Journal of Mathematical Inequalities
Volume 19, Number 1 (2025), 261–273 doi:10.7153/jmi-2025-19-17

MEAN–TYPE INEQUALITIES FOR THE NUMERICAL RADIUS AND THE OPERATOR NORM

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(*Communicated by S. Furuichi*)

Abstract. In this paper, utilizing the Hadamard product of matrices, we show several new bounds for the numerical radius in a way that extends some known bounds for the operator norm. However, the presented results treat special cases to overcome the general case, invalid for the numerical radius. As a consequence of our discussion, we find relations between the numerical radii of the Aluthge and Duggal transformations. Then, we show some bounds for the product of three Hilbert space operators, and some mean-like terms are treated using operator matrices techniques.

1. Introduction

Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. In the case when $dim\mathscr{H}=n$, we identify $\mathscr{B}(\mathscr{H})$ with the matrix algebra $\mathscr{M}_n$ of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$. Given an orthonormal basis $\{e_j\}$ of a Hilbert space $\mathscr{H}$, the Hadamard product $A\circ B$ of two operators A,B is defined by $\langle A\circ Be_i,e_j\rangle=\langle Ae_i,e_j\rangle\langle Be_i,e_j\rangle$. For matrices, one easily observes that the Hadamard product of $A=(a_{ij})$ and $B=(b_{ij})$ is $A\circ B=(a_{ij}b_{ij})$, a principal submatrix of the tensor product $A\otimes B=(a_{ij}B)_{1\leqslant i,j\leqslant n}$. If $T\in\mathscr{B}(\mathscr{H})$, the real and imaginary parts of T are defined by $\Re T=\frac{T+T^*}{2}$ and $\Im T=\frac{T-T^*}{2\mathrm{i}}$, respectively. We call a norm on operators or matrices weakly unitarily invariant if its value at operator T is not changed by replacing T by U^*TU, provided only that U is unitary.

The numerical range of an operator T in $\mathscr{B}(\mathscr{H})$ is defined as $W(T)=\{\langle Tx,x\rangle:\|x\|=1\}$. The numerical radius and the usual operator norm of an operator $T\in\mathscr{B}(\mathscr{H})$ are defined respectively as $\omega(T)=\sup\limits_{\|x\|=1}|\langle Tx,x\rangle|$ and $\|T\|=\sup\limits_{\|x\|=1}\|Tx\|$. It is well-known that $\omega(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. Namely, for $T\in\mathscr{B}(\mathscr{H})$, we have

$$\frac{1}{2}\|T\|\leqslant\omega(T)\leqslant\|T\|. \tag{1.1}$$

Other facts about the numerical radius can be found in [7].

Mathematics subject classification (2020): Primary 47A30, 47A12; Secondary 47B15, 15A60, 47A50.
Keywords and phrases: Numerical radius, operator norm, inner product, polar decomposition.

Paper JMI-19-17

The inequalities in (1.1) have been improved considerably by many authors, (see, e.g., [8, 10, 12, 13, 18, 25]). Kittaneh [16, 17] has shown the following precise estimates of $\omega(T)$ by using several norm inequalities and ingenious techniques:

$$\omega(T) \leqslant \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{\frac{1}{2}}\right), \tag{1.2}$$

and

$$\frac{1}{4}\left\||T|^2 + |T^*|^2\right\| \leqslant \omega^2(T) \leqslant \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.3}$$

In [5], Dragomir gave the following estimate of the numerical radius which refines the second inequality in (1.1): For every T,

$$\omega^2(T) \leqslant \frac{1}{2}\left(\omega\left(T^2\right) + \|T\|^2\right).$$

We refer the reader to [1, 11, 21, 23, 24] as a list of references that treated numerical radius inequalities, with attempts to sharpen the above and other bounds.

Let $T = U|T|$ be the polar decomposition of T. The Aluthge transform $\widetilde{T}$ of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ [2]. The Duggal transform T^D of T is specified by $T^D = |T|U$ which is first referred to in [6]. The mean transform $\widehat{T}$ of T is represented by $\widehat{T} = \frac{T+T^D}{2}$. This transform was first raised in [19]. A type of operator transform is the generalized mean transform $\widehat{T}(\nu)$ of T, presented recently in [3], by

$$\widehat{T}(\nu) = \frac{|T|^{\nu}U|T|^{1-\nu} + |T|^{1-\nu}U|T|^{\nu}}{2}; \quad 0 \leqslant \nu \leqslant \frac{1}{2}.$$

In this paper, we first discuss some related bounds for the numerical radius and then new types of operator norm inequalities. In particular, we present possible upper bounds for the numerical radius of Heinz-type quantities and a mean-type inequality for the numerical radius. An application will include a new relation between the Aluthge and Duggal transforms.

