \frac{48}{\pi^4}\right)\left(x - \frac{\pi}{2}\right)^2\\ &+ o\left(\left(x - \frac{\pi}{2}\right)^2\right).\end{aligned} \tag{18}$$

The first non-zero coefficient of the Taylor expansion (17) is a monotonic function with respect to $p \in (1, +\infty)$ given by $\mathfrak{c}_0^{\varphi_{p_0},0} = \frac{1}{2} - \frac{1}{2p^2} - \frac{4}{\pi^2}$.

The solution to the equation $\mathfrak{c}_0^{\varphi_{p_0},0} = \frac{1}{2} - \frac{1}{2p^2} - \frac{4}{\pi^2} = 0$ is unique with respect to $p \in (1, +\infty)$ and is given by the constant $p = p_0 = \dfrac{\pi}{\sqrt{\pi^2 - 8}} = 2.29760\ldots \in (1, +\infty)$.

The first non-zero coefficient of the Taylor expansion (18) is a monotonic function with respect to $p \in (1, +\infty)$ given by $\mathfrak{c}_1^{\varphi_{p_{\pi/2}},\pi/2} = \frac{4}{\pi^2\cos\frac{\pi}{2p}} - \frac{16}{\pi^3}$.

The solution to the equation $\mathfrak{c}_1^{\varphi_{p_{\pi/2}},\pi/2} = \frac{4}{\pi^2\cos\frac{\pi}{2p}} - \frac{16}{\pi^3} = 0$ is unique with respect to $p \in (1, +\infty)$ and is given by the constant $p = p_{\pi/2} = \dfrac{\pi}{2\arccos\frac{\pi}{4}} = 2.35340\ldots \in (1, +\infty)$.

Based on Theorem 1, it follows:

1) since the coefficient $\mathfrak{c}_2^{\varphi_{p_0},0} = -\frac{1}{24} + \frac{1}{4{p_0}^2} - \frac{5}{24{p_0}^4} < 0$, we conclude that there exists a right neighbourhood of the point 0 such that $\varphi_{p_0}(x) < 0$, and

2) since the coefficient $\mathfrak{c}_2^{\varphi_{p_{\pi/2}},\pi/2} = (-1)^2\left(\dfrac{4\sin\frac{\pi}{2p_{\pi/2}}}{p_{\pi/2}\pi^2\left(\cos\frac{\pi}{2p_{\pi/2}}\right)^2} - \dfrac{16}{\pi^3\cos\frac{\pi}{2p_{\pi/2}}} + \dfrac{48}{\pi^4}\right) > 0$, we conclude that there exists a left neighbourhood of the point $\pi/2$ such that $\varphi_{p_{\pi/2}}(x) > 0$.

In Theorem 6, we prove that $\varphi_{p_0}(x) < 0$ on the entire interval $(0, \pi/2)$, while in Theorem 7, we prove that $\varphi_{p_{\pi/2}}(x) > 0$ on the entire interval $(0, \pi/2)$.

THEOREM 6. *For* $x \in \left(0, \frac{\pi}{2}\right)$ *and* $p = p_0 = \dfrac{\pi}{\sqrt{\pi^2 - 8}}$*, it holds that*

$$\frac{4}{\pi^2} > \frac{1 - \dfrac{\cos x}{\cos\frac{x}{p}}}{x^2}.$$

The constant $p_0 = \dfrac{\pi}{\sqrt{\pi^2-8}}$ *is the best possible for the previous inequality and for the parameter* $p \in (1,+\infty)$.

Proof. For $p_0 = \frac{\pi}{\sqrt{\pi^2-8}}$, it holds that

$$\varphi_{p_0}(x) = \frac{\left(\pi^2-4x^2\right)\cos\frac{x\sqrt{\pi^2-8}}{\pi} - \pi^2\cos x}{\pi^2 x^2 \cos\frac{x\sqrt{\pi^2-8}}{\pi}}.$$

In order to examine the sign of the function

$$f(x) = \left(\pi^2-4x^2\right)\cos\frac{x\sqrt{\pi^2-8}}{\pi} - \pi^2\cos x$$

on the interval $(0,\pi/2)$, we consider the following two cases:

1. $x \in (0,1.5]$:

If we approximate the function $\cos\frac{x\sqrt{\pi^2-8}}{\pi}$ by the Maclaurin polynomial of degree 4, and the function $\cos x$ by the Maclaurin polynomial of degree 6, then the function $f(x)$ has the upward polynomial approximation

$$\begin{aligned} P_1(x) &= \left(\pi^2-4x^2\right) T_4^{\cos,0}\left(\frac{x\sqrt{\pi^2-8}}{\pi}\right) - \pi^2 T_6^{\cos,0}(x) \\ &= \left(\tfrac{\pi^2}{720} - \tfrac{1}{6} + \tfrac{8}{3\pi^2} - \tfrac{32}{3\pi^4}\right)x^6 + \left(\tfrac{4}{3} - \tfrac{40}{3\pi^2}\right)x^4 \\ &= x^4\left(\left(\tfrac{\pi^2}{720} - \tfrac{1}{6} + \tfrac{8}{3\pi^2} - \tfrac{32}{3\pi^4}\right)x^2 + \tfrac{4}{3} - \tfrac{40}{3\pi^2}\right) \end{aligned}$$

on the interval $(0,1.5]$. It is easy to prove that $P_1(x) < 0$ on the interval $(0,1.5]$. Thus,

$$f(x) < 0$$

on the interval $(0,1.5]$.

