Journal of Mathematical Inequalities
Volume 19, Number 1 (2025), 247–259 doi:10.7153/jmi-2025-19-16

SOME GENERALIZATIONS OF NUMERICAL RADIUS INEQUALITIES FOR 2×2 OPERATOR MATRICES

CHAOJUN YANG

(*Communicated by M. Krnić*)

Abstract. In this paper, we present new upper bounds of numerical radius inequalities for 2×2 operator matrices based on some quantities inequalities, which generalize and refine some known results in the literature.

1. Introduction

Let $B(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with an inner product $\langle\cdot\rangle$. For $A\in B(\mathscr{H})$, let $\|A\|$ denote the usual operator norm of A. The numerical range of A is defined by $W(A)=\{\langle Ax,x\rangle : x\in\mathscr{H}, \|x\|=1\}$. The numerical radius of A is defined by $\omega(A)=\sup\{|\lambda| : \lambda\in W(A)\}$.

It had been shown that $\omega(\cdot)$ defines a norm on $B(\mathscr{H})$. In fact, for any $A\in B(\mathscr{H})$,

$$\frac{1}{2}\|A\|\leqslant\omega(A)\leqslant\|A\|, \tag{1}$$

which indicates the usual operator norm and the numerical radius are equivalent.

The power inequality for numerical radius is an important property of numerical radius, which asserts that

$$\omega(A^n)\leqslant\omega^n(A),\quad \text{for } n=1,2,\cdots,.$$

Let $A,B,C,D\in B(\mathscr{H})$. The operator matrix $\begin{bmatrix}A & B\\ C & D\end{bmatrix}$ can be considered as an operator on $\mathscr{H}\bigoplus\mathscr{H}$, and is defined for $x=\begin{pmatrix}x_1\\ x_2\end{pmatrix}\in\mathscr{H}\bigoplus\mathscr{H}$ by $\begin{bmatrix}A & B\\ C & D\end{bmatrix}x=\begin{pmatrix}Ax_1+Bx_2\\ Cx_1+Dx_2\end{pmatrix}$.

Kittaneh [11] showed the following inequalities which improved the inequalities in (1) by using several norm inequalities and ingenious techniques:

$$\frac{1}{4}\||A|^2+|A^*|^2\|\leqslant\omega^2(A)\leqslant\frac{1}{2}\||A|^2+|A^*|^2\|,\quad A\in B(\mathscr{H}). \tag{2}$$

Mathematics subject classification (2020): 47A63, 47A30.
Keywords and phrases: Numerical radius, operator norm, operator matrices, function.

Paper JMI-19-16

El-Haddad and Kittaneh then gave the following general inequality for $A \in B(\mathscr{H})$

$$\omega^{2p}(A) \leqslant \frac{1}{2}\||A|^{2p} + |A^*|^{2p}\|, \quad p \geqslant 1. \tag{3}$$

Later in [1] Abu-Omar and Kittaneh proved that if $A \in B(\mathscr{H})$, then

$$\omega^2(A) \leqslant \frac{1}{4}\||A|^2 + |A^*|^2\| + \frac{1}{2}\omega(A^2). \tag{4}$$

In meantime [15] Sattari, Moslehian and Yamazaki showed that if $A \in B(\mathscr{H})$ and $p \geqslant 1$, then

$$\omega^{2p}(A) \leqslant \frac{1}{2}\omega^p(A^2) + \frac{1}{2}\|A\|^{2p}. \tag{5}$$

In [13], Omidvar and Moradi proved that if $A \in B(\mathscr{H})$, then

$$\omega^4(A) \leqslant \frac{3}{8}\||A|^4 + |A^*|^4\| + \frac{1}{8}\||A|^2 + |A^*|^2\|\omega(A^2). \tag{6}$$

In [14], Safshekan and Farokhinia gave the following inequality

$$\omega^{2p}(A) \leqslant \frac{1}{4}\||A|^2 + |A^*|^2\|^p + \frac{1}{4}\||A|^2 - |A^*|^2\|^p + \frac{1}{2}\omega^p(A^2), \quad p \geqslant 1. \tag{7}$$

For further information on recent developments of the numerical radius inequalities, we refer the reader to [2, 3, 4, 5, 7, 10]

In this paper, we first give a scalar inequality based on the majorization theory, then we give new upper bounds of numerical radius inequalities for 2×2 operator matrices and the sum of two operators. Our results refine and generalize the numerical radius inequalities mentioned above. Moreover, we give a numerical radius inequality involving non-negative functions.

2. Inequalities for the numerical radius

To achieve our goal of this section, we first give some lemmas.