Among many results, we show that

$$\omega\left(\widehat{T}(\nu)\right) \leqslant \omega\left(2r\widetilde{T} + (1-2r)\widehat{T}\right),$$

where $r = \min\left\{\nu, 1-\nu, \left|\frac{1}{2}-\nu\right|\right\}$. Moreover, we show an arithmetic-geometric mean inequality for the numerical radius in Theorem 2.4. As for the norm results, we show that if $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator in $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, then

$$\|BCA + AC^*B\| \leqslant \frac{\left\|A^2 + B^2\right\|}{2}\left\|\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\right\|.$$

This presents a new mean inequality for the product of three operators.

As a technical lemma, we state the following simple observation when simplifying the products.

LEMMA 1.1. *Let* $T, X \in M_n$ *be such that* $T = diag(\lambda_i)$ *is a diagonal matrix. If* $\alpha, \beta \geqslant 0$, *then* $T^\alpha X T^\beta = (\lambda_i^\alpha x_{ij} \lambda_j^\beta)$.

Proof. Letting $r_i(\cdot)$ and $c_j(\cdot)$ be the $i-$th row and $j-$th column,respectively, we have

$$\begin{aligned} r_i(T^\alpha X) &= r_i(T^\alpha)X \\ &= [\lambda_i^\alpha x_{i1}, \lambda_i^\alpha x_{i2}, \ldots, \lambda_i^\alpha x_{in}]. \end{aligned}$$

Now, it is evident that

$$r_i(T^\alpha X)c_j(T^\beta) = \lambda_i^\alpha x_{ij} \lambda_j^\beta,$$

which is the desired conclusion. □

2. Numerical radii inequalities

Before expressing the first main result of this section, recall that a continuous real-valued function f defined on an interval $J \subseteq \mathbb{R}$ is called operator monotone if $A \leqslant B$ implies that $f(A) \leqslant f(B)$ for all self-adjoint operators A, B with spectra in J.

THEOREM 2.1. *Let* $T, X \in M_n$ *such that* T *is positive definite and let* f *be an operator monotone function on* $(0, \infty)$. *Then*

$$\omega(f(T)X - Xf(T)) \leqslant \omega\left(f'(T)\right)\omega(TX - XT).$$

Proof. We focus on case $T = diag(\lambda_i) \geqslant 0$. If T is not diagonal, then using the spectral decomposition $T = U diag(\lambda_i) U^*$ and noting that $\omega(\cdot)$ is weakly unitarily invariant imply the result. Letting $Z = f(T)X - Xf(T)$, we can see that

$$\begin{aligned} z_{ij} &= \begin{cases} x_{ij}(f(\lambda_i) - f(\lambda_j)), & i \neq j \\ 0, & i = j \end{cases} \\ &= \begin{cases} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} x_{ij}(\lambda_i - \lambda_j), & i \neq j \\ f'(\lambda_i) x_{ii}(\lambda_i - \lambda_j), & i = j \end{cases}. \end{aligned}$$

Notice that this can be written as $Z = Y \circ (TX - XT)$ where $Y = f^{[1]}(T)$, i.e.,

$$y_{ij} = \begin{cases} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}; & \text{when } i \neq j \\ f'(\lambda_i); & \text{when } i = j \end{cases}.$$

By [4, Theorem V.3.4], $f^{[1]}(T) \geqslant O$. Consequently

$$\begin{aligned} \omega(f(T)X - Xf(T)) &= \omega(Y \circ (TX - XT)) \\ &\leqslant \max y_{ii} \omega(TX - XT) \\ &= \left\|f'(T)\right\| \omega(TX - XT) \\ &= \omega\left(f'(T)\right)\omega(TX - XT). \end{aligned}$$

This completes the proof. □

REMARK 2.1. It follows from Theorem 2.1 that

$$\omega\left(T^{r}X-XT^{r}\right)\leqslant r\left\|T^{r-1}\right\|\omega\left(TX-XT\right);\ 0\leqslant r\leqslant 1.$$

In particular,

$$\omega\left(|T|^{r}U-U|T|^{r}\right)\leqslant r\left\||T|^{r-1}\right\|\omega\left(T^{D}-T\right);\ 0\leqslant r\leqslant 1.$$

THEOREM 2.2. *Let $T,X\in M_n$ such that T be positive definite. Then for any $0\leqslant \nu\leqslant 1$,*

$$\omega\left(T^{\nu}XT^{1-\nu}+T^{1-\nu}XT^{\nu}\right)\leqslant\omega\left(4rT^{\frac{1}{2}}XT^{\frac{1}{2}}+(1-2r)(TX+XT)\right)$$

where $r=\min\left\{\nu,\left|\frac{1}{2}-\nu\right|,1-\nu\right\}$.