2. $x \in (1.5,\pi/2)$:

Let us prove that $f(x) < 0$ also holds on the interval $(1.5,\pi/2)$. By substitution $x = \pi/2 - t$, we obtain the function

$$\begin{aligned} g(t) = f\left(\tfrac{\pi}{2} - t\right) &= 4t\left(\pi - t\right)\cos\frac{(\pi-2t)\sqrt{\pi^2-8}}{2\pi} - \pi^2\sin t \\ &= 4t\left(\pi - t\right)\cos\frac{\sqrt{\pi^2-8}}{2}\cos\frac{t\sqrt{\pi^2-8}}{\pi} \\ &\quad + 4t\left(\pi - t\right)\sin\frac{\sqrt{\pi^2-8}}{2}\sin\frac{t\sqrt{\pi^2-8}}{\pi} - \pi^2\sin t, \end{aligned}$$

where $t \in (0,\pi/2-1.5)$. Notice that the function g is defined on the entire set $\mathbb{R}$. We will prove that $g(t)$ is negative on the interval $(0,0.1] \supset (0,\pi/2-1.5)$. If we approximate the functions $\cos\frac{t\sqrt{\pi^2-8}}{\pi}$, $\sin\frac{t\sqrt{\pi^2-8}}{\pi}$ and $\sin t$ by the Maclaurin polynomials of degrees 4, 1 and 3, respectively, then the function $g(t)$ has the upward polynomial approximation

$$\begin{aligned}
P_2(t) &= 4t\,(\pi-t)\cos\frac{\sqrt{\pi^2-8}}{2}T_4^{\cos,0}\left(\frac{t\sqrt{\pi^2-8}}{\pi}\right)\\
&\quad +4t\,(\pi-t)\sin\frac{\sqrt{\pi^2-8}}{2}T_1^{\sin,0}\left(\frac{t\sqrt{\pi^2-8}}{\pi}\right)-\pi^2T_3^{\sin,0}(t)\\
&=\left(-\frac{\cos\frac{\sqrt{\pi^2-8}}{2}}{6}+\frac{8\cos\frac{\sqrt{\pi^2-8}}{2}}{3\pi^2}-\frac{32\cos\frac{\sqrt{\pi^2-8}}{2}}{3\pi^4}\right)t^6\\
&\quad +\left(\frac{\pi\cos\frac{\sqrt{\pi^2-8}}{2}}{6}-\frac{8\cos\frac{\sqrt{\pi^2-8}}{2}}{3\pi}+\frac{32\cos\frac{\sqrt{\pi^2-8}}{2}}{3\pi^3}\right)t^5\\
&\quad +\left(2\cos\frac{\sqrt{\pi^2-8}}{2}-\frac{16\cos\frac{\sqrt{\pi^2-8}}{2}}{\pi^2}\right)t^4\\
&\quad +\left(-2\pi\cos\frac{\sqrt{\pi^2-8}}{2}+\frac{16\cos\frac{\sqrt{\pi^2-8}}{2}}{\pi}-\frac{4\sqrt{\pi^2-8}\sin\frac{\sqrt{\pi^2-8}}{2}}{\pi}+\frac{\pi^2}{6}\right)t^3\\
&\quad +\left(4\sqrt{\pi^2-8}\sin\frac{\sqrt{\pi^2-8}}{2}-4\cos\frac{\sqrt{\pi^2-8}}{2}\right)t^2\\
&\quad +\left(-\pi^2+4\pi\cos\frac{\sqrt{\pi^2-8}}{2}\right)t\\
&=-0.00463\ldots t^6+0.0145\ldots t^5+0.293\ldots t^4-0.377\ldots t^3\\
&\quad +0.353\ldots t^2-0.127\ldots t
\end{aligned}$$

on the interval $(0,0.1]$. The polynomial $P_2(t)$ has the upward polynomial approximation with rational coefficients

$$\begin{aligned}
Q_2(t) &= -0.0046t^6+0.015t^5+0.3t^4-0.37t^3+0.36t^2-0.12t\\
&= t\left(-\frac{23}{5000}t^5+\frac{3}{200}t^4+\frac{3}{10}t^3-\frac{37}{100}t^2+\frac{9}{25}t-\frac{3}{25}\right)
\end{aligned}$$

on the interval $(0,0.1]$. By applying Sturm's theorem to the polynomial

$$-\frac{23}{5000}t^5+\frac{3}{200}t^4+\frac{3}{10}t^3-\frac{37}{100}t^2+\frac{9}{25}t-\frac{3}{25}$$

on the segment $[0,0.1]$, it can be concluded that this polynomial has no zeros on the segment $[0,0.1]$. Hence, the polynomial $Q_2(t)$ has no zeros on the interval $(0,0.1]$. Considering that $Q_2(0.1)=-\frac{43699273}{5000000000}=-0.0087398\ldots<0$, it holds that $Q_2(t)<0$ on the interval $(0,0.1]$. Therefore,

$$g(t)<0$$

on the interval $(0,0.1]$ since $g(t)<P_2(t)<Q_2(t)<0$ on the interval $(0,0.1]$. Thus,

$$f(x)<0$$

on the interval $[\pi/2-0.1,\pi/2)$, and consequently on the interval $(1.5,\pi/2)$.