LEMMA 2.1. (see [9]) *Let $x, y, z \in \mathscr{H}$. Then*

$$|\langle y, x\rangle|^2 + |\langle x, z\rangle|^2 \leqslant \|x\|^2\left(\max\{\|y\|^2, \|z\|^2\} + |\langle y, z\rangle|\right).$$

LEMMA 2.2. *Let $x, y, z \in \mathscr{H}$ and $p \geqslant 1$. Then*

$$|\langle y, x\rangle|^{2p} + |\langle x, z\rangle|^{2p} \leqslant \|x\|^{2p}\left(\max\{\|y\|^{2p}, \|z\|^{2p}\} + |\langle y, z\rangle|^p\right).$$

Proof. Let us consider two vectors $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ with components

$$X_1 = |\langle y, x\rangle|^2, \quad X_2 = |\langle x, z\rangle|^2,$$

and

$$Y_1 = \|x\|^2\big(\max\{\|y\|^2, \|z\|^2\}\big), \quad Y_2 = \|x\|^2 |\langle y,z\rangle|.$$

It is easy to infer that $X_1 \leqslant Y_1$ and $X_2 \leqslant Y_1$, that is to say $X_1^* \leqslant Y_1^*$, where $X^* = (X_1^*, X_2^*)$ and $Y^* = (Y_1^*, Y_2^*)$ are the components of X and Y rearranged in decreasing order, respectively. From Lemma 2.1 we know that

$$X_1^* + X_2^* \leqslant Y_1^* + Y_2^*.$$

By two inequalities above, we thus get that X is weakly sub-majorized by Y. Given the function $f(x) = x^t, t \geqslant 1$ is continuously increasing convex on $[0,\infty)$, by [12, p. 13] we thus have $X_1^p + X_2^p \leqslant Y_1^p + Y_2^p$, which completes the proof. □

THEOREM 2.3. *Let $A, B \in B(\mathscr{H})$ and $p \geqslant 1$. Then*

$$2^{1-2p}\omega^{2p}(A+B) \leqslant \max\{\|A\|^{2p}, \|B\|^{2p}\} + \omega^p(BA).$$

Proof. Let x be a unit vector in $\mathscr{H}$. Then

$$\begin{aligned}
2^{1-2p}|\langle (A+B)x,x\rangle|^{2p} &= 2^{1-2p}|\langle Ax,x\rangle + \langle Bx,x\rangle|^{2p}\\
&\leqslant 2^{1-2p}(|\langle Ax,x\rangle| + |\langle Bx,x\rangle|)^{2p}\\
&\leqslant |\langle Ax,x\rangle|^{2p} + |\langle Bx,x\rangle|^{2p}\\
&= |\langle Ax,x\rangle|^{2p} + |\langle x,B^*x\rangle|^{2p}\\
&\leqslant \max\{\|Ax\|^{2p}, \|B^*x\|^{2p}\} + |\langle Ax,B^*x\rangle|^p \quad \text{(by Lemma 2.2)}\\
&\leqslant \max\{\|A\|^{2p}, \|B\|^{2p}\} + |\langle Ax,B^*x\rangle|^p\\
&= \max\{\|A\|^{2p}, \|B\|^{2p}\} + |\langle BAx,x\rangle|^p\\
&\leqslant \max\{\|A\|^{2p}, \|B\|^{2p}\} + \omega^p(BA).
\end{aligned}$$

By taking the supremum over unit vector x, we obtain

$$2^{1-2p}\omega^{2p}(A+B) \leqslant \max\{\|A\|^{2p}, \|B\|^{2p}\} + \omega^p(BA). \quad \square$$

REMARK 2.4. If we take $A = B$ in Theorem 2.3, then we reobtain inequality (5).

THEOREM 2.5. *Let $A, B, C, D \in B(\mathscr{H})$ and $p \geqslant 1$. Then*

$$\omega^{2p}\left(\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right) \leqslant 2^{p-1}\Big(\max\{\||A|^2 + |B^*|^2\|, \||D|^2 + |C^*|^2\|\} + \max\{\||A|^2 - |B^*|^2\|, \||D|^2 - |C^*|^2\|\}\Big)^p + 2^{2p-1}\omega^p\left(\begin{bmatrix} 0 & BD\\ CA & 0\end{bmatrix}\right).$$

Proof. Let x be a unit vector in $\mathscr{H}\bigoplus\mathscr{H}$, and let $E=\begin{bmatrix}A&0\\0&D\end{bmatrix}$, $F=\begin{bmatrix}0&B\\C&0\end{bmatrix}$. Then