Proof. First, we consider the case $0\leqslant\nu\leqslant\frac{1}{2}$. Let $T=diag(\lambda_i)\geqslant 0$. Of course, if T is not diagonal, then using the spectral decomposition $T=Udiag(\lambda_i)U^{*}$ and noting that $\omega(\cdot)$ is weakly unitarily invariant imply the result. By Lemma 1.1, we conclude that for $0\leqslant r\leqslant\frac{1}{4}$,

$$\begin{aligned}
&\left(T^{\nu}XT^{1-\nu}+T^{1-\nu}XT^{\nu}\right)_{ij}\\
&=\lambda_i^{\nu}x_{ij}\lambda_j^{1-\nu}+\lambda_i^{1-\nu}x_{ij}\lambda_j^{\nu}\\
&=\lambda_i^{\nu}\left(\lambda_j^{1-2\nu}+\lambda_i^{1-2\nu}\right)\lambda_j^{\nu}x_{ij}\\
&=\frac{\lambda_i^{\nu}\left(\lambda_j^{1-2\nu}+\lambda_i^{1-2\nu}\right)\lambda_j^{\nu}}{4r\lambda_i^{\frac{1}{2}}\lambda_j^{\frac{1}{2}}+(1-2r)(\lambda_i+\lambda_j)}\left(4r\lambda_i^{\frac{1}{2}}\lambda_j^{\frac{1}{2}}+(1-2r)(\lambda_i+\lambda_j)\right)x_{ij}\\
&=\frac{\lambda_i^{\nu}\left(\lambda_j^{1-2\nu}+\lambda_i^{1-2\nu}\right)\lambda_j^{\nu}}{4r\lambda_i^{\frac{1}{2}}\lambda_j^{\frac{1}{2}}+(1-2r)(\lambda_i+\lambda_j)}\left(4r\lambda_i^{\frac{1}{2}}x_{ij}\lambda_j^{\frac{1}{2}}+(1-2r)(\lambda_ix_{ij}+x_{ij}\lambda_j)\right).
\end{aligned}$$

This means that

$$T^{\nu}XT^{1-\nu}+T^{1-\nu}XT^{\nu}=W\circ\left(4rT^{\frac{1}{2}}XT^{\frac{1}{2}}+(1-2r)(TX+XT)\right)$$

where W is a Hermitian matrix with entries

$$w_{ij}=\begin{cases}\dfrac{\lambda_i^{\nu}\left(\lambda_i^{1-2\nu}+\lambda_j^{1-2\nu}\right)\lambda_j^{\nu}}{4r\lambda_i^{\frac{1}{2}}\lambda_j^{\frac{1}{2}}+(1-2r)\left(\lambda_i+\lambda_j\right)}; & \text{when } i\neq j\\ 1; & \text{when } i=j\end{cases}.$$

From [14, Theorem 4.4], we know that $W\geqslant O$ whenever $0\leqslant r\leqslant\frac{1}{4}$. Therefore,

$$\begin{aligned}
\omega\left(T^{\nu}XT^{1-\nu}+T^{1-\nu}XT^{\nu}\right)&=\omega\left(W\circ\left(4rT^{\frac{1}{2}}XT^{\frac{1}{2}}+(1-2r)(TX+XT)\right)\right)\\
&\leqslant\omega\left(4rT^{\frac{1}{2}}XT^{\frac{1}{2}}+(1-2r)(TX+XT)\right),
\end{aligned}$$

which completes the proof for the case $0 \leqslant \nu \leqslant \frac{1}{2}$. For the case $\frac{1}{2} \leqslant \nu \leqslant 1$, replacing ν by $1-\nu$ in the first case implies the desired conclusion. □

REMARK 2.2. It observes from Theorem 2.2 that

$$\omega\left(|T|^{\nu}U|T|^{1-\nu}+|T|^{1-\nu}U|T|^{\nu}\right) \leqslant \omega\left(4r|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}+(1-2r)(|T|U+U|T|)\right)$$

which can be written as

$$\begin{aligned}2\omega\left(\widehat{T}(\nu)\right) &\leqslant \omega\left(4r\widehat{T}\left(\frac{1}{2}\right)+2(1-2r)\widehat{T}(0)\right)\\ &= \omega\left(4r\widetilde{T}+2(1-2r)\widehat{T}\right)\end{aligned}$$

i.e.,

$$\omega\left(\widehat{T}(\nu)\right) \leqslant \omega\left(2r\widetilde{T}+(1-2r)\widehat{T}\right).$$

Integral inequalities have attracted several researchers' attention in operator theory, as found in [14]. In the following result, we present possible bounds for the numerical radius of the integral of the Heinz means.