Based on the previous two cases, it is evident that $\varphi_{p_0}(x)<0$ on the interval $(0,\pi/2)$.

Based on Theorem 2 and Lemma 1, the constant $p_0=\dfrac{\pi}{\sqrt{\pi^2-8}}$ is the best possible for $p\in(1,+\infty)$. $\square$

THEOREM 7. *For $x\in\left(0,\dfrac{\pi}{2}\right)$ and $p=p_{\pi/2}=\dfrac{\pi}{2\arccos\frac{\pi}{4}}$, it holds that*

$$\frac{4}{\pi^2}<\frac{1-\dfrac{\cos x}{\cos\frac{x}{p}}}{x^2}.$$

The constant $p_{\pi/2}=\dfrac{\pi}{2\arccos\frac{\pi}{4}}$ is the best possible for the previous inequality and for the parameter $p\in(1,+\infty)$.

Proof. For $p_{\pi/2}=\dfrac{\pi}{2\arccos\frac{\pi}{4}}$, it holds that

$$\varphi_{p_{\pi/2}}(x)=\frac{\left(\pi^2-4x^2\right)\cos\dfrac{2x\arccos\frac{\pi}{4}}{\pi}-\pi^2\cos x}{\pi^2x^2\cos\dfrac{2x\arccos\frac{\pi}{4}}{\pi}}.$$

In order to examine the sign of the function

$$f(x)=\left(\pi^2-4x^2\right)\cos\frac{2x\arccos\frac{\pi}{4}}{\pi}-\pi^2\cos x$$

on the interval $(0,\pi/2)$, we consider the following two cases:

1. $x\in(0,0.9]$:

If we approximate the function $\cos\dfrac{2x\arccos\frac{\pi}{4}}{\pi}$ by the Maclaurin polynomial of degree 2, and the function $\cos x$ by the Maclaurin polynomial of degree 4, then the function $f(x)$ has the downward polynomial approximation

$$\begin{aligned}P_1(x)&=\left(\pi^2-4x^2\right)T_2^{\cos,0}\left(\frac{2x\arccos\frac{\pi}{4}}{\pi}\right)-\pi^2T_4^{\cos,0}(x)\\&=\left(-\tfrac{\pi^2}{24}+\tfrac{8\left(\arccos\frac{\pi}{4}\right)^2}{\pi^2}\right)x^4+\left(\tfrac{\pi^2}{2}-4-2\left(\arccos\tfrac{\pi}{4}\right)^2\right)x^2\\&=x^2\left(\left(-\tfrac{\pi^2}{24}+\tfrac{8\left(\arccos\frac{\pi}{4}\right)^2}{\pi^2}\right)x^2+\tfrac{\pi^2}{2}-4-2\left(\arccos\tfrac{\pi}{4}\right)^2\right)\end{aligned}$$

on the interval $(0,0.9]$. It is easy to prove that $P_1(x)>0$ on the interval $(0,0.9]$. Thus,

$$f(x)>0$$

on the interval $(0,0.9]$.

2. $x\in(0.9,\pi/2)$:

Let us prove that $f(x)>0$ also holds on the interval $(0.9,\pi/2)$. By substitution $x=\pi/2-t$, we obtain the function

$$\begin{aligned} g(t)=f\left(\tfrac{\pi}{2}-t\right) &= 4t\,(\pi-t)\cos\tfrac{(\pi-2t)\arccos\frac{\pi}{4}}{\pi}-\pi^2\sin t\\ &= t\,(\pi-t)\,\pi\cos\tfrac{2t\arccos\frac{\pi}{4}}{\pi}\\ &\quad +t\,(\pi-t)\sqrt{16-\pi^2}\sin\tfrac{2t\arccos\frac{\pi}{4}}{\pi}-\pi^2\sin t, \end{aligned}$$

where $t\in(0,\pi/2-0.9)$. Notice that the function g is defined on the entire set $\mathbb{R}$. We will prove that $g(t)$ is negative on the interval $(0,0.7]\supset(0,\pi/2-0.9)$. If we approximate the functions $\cos\frac{2t\arccos\frac{\pi}{4}}{\pi}$, $\sin\frac{2t\arccos\frac{\pi}{4}}{\pi}$ and $\sin t$ by the Maclaurin polynomials of degrees 2, 3 and 5, respectively, then the function $g(t)$ has the downward polynomial approximation

$$\begin{aligned} P_2(t) &= t\,(\pi-t)\,\pi T_2^{\cos,0}\left(\tfrac{2t\arccos\frac{\pi}{4}}{\pi}\right)\\ &\quad +t\,(\pi-t)\sqrt{16-\pi^2}\,T_3^{\sin,0}\left(\tfrac{2t\arccos\frac{\pi}{4}}{\pi}\right)-\pi^2T_5^{\sin,0}(t)\\ &= \left(\tfrac{4\sqrt{16-\pi^2}\left(\arccos\frac{\pi}{4}\right)^3}{3\pi^3}-\tfrac{\pi^2}{120}\right)t^5\\ &\quad +\left(\tfrac{2\left(\arccos\frac{\pi}{4}\right)^2}{\pi}-\tfrac{4\sqrt{16-\pi^2}\left(\arccos\frac{\pi}{4}\right)^3}{3\pi^2}\right)t^4\\ &\quad +\left(-2\left(\arccos\tfrac{\pi}{4}\right)^2-\tfrac{2\sqrt{16-\pi^2}\arccos\frac{\pi}{4}}{\pi}+\tfrac{\pi^2}{6}\right)t^3\\ &\quad +\left(2\sqrt{16-\pi^2}\arccos\tfrac{\pi}{4}-\pi\right)t^2\\ &= -0.0505\ldots t^5+0.184\ldots t^4-0.298\ldots t^3+0.163\ldots t^2 \end{aligned}$$