$$\begin{aligned}
&2^{1-2p}\left|\left\langle\begin{bmatrix}A&B\\C&D\end{bmatrix}x,x\right\rangle\right|^{2p}\\
&=2^{1-2p}(|\langle Ex,x\rangle+\langle Fx,x\rangle|)^{2p}\leqslant 2^{1-2p}(|\langle Ex,x\rangle|+|\langle Fx,x\rangle|)^{2p}\\
&\leqslant|\langle Ex,x\rangle|^{2p}+|\langle Fx,x\rangle|^{2p}=|\langle Ex,x\rangle|^{2p}+|\langle x,F^*x\rangle|^{2p}\\
&\leqslant(\max\{\|Ex\|^2,\|F^*x\|^2\})^p+|\langle Ex,F^*x\rangle|^p\quad\text{(by Lemma 2.2)}\\
&=\frac{1}{2^p}(\|Ex\|^2+\|F^*x\|^2+|\|Ex\|^2-\|F^*x\|^2|)^p+|\langle FEx,x\rangle|^p\\
&=\frac{1}{2^p}(\langle(|E|^2+|F^*|^2)x,x\rangle+|\langle(|E|^2-|F^*|^2)x,x\rangle|)^p+|\langle FEx,x\rangle|^p\\
&=\frac{1}{2^p}\left(\left\langle\begin{bmatrix}|A|^2+|B^*|^2&0\\0&|D|^2+|C^*|^2\end{bmatrix}x,x\right\rangle\right.\\
&\qquad\left.+\left|\left\langle\begin{bmatrix}|A|^2-|B^*|^2&0\\0&|D|^2-|C^*|^2\end{bmatrix}x,x\right\rangle\right|\right)^p+\left|\left\langle\begin{bmatrix}0&BD\\CA&0\end{bmatrix}x,x\right\rangle\right|^p\\
&\leqslant\frac{1}{2^p}\Bigg(\max\{\||A|^2+|B^*|^2\|,\||D|^2+|C^*|^2\|\}\\
&\qquad+\max\{\||A|^2-|B^*|^2\|,\||D|^2-|C^*|^2\|\}\Bigg)^p+\omega^p\left(\begin{bmatrix}0&BD\\CA&0\end{bmatrix}\right).
\end{aligned}$$

By taking the supremum over unit vector x, we obtain

$$\begin{aligned}
\omega^{2p}\left(\begin{bmatrix}A&B\\C&D\end{bmatrix}\right)\leqslant 2^{p-1}\Bigg(&\max\{\||A|^2+|B^*|^2\|,\||D|^2+|C^*|^2\|\}\\
&+\max\{\||A|^2-|B^*|^2\|,\||D|^2-|C^*|^2\|\}\Bigg)^p\\
&+2^{2p-1}\omega^p\left(\begin{bmatrix}0&BD\\CA&0\end{bmatrix}\right).\quad\square
\end{aligned}$$

REMARK 2.6. If we take $A=B=C=D$ in Theorem 2.5, we have the following chains of inequalities

$$\begin{aligned}
\omega^{2p}(A)&\leqslant 2^{-p-1}(\||A|^2+|A^*|^2\|+\||A|^2-|A^*|^2\|)^p+\frac{1}{2}\omega^p(A^2)\\
&=\frac{1}{2}\left(\frac{\||A|^2+|A^*|^2\|+\||A|^2-|A^*|^2\|}{2}\right)^p+\frac{1}{2}\omega^p(A^2)\\
&\leqslant\frac{1}{4}\left(\||A|^2+|A^*|^2\|^p+\||A|^2-|A^*|^2\|^p\right)+\frac{1}{2}\omega^p(A^2),
\end{aligned}$$

which refines inequality (7).

LEMMA 2.7. (see [8]) *Let* $x,y,z\in\mathscr{H}$ *with* $\|x\|=1$ *and* $t\in\mathbb{C}\setminus\{0\}$. *Then*

$$|\langle y,x\rangle\langle x,z\rangle|\leqslant\frac{1}{|t|}(\max\{1,|t-1|\}\|y\|\|z\|+|\langle y,z\rangle|). \tag{8}$$

Next, we give an operator norm inequality for the sum of two operators.

THEOREM 2.8. *Let* $A,B\in B(\mathscr{H})$ *and* $t\in\mathbb{C}\setminus\{0\}$. *Then*

$$\|A+B\|^2\leqslant\left(1+\frac{1}{|t|}\max\{1,|t-1|\}\right)\||A|^2+|B|^2\|+\frac{2}{|t|}\omega(B^*A).$$

Proof. Let x and y be unit vectors in $\mathscr{H}$. Then

$$\begin{aligned}
&|\langle(A+B)x,y\rangle|^2\\
&\leqslant(|\langle Ax,y\rangle|+|\langle Bx,y\rangle|)^2\\
&=|\langle Ax,y\rangle|^2+|\langle Bx,y\rangle|^2+2|\langle Ax,y\rangle||\langle Bx,y\rangle|\\
&\leqslant|\langle Ax,y\rangle|^2+|\langle Bx,y\rangle|^2+\frac{2}{|t|}\Big((\max\{1,|t-1|\})\|Ax\|\|Bx\|+|\langle Ax,Bx\rangle|\Big)\\
&\quad\text{(by Lemma 2.7)}\\
&\leqslant\|Ax\|^2+\|Bx\|^2+\frac{2}{|t|}\Big((\max\{1,|t-1|\})\sqrt{\langle Ax,Ax\rangle\langle Bx,Bx\rangle}+\langle Ax,Bx\rangle\Big)\\
&=\left\langle|A|^2x,x\right\rangle+\left\langle|B|^2x,x\right\rangle+\frac{2}{|t|}\Big((\max\{1,|t-1|\})\sqrt{\langle|A|^2x,x\rangle\langle|B|^2x,x\rangle}+|\langle B^*Ax,x\rangle|\Big)\\
&=\left\langle(|A|^2+|B|^2)x,x\right\rangle+\frac{2}{|t|}\Big((\max\{1,|t-1|\})\sqrt{\langle|A|^2x,x\rangle\langle|B|^2x,x\rangle}+|\langle B^*Ax,x\rangle|\Big)\\
&\leqslant\left\langle(|A|^2+|B|^2)x,x\right\rangle+\frac{2}{|t|}\left(\frac{(\max\{1,|t-1|\})}{2}\left\langle(|A|^2+|B|^2)x,x\right\rangle+|\langle B^*Ax,x\rangle|\right)\\
&\leqslant\left(1+\frac{1}{|t|}\max\{1,|t-1|\}\right)\left\langle(|A|^2+|B|^2)x,x\right\rangle+\frac{2}{|t|}|\langle B^*Ax,x\rangle|\\
&\leqslant\left(1+\frac{1}{|t|}\max\{1,|t-1|\}\right)\||A|^2+|B|^2\|+\frac{2}{|t|}\omega(B^*A).
\end{aligned}$$