THEOREM 2.3. *Let $T,X \in M_n$ such that T be positive definite. Then for any $\alpha,\beta \in \mathbb{R}$,*

$$\begin{aligned}&\omega\left(T^{\frac{\alpha+\beta}{2}}XT^{1-\frac{\alpha+\beta}{2}}+T^{1-\frac{\alpha+\beta}{2}}XT^{\frac{\alpha+\beta}{2}}\right)\\ &\leqslant \frac{1}{|\beta-\alpha|}\omega\left(\int_{\alpha}^{\beta}\left(T^{\nu}XT^{1-\nu}+T^{1-\nu}XT^{\nu}\right)d\nu\right)\\ &\leqslant \frac{1}{2}\omega\left(T^{\alpha}XT^{1-\alpha}+T^{1-\alpha}XT^{\alpha}+T^{\beta}XT^{1-\beta}+T^{1-\beta}XT^{\beta}\right).\end{aligned}$$

Proof. Without loss of generality, assume that $\alpha<\beta$. Let $T=diag(\lambda_i)\geqslant 0$. Of course, if T is not diagonal, then using the spectral decomposition $T=Udiag(\lambda_i)U^*$ and noting that $\omega(\cdot)$ is weakly unitarily invariant imply the result.

Lemma 1.1 implies that, for $i \neq j$,

$$\begin{aligned}\left(T^{\frac{\alpha+\beta}{2}}XT^{1-\frac{\alpha+\beta}{2}}+T^{1-\frac{\alpha+\beta}{2}}XT^{\frac{\alpha+\beta}{2}}\right)_{ij} &= \lambda_i^{\frac{\alpha+\beta}{2}}x_{ij}\lambda_j^{1-\frac{\alpha+\beta}{2}}+\lambda_i^{1-\frac{\alpha+\beta}{2}}x_{ij}\lambda_j^{\frac{\alpha+\beta}{2}}\\ &= \frac{\lambda_i^{\frac{\beta-\alpha}{2}}(\log\lambda_i-\log\lambda_j)\lambda_j^{\frac{\beta-\alpha}{2}}}{\lambda_i^{\beta-\alpha}-\lambda_j^{\beta-\alpha}}\eta_{ij},\end{aligned}$$

where

$$\begin{aligned}\eta_{ij} &= \frac{x_{ij}}{\log\lambda_i - \log\lambda_j}\left(-\lambda_i^{\beta}\lambda_j^{1-\beta} + \lambda_i^{1-\beta}\lambda_j^{\beta} + \lambda_i^{\alpha}\lambda_j^{1-\alpha} - \lambda_i^{1-\alpha}\lambda_j^{\alpha}\right)\\ &= x_{ij}\int_{\alpha}^{\beta}\left(\lambda_i^{v}\lambda_j^{1-v} + \lambda_i^{1-v}\lambda_j^{v}\right)dv\\ &= \int_{\alpha}^{\beta}\left(\lambda_i^{v}x_{ij}\lambda_j^{1-v} + \lambda_i^{1-v}x_{ij}\lambda_j^{v}\right)dv.\end{aligned}$$

This means that

$$T^{\frac{\alpha+\beta}{2}}XT^{1-\frac{\alpha+\beta}{2}} + T^{1-\frac{\alpha+\beta}{2}}XT^{\frac{\alpha+\beta}{2}} = Y\circ\left(\int_{\alpha}^{\beta}\left(T^{v}XT^{1-v} + T^{1-v}XT^{v}\right)dv\right)$$

where Y is the Hermitian matrix with entries

$$y_{ij} = \begin{cases} \dfrac{\lambda_i^{\frac{\beta-\alpha}{2}}\left(\log\lambda_i - \log\lambda_j\right)\lambda_j^{\frac{\beta-\alpha}{2}}}{\lambda_i^{\beta-\alpha} - \lambda_j^{\beta-\alpha}}; & \text{when } i \neq j\\ \dfrac{1}{\beta-\alpha}; & \text{when } i = j\end{cases}.$$