on the interval $(0,0.7]$. The polynomial $P_2(t)$ has the downward polynomial approximation with rational coefficients

$$\begin{aligned} Q_2(t) &= -0.051t^5+0.18t^4-0.3t^3+0.16t^2\\ &= -t^2\left(\frac{51}{1000}t^3-\frac{9}{50}t^2+\frac{3}{10}t-\frac{4}{25}\right) \end{aligned}$$

on the interval $(0,0.7]$. By applying Sturm's theorem to the polynomial

$$\frac{51}{1000}t^3-\frac{9}{50}t^2+\frac{3}{10}t-\frac{4}{25}$$

on the segment $[0,0.7]$, it can be concluded that this polynomial has no zeros on the segment $[0,0.7]$. Hence, the polynomial $Q_2(t)$ has no zeros on the interval $(0,0.7]$.

Considering that $Q_2(0.7) = \frac{1014643}{100000000} = 0.010146\ldots > 0$, it holds that $Q_2(t) > 0$ on the interval $(0, 0.7]$. Therefore,

$$g(t) > 0$$

on the interval $(0, 0.7]$ since $g(t) > P_2(t) > Q_2(t) > 0$ on the interval $(0, 0.7]$. Thus,

$$f(x) > 0$$

on the interval $[\pi/2 - 0.7, \pi/2)$, and consequently on the interval $(0.9, \pi/2)$.

Based on the previous two cases, it is evident that $\varphi_{p_{\pi/2}}(x) > 0$ on the interval $(0, \pi/2)$.

Based on Theorem 2 and Lemma 1, the constant $p_{\pi/2} = \dfrac{\pi}{2\arccos\frac{\pi}{4}}$ is the best possible for $p \in (1, +\infty)$. $\square$

Therefore, based on the previous analysis, the following general statement holds.

THEOREM 8. *Let:*

$$A = p_0 = \frac{\pi}{\sqrt{\pi^2 - 8}} \quad \textit{and} \quad B = p_{\pi/2} = \frac{\pi}{2\arccos\frac{\pi}{4}}.$$

Then, it holds:

(i) If $p \in (1, A)$, then

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \ \frac{1 - \dfrac{\cos x}{\cos\frac{x}{p}}}{x^2} < \frac{1 - \dfrac{\cos x}{\cos\frac{x}{A}}}{x^2} < \frac{4}{\pi^2}$$

and the constant A is the best possible.

(ii) If $p \in (B, +\infty)$, then

$$\left(\forall x \in \left(0, \frac{\pi}{2}\right)\right) \ \frac{4}{\pi^2} < \frac{1 - \dfrac{\cos x}{\cos\frac{x}{B}}}{x^2} < \frac{1 - \dfrac{\cos x}{\cos\frac{x}{p}}}{x^2}$$

and the constant B is the best possible.

Figure 1 illustrates the stratified family of functions defined by (16), with particular emphasis on the cases when $p = 2$, $p = 3$, $p = 4$ and $p = 5$ from Theorem 5, as well as when the parameter p is equal to the best constants according to Theorem 8.

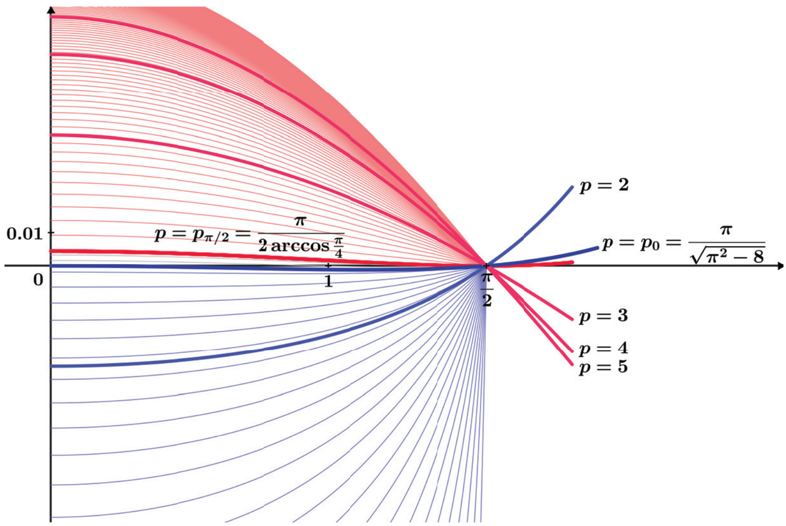

Figure 1: *Stratified family of functions defined by* (16)

4.2. Extension of the right side of the inequality (14)

Based on the right side of the inequality (14), we introduce the family of functions $\{\varphi_p(x)\}_{p\in\mathscr{P}}$, where

$$\varphi_p(x) = \begin{cases} \dfrac{1-\dfrac{\cos x}{\cos\frac{x}{p}}}{x^2} - \dfrac{p^2-1}{2p^2}, & x\in(0,\pi/2] \\ 0, & x=0, \end{cases} \tag{19}$$

which is defined for each $x\in[0,\pi/2]$ and the parameter $p\in\mathscr{P}=\mathbb{R}\setminus[-1,1]$.