By taking the supremum over unit vectors x,y, we obtain

$$\|A+B\|^2\leqslant(1+\frac{1}{|t|}\max\{1,|t-1|\})\||A|^2+|B|^2\|+\frac{2}{|t|}\omega(B^*A).\quad\square$$

We note that Theorem 2.8 is a generalization of Theorem 2.9 in [16].

LEMMA 2.9. *Let* $x,y,z\in\mathscr{H}$ *with* $\|x\|=1$ *and* $t\in\mathbb{C}\setminus\{0\}$, $s=\max\{1,|t-1|\}$. *Then*

$$|\langle y,x\rangle\langle x,z\rangle|^2\leqslant\frac{s^2}{|t|^2}\|y\|^2\|z\|^2+\frac{1+2s}{|t|^2}\|y\|\|z\|\langle y,z\rangle|.$$

Proof. Utilizing Lemma 2.7 we have

$$\begin{aligned}
|\langle y,x\rangle\langle x,z\rangle|^2 &\leqslant \frac{1}{|t|^2}(s^2\|y\|^2\|z\|^2+|\langle y,z\rangle|^2+2s\|y\|\|z\||\langle y,z\rangle|)\\
&= \frac{s^2}{|t|^2}\|y\|^2\|z\|^2+\frac{1}{|t|^2}|\langle y,z\rangle|^2+\frac{2s}{|t|^2}\|y\|\|z\||\langle y,z\rangle|\\
&\leqslant \frac{s^2}{|t|^2}\|y\|^2\|z\|^2+\frac{1}{|t|^2}\|y\|\|z\|\langle y,z\rangle|+\frac{2s}{|t|^2}|\|y\|\|z\|\langle y,z\rangle|\\
&= \frac{s^2}{|t|^2}\|y\|^2\|z\|^2+\frac{1+2s}{|t|^2}\|y\|\|z\|\langle y,z\rangle|. \quad \square
\end{aligned}$$

THEOREM 2.10. *Let* $A,B,C,D\in B(\mathscr{H})$ *and* $t\in\mathbb{C}\setminus\{0\}$, $s=\max\{1,|t-1|\}$. *Then*

$$\begin{aligned}
\omega^4\left(\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right) &\leqslant 8\max\{\omega^4(A),\omega^4(D)\}+\frac{4s^2}{|t|^2}\max\{\||C|^4+|B^*|^4\|,\||B|^4+|C^*|^4\|\}\\
&\quad+\frac{4+8s}{|t|^2}\max\{\||C|^2+|B^*|^2\|,\||B|^2+|C^*|^2\|\}\max\{\omega(BC),\omega(CB)\}.
\end{aligned}$$

Proof. Let x be a unit vector in $\mathscr{H}\bigoplus\mathscr{H}$, and let $E=\begin{bmatrix}A & 0\\ 0 & D\end{bmatrix}$, $F=\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}$. Then

$$\begin{aligned}
\left|\left\langle\begin{bmatrix}A & B\\ C & D\end{bmatrix}x,x\right\rangle\right|^4 &= (|\langle Ex,x\rangle+\langle Fx,x\rangle|)^4\\
&\leqslant (|\langle Ex,x\rangle|+|\langle Fx,x\rangle|)^4\\
&= \Big(\frac{2|\langle Ex,x\rangle|+2|\langle Fx,x\rangle|}{2}\Big)^4\\
&\leqslant 8(|\langle Ex,x\rangle|^4+|\langle Fx,x\rangle|^4)\\
&= 8(|\langle Ex,x\rangle|^4+|\langle Fx,x\rangle\langle x,F^*x\rangle|^2)\\
&\leqslant 8|\langle Ex,x\rangle|^4+8\Big(\frac{s^2}{|t|^2}\|Fx\|^2\|F^*x\|^2+\frac{1+2s}{|t|^2}\|Fx\|\|F^*x\|\langle Fx,F^*x\rangle|\Big)\\
&\quad \text{(by Lemma 2.9)}\\
&= 8|\langle Ex,x\rangle|^4+8\Big(\frac{s^2}{|t|^2}\langle|F|^2x,x\rangle\langle|F^*|^2x,x\rangle\\
&\quad+\frac{1+2s}{|t|^2}|\sqrt{\langle|F|^2x,x\rangle\langle|F^*|^2x,x\rangle}\langle F^2x,x\rangle|\Big)\\
&\leqslant 8|\langle Ex,x\rangle|^4+\frac{4s^2}{|t|^2}\langle(|F|^4+|F^*|^4)x,x\rangle\\
&\quad+\frac{4+8s}{|t|^2}|\langle(|F|^2+|F^*|^2)x,x\rangle\langle F^2x,x\rangle|
\end{aligned}$$