Due to the evenness of all functions $\varphi_p(x)$, it is sufficient to analyse the aforementioned family for $p\in(1,+\infty)$. Additionally, for this family, the following assertion holds.

LEMMA 2. *The family of functions* $\{\varphi_p(x)\}_{p\in\mathbb{P}}$, $\mathbb{P}=\left[\frac{\sqrt{15}}{3},+\infty\right)$, *is decreasingly stratified on the interval* $(0,\pi/2)$.

Proof. It holds that

$$\frac{\partial\varphi_p(x)}{\partial p} = \frac{p\cos x\sin\frac{x}{p} - x\left(\cos\frac{x}{p}\right)^2}{xp^3\left(\cos\frac{x}{p}\right)^2}.$$

Let us introduce the family of functions $\{f_p(x)\}_{p\in\mathbb{P}}$, defined by

$$f_p(x) = p\cos x \sin\frac{x}{p} - x\left(\cos\frac{x}{p}\right)^2, \tag{20}$$

on the interval $[0,\pi/2]$. The Taylor expansion of $f_p(x)$ at the point 0 is

$$f_p(x) = \left(-\frac{1}{2}+\frac{5}{6p^2}\right)x^3 + \left(\frac{1}{24}+\frac{1}{12p^2}-\frac{13}{40p^4}\right)x^5 + o\left(x^5\right). \tag{21}$$

Therefore, based on Theorem 1, there exists a right neighbourhood of the point 0 such that $f_{\sqrt{15}/3}(x) < 0$ ($\sqrt{15}/3 = 1.29099\ldots$).

Let us prove that $f_p(x) < 0$ for $p \geqslant \sqrt{15}/3$, when $x\in(0,\pi/2)$. By introducing substitution $x = t\,p$ in (20), we obtain:

$$f_p(x) = f_p(t\,p) = p\cos(t\,p)\sin t - t\,p\,(\cos t)^2 = p\left(\cos(t\,p)\sin t - t\,(\cos t)^2\right).$$

Let us observe the family of functions $\{g_p(t)\}_{p\in\mathbb{P}}$, where

$$g_p(t) = \cos(t\,p)\sin t - t\,(\cos t)^2, \quad \text{for } p \geqslant \sqrt{15}/3 \text{ and } t = \frac{x}{p} \in \left(0, \tfrac{3\pi}{2\sqrt{15}}\right) \subset (0,\pi/2).$$

It holds that

$$\frac{\partial g_p(t)}{\partial p} = -t\sin(t)\sin(t\,p) < 0$$

given that $t\,p\in(0,\pi/2)$. Thus, the family of functions $\{g_p(t)\}_{p\in\mathbb{P}}$ is decreasingly stratified for $p \geqslant \sqrt{15}/3$ and $t\in(0,\pi/2)$. Let us prove that the function

$$\begin{aligned} g_{\frac{\sqrt{15}}{3}}(t) &= \cos\tfrac{t\sqrt{15}}{3}\sin t - t\,(\cos t)^2 \\ &= \tfrac{1}{2}\sin\tfrac{t\left(\sqrt{15}+3\right)}{3} - \tfrac{1}{2}\sin\tfrac{t\left(\sqrt{15}-3\right)}{3} - \tfrac{1}{2}t\cos(2t) - \tfrac{1}{2}t \end{aligned}$$

is negative for $t\in(0,\pi/2)$. If we approximate the functions $\sin\tfrac{t\left(\sqrt{15}+3\right)}{3}$, $\sin\tfrac{t\left(\sqrt{15}-3\right)}{3}$ and $\cos(2t)$ by the Maclaurin polynomials of degrees 9, 3 and 6, respectively, then the function $g_{\frac{\sqrt{15}}{3}}(t)$ has the upward polynomial approximation

$$\begin{aligned} P(t) &= \tfrac{1}{2}T_9^{\sin,0}\left(\tfrac{t\left(\sqrt{15}+3\right)}{3}\right) - \tfrac{1}{2}T_3^{\sin,0}\left(\tfrac{t\left(\sqrt{15}-3\right)}{3}\right) - \tfrac{1}{2}t\,T_6^{\cos,0}(2t) - \tfrac{1}{2}t \\ &= \left(\tfrac{487\sqrt{15}}{1574640}+\tfrac{163}{136080}\right)t^9 + \left(-\tfrac{433\sqrt{15}}{102060}+\tfrac{953}{34020}\right)t^7 + \left(\tfrac{11\sqrt{15}}{324}-\tfrac{109}{540}\right)t^5 \\ &= t^5\left(\left(\tfrac{487\sqrt{15}}{1574640}+\tfrac{163}{136080}\right)t^4 + \left(-\tfrac{433\sqrt{15}}{102060}+\tfrac{953}{34020}\right)t^2 + \tfrac{11\sqrt{15}}{324}-\tfrac{109}{540}\right). \end{aligned}$$

for $t\in(0,\pi/2)$. It is easy to prove that $P(t) < 0$ for $t\in(0,\pi/2)$. Thus,

$$g_{\frac{\sqrt{15}}{3}}(t) < 0$$

for $t \in (0,\pi/2)$. Based on the decreasing stratification of the family of functions $\{g_p(t)\}_{p\in\mathbb{P}}$, it follows that

$$g_p(t) < 0 \;\text{ for }\; p \geqslant \tfrac{\sqrt{15}}{3} \;\text{ and }\; t \in (0,\pi/2).$$