$$= 8\left|\left\langle \begin{bmatrix} A & 0\\ 0 & D\end{bmatrix} x,x\right\rangle\right|^4 + \frac{4s^2}{|t|^2}\left\langle \begin{bmatrix} |C|^4+|B^*|^4 & 0\\ 0 & |B|^4+|C^*|^4\end{bmatrix} x,x\right\rangle$$
$$+\frac{4+8s}{|t|^2}\left\langle \begin{bmatrix} |C|^2+|B^*|^2 & 0\\ 0 & |B|^2+|C^*|^2\end{bmatrix} x,x\right\rangle \left|\left\langle \begin{bmatrix} BC & 0\\ 0 & CB\end{bmatrix} x,x\right\rangle\right|.$$

By taking the supremum over unit vector x, we obtain

$$\omega^4\left(\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right) \leqslant 8\max\{\omega^4(A),\omega^4(D)\} + \frac{4s^2}{|t|^2}\max\{\||C|^4+|B^*|^4\|,\||B|^4+|C^*|^4\|\}$$
$$+\frac{4+8s}{|t|^2}\max\{\||C|^2+|B^*|^2\|,\||B|^2+|C^*|^2\|\}\max\{\omega(BC),\omega(CB)\}.$$

□

REMARK 2.11. If we take $A=B=C=D$, $t=2$ (hence $s=1$) in Theorem 2.10, we have the following chains of inequalities

$$\begin{aligned}
\omega^4(A) &\leqslant \frac{1}{8}\||A|^4+|A^*|^4\| + \frac{3}{8}\||A|^2+|A^*|^2\|\omega(A^2)\\
&\leqslant \frac{1}{8}\||A|^4+|A^*|^4\| + \frac{3}{8}\||A|^2+|A^*|^2\|\omega^2(A)\\
&\leqslant \frac{1}{8}\||A|^4+|A^*|^4\| + \frac{3}{16}\||A|^2+|A^*|^2\|^2\\
&= \frac{1}{8}\||A|^4+|A^*|^4\| + \frac{3}{4}\left\|\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right\|\\
&\leqslant \frac{1}{8}\||A|^4+|A^*|^4\| + \frac{3}{8}\||A|^4+|A^*|^4\|\\
&= \frac{1}{2}\||A|^4+|A^*|^4\|,
\end{aligned}$$

which refines inequality (3) when $p=2$. From the last inequalities, we also obtain that

$$\begin{aligned}
\omega^4(A) &\leqslant \frac{1}{8}\||A|^4+|A^*|^4\| + \frac{3}{8}\||A|^2+|A^*|^2\|\omega(A^2)\\
&\leqslant \frac{3}{8}\||A|^4+|A^*|^4\| + \frac{1}{8}\||A|^2+|A^*|^2\|\omega(A^2).
\end{aligned}$$

This is because

$$\begin{aligned}
\||A|^2+|A^*|^2\|\omega(A^2) &\leqslant \||A|^2+|A^*|^2\|\omega^2(A)\\
&\leqslant \frac{1}{2}\||A|^2+|A^*|^2\|^2\\
&= 2\left\|\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right\|\\
&\leqslant \||A|^4+|A^*|^4\|.
\end{aligned}$$

Thus Theorem 2.10 also refines inequality (6).

LEMMA 2.12. (see [17]) *Let* $x,y,z\in\mathscr{H}$. *Then*

$$|\langle x,y\rangle|^2+|\langle x,z\rangle|^2\leqslant\|x\|^2(|\langle y,y\rangle|^2+|\langle z,z\rangle|^2+2|\langle y,z\rangle|^2)^{\frac{1}{2}}.$$

THEOREM 2.13. *Let* $A\in B(\mathscr{H})$ *and* $p\geqslant 2$. *Then*

$$\omega^{2p}(A)\leqslant\frac{1}{4}\||A|^{2p}+|A^*|^{2p}\|+\frac{1}{2}\omega^p(A^2).$$

Proof. Let x be a unit vector in $\mathscr{H}$. Then

$$\begin{aligned}
\frac{1}{2^p}|\langle Ax,x\rangle|^{2p}&=\left(\frac{2|\langle Ax,x\rangle|^2}{4}\right)^p\\
&=\left(\frac{|\langle Ax,x\rangle|^2+|\langle A^*x,x\rangle|^2}{4}\right)^p\\
&\leqslant\left(\frac{1}{4}\right)^{\frac{p}{2}}\left(\frac{|\langle Ax,Ax\rangle|^2+|\langle A^*x,A^*x\rangle|^2+2|\langle Ax,A^*x\rangle|^2}{4}\right)^{\frac{p}{2}}\\
&\quad\text{(by Lemma 2.12)}\\
&\leqslant\left(\frac{1}{4}\right)^{\frac{p}{2}}\left(\frac{1}{4}|\langle Ax,Ax\rangle|^p+\frac{1}{4}|\langle A^*x,A^*x\rangle|^p+\frac{1}{2}|\langle Ax,A^*x\rangle|^p\right)\\
&=\left(\frac{1}{4}\right)^{\frac{p}{2}}\left(\frac{1}{4}|\left\langle|A|^2x,x\right\rangle|^p+\frac{1}{4}|\left\langle|A^*|^2x,x\right\rangle|^p+\frac{1}{2}|\left\langle A^2x,x\right\rangle|^p\right)\\
&\leqslant\left(\frac{1}{4}\right)^{\frac{p}{2}}\left(\frac{1}{4}\left\langle|A|^{2p}x,x\right\rangle+\frac{1}{4}\left\langle|A^*|^{2p}x,x\right\rangle+\frac{1}{2}|\left\langle A^2x,x\right\rangle|^p\right)\\
&=\left(\frac{1}{4}\right)^{\frac{p}{2}}\left(\frac{1}{4}\left\langle(|A|^{2p}+|A^*|^{2p})x,x\right\rangle+\frac{1}{2}|\left\langle A^2x,x\right\rangle|^p\right),
\end{aligned}$$

which is equivalent to

$$|\langle Ax,x\rangle|^{2p}\leqslant\frac{1}{4}\left\langle(|A|^{2p}+|A^*|^{2p})x,x\right\rangle+\frac{1}{2}|\left\langle A^2x,x\right\rangle|^p.$$

By taking the supremum over unit vector x, we obtain

$$\omega^{2p}(A)\leqslant\frac{1}{4}\||A|^{2p}+|A^*|^{2p}\|+\frac{1}{2}\omega^p(A^2).\quad\square$$

REMARK 2.14. From Theorem 1.6.9 in [6] we know that $\|A^p+B^p\|\leqslant\|(A+B)^p\|$ for two positive operators $A,B\in\mathscr{H}$ when $p\geqslant 2$. Thus by Theorem 2.13 and the preceding inequality we can obtain

$$\begin{aligned}
\omega^{2p}(A)&\leqslant\frac{1}{4}\||A|^{2p}+|A^*|^{2p}\|+\frac{1}{2}\omega^p(A^2)\\
&\leqslant\frac{1}{4}\|(|A|^2+|A^*|^2)^p\|+\frac{1}{2}\omega^p(A^2)
\end{aligned}$$

$$= \frac{1}{4}\||A|^2+|A^*|^2\|^p+\frac{1}{2}\omega^p(A^2)$$
$$\leqslant \frac{1}{4}\||A|^2+|A^*|^2\|^p+\frac{1}{4}\||A|^2-|A^*|^2\|^p+\frac{1}{2}\omega^p(A^2),$$

which significantly improved inequality (7). Moreover one can check that Theorem 2.13 also refines inequality (3).

The following lemma is a direct consequence of Lemma 2.12 by the arithmetic-geometric mean inequality for scalars.

LEMMA 2.15. *Let $x,y,z\in\mathscr{H}$. Then*

$$|\langle y,x\rangle\langle x,z\rangle|^2 \leqslant \frac{\|x\|^4}{4}(|\langle y,y\rangle|^2+|\langle z,z\rangle|^2+2|\langle y,z\rangle|^2).$$

THEOREM 2.16. *Let $A,B,C,D\in B(\mathscr{H})$. Then*

$$\omega^4\left(\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right) \leqslant 8\max\{\omega^4(A),\omega^4(D)\}+2\max\{\||C|^4+|B^*|^4\|,\||B|^4+|C^*|^4\|\}$$
$$+4\max\{\omega^2(BC),\omega^2(CB)\}.$$

Proof. Let x be a unit vector in $\mathscr{H}\bigoplus\mathscr{H}$, and let $E=\begin{bmatrix}A & 0\\ 0 & D\end{bmatrix}$, $F=\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}$. Then

$$\begin{aligned}
\left|\left\langle\begin{bmatrix}A & B\\ C & D\end{bmatrix}x,x\right\rangle\right|^4 &= (|\langle Ex,x\rangle+\langle Fx,x\rangle|)^4\\
&\leqslant (|\langle Ex,x\rangle|+|\langle Fx,x\rangle|)^4\\
&= \left(\frac{2|\langle Ex,x\rangle|+2|\langle Fx,x\rangle|}{2}\right)^4\\
&\leqslant 8(|\langle Ex,x\rangle|^4+|\langle Fx,x\rangle|^4)\\
&= 8(|\langle Ex,x\rangle|^4+|\langle Fx,x\rangle\langle x,F^*x\rangle|^2)\\
&\leqslant 8|\langle Ex,x\rangle|^4+2\Big(|\langle Fx,Fx\rangle|^2+|\langle F^*x,F^*x\rangle|^2+2|\langle Fx,F^*x\rangle|^2\Big)\\
&\quad \text{(by Lemma 2.15)}\\
&= 8|\langle Ex,x\rangle|^4+2\Big(|\langle |F|^2x,x\rangle|^2+|\langle |F^*|^2x,x\rangle|^2+2|\langle F^2x,x\rangle|^2\Big)\\
&\leqslant 8|\langle Ex,x\rangle|^4+2\Big(\langle |F|^4x,x\rangle+\langle |F^*|^4x,x\rangle+2|\langle F^2x,x\rangle|^2\Big)\\
&= 8|\langle Ex,x\rangle|^4+2\langle(|F|^4+|F^*|^4)x,x\rangle+4|\langle F^2x,x\rangle|^2\\
&= 8\left|\left\langle\begin{bmatrix}A & 0\\ 0 & D\end{bmatrix}x,x\right\rangle\right|^4+2\left\langle\begin{bmatrix}|C|^4+|B^*|^4 & 0\\ 0 & |B|^4+|C^*|^4\end{bmatrix}x,x\right\rangle\\
&\quad +4\left|\left\langle\begin{bmatrix}BC & 0\\ 0 & CB\end{bmatrix}x,x\right\rangle\right|^2.
\end{aligned}$$