Since $f_p(t\,p) = p \cdot g_p(t)$, it holds that

$$f_p(x) = f_p(t\,p) < 0$$

for $p \geqslant \frac{\sqrt{15}}{3}$ and $x \in (0,\pi/2)$. Hence, it holds that

$$\frac{\partial \varphi_p(x)}{\partial p} = \frac{f_p(x)}{xp^3\left(\cos\frac{x}{p}\right)^2} < 0$$

for $p \in \left[\frac{\sqrt{15}}{3}, +\infty\right)$ on the interval $(0,\pi/2)$. □

For $p \in \left(1, \sqrt{15}/3\right)$, the family of functions $\{\varphi_p(x)\}_{p\in(1,+\infty)}$ defined by (19) is not stratified on the interval $(0,\pi/2)$, which we prove at the end of this subsection, in Lemma 3.

Further, we consider the family $\{\varphi_p(x)\}_{p\in\mathbb{P}}$ for $\mathbb{P} = [\sqrt{15}/3, +\infty)$ and $x \in [0,\pi/2]$. The Taylor expansion of $\varphi_p(x)$ at the point 0 is

$$\varphi_p(x) = \left(-\frac{1}{24} + \frac{1}{4p^2} - \frac{5}{24p^4}\right)x^2 + \left(\frac{1}{720} - \frac{1}{48p^2} + \frac{5}{48p^4} - \frac{61}{720p^6}\right)x^4 + o\left(x^4\right) \quad (22)$$

and the Taylor expansion of $\varphi_p(x)$ at the point $\pi/2$ is

$$\varphi_p(x) = \left(-\frac{1}{2} + \frac{1}{2p^2} + \frac{4}{\pi^2}\right) + \left(\frac{4}{\pi^2\cos\frac{\pi}{2p}} - \frac{16}{\pi^3}\right)\left(x - \frac{\pi}{2}\right) + o\left(\left(x - \frac{\pi}{2}\right)\right). \quad (23)$$

The first non-zero coefficient of the Taylor expansion (22) is a monotonic function with respect to $p \in [\sqrt{15}/3, +\infty)$ given by $\mathfrak{c}_2^{\varphi_{p_0},0} = -\frac{1}{24} + \frac{1}{4p^2} - \frac{5}{24p^4}$.

The solution to the equation $\mathfrak{c}_2^{\varphi_{p_0},0} = -\frac{1}{24} + \frac{1}{4p^2} - \frac{5}{24p^4} = 0$ is unique with respect to $p \in [\sqrt{15}/3, +\infty)$ and is given by the constant $p = p_0 = \sqrt{5} = 2.23606\ldots \in [\sqrt{15}/3, +\infty)$.

The first non-zero coefficient of the Taylor expansion (23) is a monotonic function with respect to $p \in [\sqrt{15}/3, +\infty)$ given by $\mathfrak{c}_0^{\varphi_{p_{\pi/2}},\pi/2} = -\frac{1}{2} + \frac{1}{2p^2} + \frac{4}{\pi^2}$.

The solution to the equation $\mathfrak{c}_0^{\varphi_{p_{\pi/2}},\pi/2} = -\frac{1}{2} + \frac{1}{2p^2} + \frac{4}{\pi^2} = 0$ is unique with respect to $p \in [\sqrt{15}/3, +\infty)$ and is given by the constant $p = p_{\pi/2} = \dfrac{\pi}{\sqrt{\pi^2 - 8}} = 2.29760\ldots \in [\sqrt{15}/3, +\infty)$.

Based on Theorem 1, it follows:

1) since the coefficient $\mathfrak{c}_4^{\varphi_{p_0},0} = \frac{1}{720} - \frac{1}{48p_0^2} + \frac{5}{48p_0^4} - \frac{61}{720p_0^6} > 0$, we conclude that there exists a right neighbourhood of the point 0 such that $\varphi_{p_0}(x) > 0$.

2) since the coefficient $\mathfrak{c}_1^{\varphi_{p_{\pi/2}},\pi/2} = (-1)^1 \left(\frac{4}{\pi^2 \cos \frac{\pi}{2p_{\pi/2}}} - \frac{16}{\pi^3} \right) < 0$, we conclude that there exists a left neighbourhood of the point $\pi/2$ such that $\varphi_{p_{\pi/2}}(x) < 0$.

In Theorem 9, we prove that $\varphi_{p_{\pi/2}}(x) < 0$ on the entire interval $(0,\pi/2)$, while in Theorem 10, we prove that $\varphi_{p_0}(x) > 0$ on the entire interval $(0,\pi/2)$.

THEOREM 9. *For $x \in \left(0, \frac{\pi}{2}\right)$ and $p = p_{\pi/2} = \dfrac{\pi}{\sqrt{\pi^2-8}}$, it holds that*

$$\frac{1 - \dfrac{\cos x}{\cos \frac{x}{p}}}{x^2} < \frac{p^2-1}{2p^2}.$$

The constant $p_{\pi/2} = \dfrac{\pi}{\sqrt{\pi^2-8}}$ is the best possible for the previous inequality and for the parameter $p \in [\sqrt{15}/3, +\infty)$.

Proof. For $p_{\pi/2} = \dfrac{\pi}{\sqrt{\pi^2-8}}$, it holds that

$$\varphi_{p_{\pi/2}}(x) = \frac{\left(\pi^2 - 4x^2\right) \cos \dfrac{x\sqrt{\pi^2-8}}{\pi} - \pi^2 \cos x}{\pi^2 x^2 \cos \dfrac{x\sqrt{\pi^2-8}}{\pi}}.$$

In Theorem 6, it has already been proved that the aforementioned function is negative on the interval $(0,\pi/2)$. Based on Theorem 2 and Lemma 2, the constant $p_{\pi/2} = \dfrac{\pi}{\sqrt{\pi^2-8}}$ is the best possible for $p \in [\sqrt{15}/3, +\infty)$. □

THEOREM 10. *For $x \in \left(0, \frac{\pi}{2}\right)$ and $p = p_0 = \sqrt{5}$, it holds that*

$$\frac{1 - \dfrac{\cos x}{\cos \frac{x}{p}}}{x^2} > \frac{p^2-1}{2p^2}.$$

The constant $p_0 = \sqrt{5}$ is the best possible for the previous inequality and for the parameter $p \in [\sqrt{15}/3, +\infty)$.

Proof. For $p_0 = \sqrt{5}$, it holds that

$$\varphi_{p_0}(x) = \frac{-2x^2 \cos \frac{x\sqrt{5}}{5} + 5 \cos \frac{x\sqrt{5}}{5} - 5 \cos x}{5x^2 \cos \frac{x\sqrt{5}}{5}}.$$

Let us examine the sign of the function

$$f(x) = -2x^2\cos\frac{x\sqrt{5}}{5} + 5\cos\frac{x\sqrt{5}}{5} - 5\cos x \tag{24}$$

on the interval $(0,\pi/2)$. If we approximate the function $\cos\frac{x\sqrt{5}}{5}$ by the Maclaurin polynomial of degree 4 in the first addend of the function (24) and by the Maclaurin polynomial of degree 6 in the second addend of the function (24), and the function $\cos x$ by the Maclaurin polynomial of degree 8, then the function $f(x)$ has the downward polynomial approximation

$$\begin{aligned} P(x) &= -2x^2T_4^{\cos,0}\left(\tfrac{x\sqrt{5}}{5}\right) + 5T_6^{\cos,0}\left(\tfrac{x\sqrt{5}}{5}\right) - 5T_8^{\cos,0}(x) \\ &= -\tfrac{1}{8064}x^8 + \tfrac{4}{1125}x^6 = x^6\left(-\tfrac{1}{8064}x^2 + \tfrac{4}{1125}\right) \end{aligned}$$

on the interval $(0,\pi/2)$. It is easy to prove that $P(x) > 0$ on the interval $(0,\pi/2)$. Thus,

$$f(x) > 0$$

on the interval $(0,\pi/2)$. Therefore, it is evident that $\varphi_{p_0}(x) > 0$ on the interval $(0,\pi/2)$. Based on Theorem 2 and Lemma 2, the constant $p_0 = \sqrt{5}$ is the best possible for $p \in [\sqrt{15}/3, +\infty)$. $\square$

Therefore, based on the previous analysis, the following general statement holds.

THEOREM 11. *Let:*

$$C = \frac{\sqrt{15}}{3}, \quad A = p_0 = \sqrt{5} \quad \text{and} \quad B = p_{\pi/2} = \frac{\pi}{\sqrt{\pi^2-8}}.$$

Then, it holds:

(i) If $p \in [C,A)$, then

$$\left(\forall x \in \left(0,\frac{\pi}{2}\right)\right)\ 0 < \frac{1-\dfrac{\cos x}{\cos\frac{x}{A}}}{x^2} - \frac{A^2-1}{2A^2} < \frac{1-\dfrac{\cos x}{\cos\frac{x}{p}}}{x^2} - \frac{p^2-1}{2p^2}$$

and the constant A is the best possible.

(ii) If $p \in (B,+\infty)$, then

$$\left(\forall x \in \left(0,\frac{\pi}{2}\right)\right)\ \frac{1-\dfrac{\cos x}{\cos\frac{x}{p}}}{x^2} - \frac{p^2-1}{2p^2} < \frac{1-\dfrac{\cos x}{\cos\frac{x}{B}}}{x^2} - \frac{B^2-1}{2B^2} < 0$$

and the constant B is the best possible.

Figure 2 illustrates the stratified family of functions defined by (19) for some values of the parameter $p \in [\sqrt{15}/3, +\infty)$ for which this family is stratified, with particular emphasis on the cases when $p=2$, $p=3$, $p=4$ and $p=5$ from Theorem 5, as well as when the parameter p is equal to the best constants according to Theorem 11.