By taking the supremum over unit vector x, we obtain

$$\omega^4\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant 8\max\{\omega^4(A),\omega^4(D)\} + 2\max\{\||C|^4+|B^*|^4\|,\||B|^4+|C^*|^4\|\} \\ +4\max\{\omega^2(BC),\omega^2(CB)\}. \quad \square$$

REMARK 2.17. If we take $A=B=C=D$ in Theorem 2.16, we have the following chains of inequalities

$$\begin{aligned}\omega^4(A) &\leqslant \frac{1}{4}\||A|^4+|A^*|^4\| + \frac{1}{2}\omega^2(A^2) \\ &\leqslant \frac{1}{4}\||A|^4+|A^*|^4\| + \frac{1}{2}\omega^4(A) \\ &\leqslant \frac{1}{4}\||A|^4+|A^*|^4\| + \frac{1}{4}\||A|^4+|A^*|^4\| \\ &= \frac{1}{2}\||A|^4+|A^*|^4\|,\end{aligned}$$

which refines inequality (3) when $p=2$.

THEOREM 2.18. *Let $A,B,C,D \in B(\mathscr{H})$ and $p \geqslant 1$. Then*

$$\omega^{4p}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leqslant \frac{1}{4}\max\{\||C|^{4p}+|B^*|^{4p}\|,\||B|^{4p}+|C^*|^{4p}\|\} \\ +\frac{1}{2}\max\{\omega^{2p}(BC),\omega^{2p}(CB)\}.$$

Proof. Let x be a unit vector in $\mathscr{H}\bigoplus\mathscr{H}$, and let $T=\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Then

$$\begin{aligned}&\left|\left\langle \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} x,x\right\rangle\right|^{4p} \\ &= (|\langle Tx,x\rangle|)^{4p} \\ &= \frac{1}{4^p}(|\langle x,Tx\rangle|^2+|\langle x,T^*x\rangle|^2)^{2p} \\ &\leqslant \frac{1}{4^p}(|\langle Tx,Tx\rangle|^2+|\langle T^*x,T^*x\rangle|^2+2|\langle Tx,T^*x\rangle|^2)^p \\ &= \frac{1}{4^p}(\langle |T|^2x,x\rangle^2+\langle |T^*|^2x,x\rangle^2+2|\langle T^2x,x\rangle|^2)^p \\ &\leqslant \frac{1}{4^p}(\langle |T|^4x,x\rangle+\langle |T^*|^4x,x\rangle+2|\langle T^2x,x\rangle|^2)^p \quad \text{(by Lemma 2.12)} \\ &\leqslant \frac{1}{4}\langle |T|^4x,x\rangle^p+\frac{1}{4}\langle |T^*|^4x,x\rangle^p+\frac{1}{2}|\langle T^2x,x\rangle|^{2p}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{4}\left\langle |T|^{4p}x,x\right\rangle+\frac{1}{4}\left\langle |T^*|^{4p}x,x\right\rangle+\frac{1}{2}|\left\langle T^2x,x\right\rangle|^{2p}\\
&=\frac{1}{4}\left\langle (|T|^{4p}+|T^*|^{4p})x,x\right\rangle+\frac{1}{2}|\left\langle T^2x,x\right\rangle|^{2p}\\
&=\frac{1}{4}\left\langle \begin{bmatrix}|C|^{4p}+|B^*|^{4p} & 0\\ 0 & |B|^{4p}+|C^*|^{4p}\end{bmatrix}x,x\right\rangle+\frac{1}{2}\left|\left\langle \begin{bmatrix}BC & 0\\ 0 & CB\end{bmatrix}x,x\right\rangle\right|^{2p}\\
&\leqslant \frac{1}{4}\left\|\begin{bmatrix}|C|^{4p}+|B^*|^{4p} & 0\\ 0 & |B|^{4p}+|C^*|^{4p}\end{bmatrix}\right\|+\frac{1}{2}\omega^{2p}\left(\begin{bmatrix}BC & 0\\ 0 & CB\end{bmatrix}\right)\\
&\leqslant \frac{1}{4}\max\{\||C|^{4p}+|B^*|^{4p}\|,\||B|^{4p}+|C^*|^{4p}\|\}+\frac{1}{2}\max\{\omega^{2p}(BC),\omega^{2p}(CB)\}.
\end{aligned}
$$

By taking the supremum over unit vector x, we obtain

$$
\begin{aligned}
\omega^{4p}\left(\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}\right) &\leqslant \frac{1}{4}\max\{\||C|^{4p}+|B^*|^{4p}\|,\||B|^{4p}+|C^*|^{4p}\|\}\\
&\quad+\frac{1}{2}\max\{\omega^{2p}(BC),\omega^{2p}(CB)\}. \quad \square
\end{aligned}
$$

REMARK 2.19. If we take $B=C$ in Theorem 2.18, we have the following chains of inequalities

$$
\begin{aligned}
\omega^{4p}(A) &\leqslant \frac{1}{4}\||A|^{4p}+|A^*|^{4p}\|+\frac{1}{2}\omega^{2p}(A^2)\\
&\leqslant \frac{1}{4}\||A|^{4p}+|A^*|^{4p}\|+\frac{1}{2}\omega^{4p}(A)\\
&\leqslant \frac{1}{4}\||A|^{4p}+|A^*|^{4p}\|+\frac{1}{4}\||A|^{4p}+|A^*|^{4p}\|\\
&=\frac{1}{2}\||A|^{4p}+|A^*|^{4p}\|,
\end{aligned}
$$

which refines inequality (3).

PROPOSITION 2.20. *Let $A,B\in B(\mathscr{H})$ with $A,B\geqslant 0$ and f,g be two non-negative functions on $[0.\infty)$. Then*

$$
\omega^p(f(A)g(B)+f(B)g(A))\leqslant \frac{1}{2}\|(f^2(A)+f^2(B))^p+(g^2(A)+g^2(B))^p\|,
$$

where $p\geqslant 1$.

Proof. Let $X=\begin{pmatrix}f(A)\\ g(B)\end{pmatrix}$ and $Y=\begin{pmatrix}f(B)\\ g(A)\end{pmatrix}$. Then it is easy to deduce that

$$
\begin{aligned}
XX^*+YY^* &= \begin{bmatrix}f^2(A) & f(A)g(B)\\ g(B)f(A) & g^2(B)\end{bmatrix}+\begin{bmatrix}f^2(B) & f(B)g(A)\\ g(A)f(B) & g^2(A)\end{bmatrix}\\
&=\begin{bmatrix}f^2(A)+f^2(B) & f(A)g(B)+f(B)g(A)\\ g(B)f(A)+g(A)f(B) & g^2(B)+g^2(A)\end{bmatrix}\geqslant 0
\end{aligned}
$$

By Proposition1.3.2 in [6], one can obtain a contraction W satisfying $f(A)g(B)+f(B)g(A)=(f^2(A)+f^2(B))^{\frac{1}{2}}W(g^2(B)+g^2(A))^{\frac{1}{2}}$. Setting $N=(f^2(A)+f^2(B))^{\frac{1}{2}}$ and $L=(g^2(B)+g^2(A))^{\frac{1}{2}}$.

Let x be a unit vector in $\mathscr{H}$. Then

$$\begin{aligned}
|\langle (f(A)g(B)+f(B)g(A))x,x\rangle|^p &= |\langle NWLx,x\rangle|^p \\
&= |\langle WLx,Nx\rangle|^p \\
&\leqslant \|WLx\|^p\|Nx\|^p \\
&\leqslant \|W\|^p\|Lx\|^p\|Nx\|^p \\
&\leqslant \|Lx\|^p\|Nx\|^p \\
&= \langle Lx,Lx\rangle^{\frac{p}{2}}\langle Nx,Nx\rangle^{\frac{p}{2}} \\
&= \left\langle L^2x,x\right\rangle^{\frac{p}{2}}\left\langle N^2x,x\right\rangle^{\frac{p}{2}} \\
&\leqslant \left\langle L^{2p}x,x\right\rangle^{\frac{1}{2}}\left\langle N^{2p}x,x\right\rangle^{\frac{1}{2}} \\
&\leqslant \frac{\left\langle (L^{2p}+N^{2p})x,x\right\rangle}{2} \\
&\leqslant \frac{1}{2}\|L^{2p}+N^{2p}\| \\
&= \frac{1}{2}\|(f^2(A)+f^2(B))^p+(g^2(A)+g^2(B))^p\|.
\end{aligned}$$

By taking the supremum over unit vector x, we obtain

$$\omega^p(f(A)g(B)+f(B)g(A)) \leqslant \frac{1}{2}\|(f^2(A)+f^2(B))^p+(g^2(A)+g^2(B))^p\|. \quad \square$$

Availability of data and materials. Not applicable.

Competing interests. The author declares that he has no competing interests.

Funding. This project was funded by National Natural Science Foundation of P. R. China (No. 12301147).

Acknowledgement. The author is grateful to the referee and editor for their helpful comments and suggestions.

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(Received August 27, 2024)

Chaojun Yang
Department of Mathematics
Changzhou University
Changzhou, 213000, P. R. China
e-mail: cjyangmath@163.com