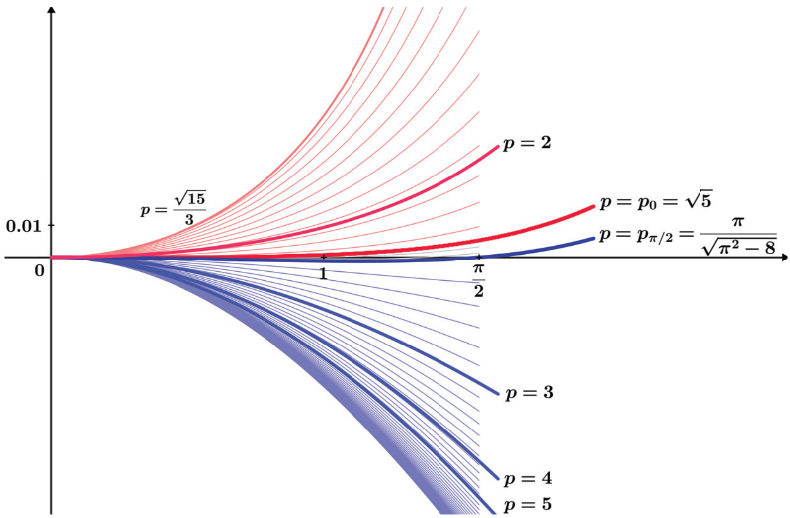

Figure 2: *The stratified part of family of functions defined by* (19)

Lastly, we prove that the considered family of functions $\{\varphi_p(x)\}_{p\in(1,+\infty)}$ defined by (19) is not stratified for $p \in \left(1, \sqrt{15}/3\right)$ on the interval $(0, \pi/2)$.

LEMMA 3. *For the family of functions* $\{\varphi_p(x)\}_{p\in(1,+\infty)}$, $x \in [0, \pi/2]$, *and for* $p \in \left(1, \sqrt{15}/3\right)$, *it holds:*

$$\frac{\partial \varphi_p(x)}{\partial p} > 0 \qquad \text{in some right neighbourhood of the point } 0 \tag{25}$$

and

$$\frac{\partial \varphi_p(x)}{\partial p} < 0 \qquad \text{in some left neighbourhood of the point } \pi/2. \tag{26}$$

Proof. Based on Theorem 1 and the Taylor expansion (21), for $p \in \left(1, \sqrt{15}/3\right)$, there exists a right neighbourhood of the point 0 such that

$$f_p(x) > 0;$$

thus,

$$\frac{\partial \varphi_p(x)}{\partial p} = \frac{f_p(x)}{xp^3\left(\cos\frac{x}{p}\right)^2} > 0.$$

Based on the Taylor expansion of $f_p(x)$ at the point $\pi/2$

$$f_p(x)=-\frac{\pi}{2}\left(\cos\frac{\pi}{2p}\right)^2+\left(-p\sin\frac{\pi}{2p}-\left(\cos\frac{\pi}{2p}\right)^2+\frac{\pi}{p}\cos\frac{\pi}{2p}\sin\frac{\pi}{2p}\right)\left(x-\frac{\pi}{2}\right)+o\left(x-\frac{\pi}{2}\right),$$

we conclude that there exists a left neighbourhood of the point $\pi/2$ such that

$$f_p(x)<0$$

for $p>1$. Therefore,

$$\frac{\partial\varphi_p(x)}{\partial p}=\frac{f_p(x)}{xp^3\left(\cos\frac{x}{p}\right)^2}<0$$

in a left neighbourhood of the point $\pi/2$ for $p\in\left(1,\sqrt{15}/3\right)$. □

Figure 3 illustrates some functions from the family of functions defined by (19) for some values of the parameter $p\in(1,\sqrt{15}/3)$ for which this family is not stratified on the interval $(0,\pi/2)$. The conditions (25) and (26) provide a self-intersection of the functions from this family, as shown in the figure.

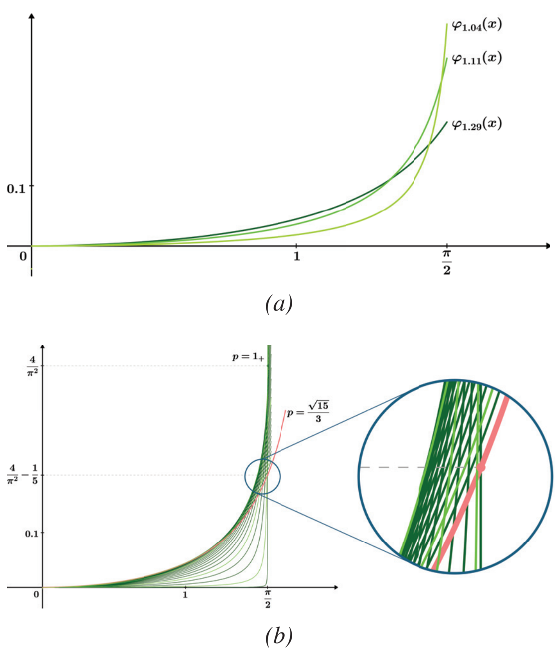

Figure 3: *(a) Functions* $\varphi_{1.04}(x)$, $\varphi_{1.11}(x)$ *and* $\varphi_{1.29}(x)$ *on the interval* $(0,\pi/2)$*; (b) The unstratified part of family of functions defined by* (19)

5. Conclusion

There are numerous well-known inequalities for which the best constants are obtained [1–3, 5–8, 10, 16, 25, 27–35, 37, 38, 42–44, 46–50]. Closely related to any consideration of inequalities of the type (1) is the question of the existence and determination of the best constants. In this paper, we gave a new approach for selecting the best real constants for some inequalities of the type (1).

By applying the proposed method, an improvement of the double inequality (14) was obtained. It has been shown how, by applying the stratification of families of functions, such inequalities can be generalised, i.e. considered for real values of parameters, with selecting the best values for those parameters.

We propose the extension of the double inequality (15) to real values of the parameter, along with selecting the best constants, as an open problem.

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(Received August 23, 2024)

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