Notice that $\frac{1}{\beta-\alpha}$ follows from the above integral when $i = j$, up to a scalar factor. It has been shown in [14, Theorem 4.1] that $Y \geqslant O$. Thus,

$$\begin{aligned}\omega\left(T^{\frac{\alpha+\beta}{2}}XT^{1-\frac{\alpha+\beta}{2}} + T^{1-\frac{\alpha+\beta}{2}}XT^{\frac{\alpha+\beta}{2}}\right) &= \omega\left(Y\circ\left(\int_{\alpha}^{\beta}\left(T^{v}XT^{1-v} + T^{1-v}XT^{v}\right)dv\right)\right)\\ &\leqslant \frac{1}{\beta-\alpha}\omega\left(\int_{\alpha}^{\beta}\left(T^{v}XT^{1-v} + T^{1-v}XT^{v}\right)dv\right).\end{aligned}$$

So, for arbitrary α,β,

$$\omega\left(T^{\frac{\alpha+\beta}{2}}XT^{1-\frac{\alpha+\beta}{2}} + T^{1-\frac{\alpha+\beta}{2}}XT^{\frac{\alpha+\beta}{2}}\right) \leqslant \frac{1}{|\beta-\alpha|}\omega\left(\int_{\alpha}^{\beta}\left(T^{v}XT^{1-v} + T^{1-v}XT^{v}\right)dv\right).$$

To prove the second inequality, by Lemma 1.1 and an argument similar to that in the proof of the first inequality, we have

$$\int_{\alpha}^{\beta}\left(T^{v}XT^{1-v} + T^{1-v}XT^{v}\right)dv = Z\circ\left(T^{\alpha}XT^{1-\alpha} + T^{1-\alpha}XT^{\alpha} + T^{\beta}XT^{1-\beta} + T^{1-\beta}XT^{\beta}\right)$$

where Z is the Hermitian matrix with entries

$$z_{ij} = \begin{cases} \dfrac{\lambda_i^{\beta-\alpha} - \lambda_j^{\beta-\alpha}}{\left(\log\lambda_i - \log\lambda_j\right)\left(\lambda_i^{\beta-\alpha} + \lambda_j^{\beta-\alpha}\right)}; & \text{when } i \neq j\\ \dfrac{\beta-\alpha}{2}; & \text{when } i = j\end{cases}.$$

From [14, Theorem 4.1] we know that $Z \geqslant O$. Thus,

$$\begin{aligned}
&\omega\left(\int_{\alpha}^{\beta}\left(T^{\nu}XT^{1-\nu}+T^{1-\nu}XT^{\nu}\right)d\nu\right)\\
&=\omega\left(Z\circ\left(T^{\alpha}XT^{1-\alpha}+T^{1-\alpha}XT^{\alpha}+T^{\beta}XT^{1-\beta}+T^{1-\beta}XT^{\beta}\right)\right)\\
&\leqslant\frac{\beta-\alpha}{2}\omega\left(T^{\alpha}XT^{1-\alpha}+T^{1-\alpha}XT^{\alpha}+T^{\beta}XT^{1-\beta}+T^{1-\beta}XT^{\beta}\right)
\end{aligned}$$

as required. □

REMARK 2.3. It follows from Theorem 2.3 that

$$\omega\left(\widehat{T}\left(\frac{\alpha+\beta}{2}\right)\right)\leqslant\frac{1}{|\beta-\alpha|}\omega\left(\int_{\alpha}^{\beta}\widehat{T}(\nu)\,d\nu\right)\leqslant\frac{\omega\left(\widehat{T}(\alpha)\right)+\omega\left(\widehat{T}(\beta)\right)}{2}.$$

A possible arithmetic-geometric mean inequality for the numerical radius can be stated as follows. We should remark that, in the next result, a similar bound for $\omega(TSX)$ cannot be found similarly. This idea was discussed in [20].

THEOREM 2.4. *Let $T,X \in M_n$. Then for any $t \geqslant 0$,*

$$\begin{aligned}
(2+t)\,\omega\left(TXT^{*}\right)&\leqslant\omega\left(|T|^{2}X+t\,|T|\,X\,|T|+X|T|^{2}\right)\\
&\leqslant\frac{2+t}{2}\omega\left(|T|^{2}X+X|T|^{2}\right).
\end{aligned}$$

Proof. Let $T=diag\,(\lambda_i)\geqslant 0$. By Lemma 1.1, we have for $r\geqslant\nu>0$ or $r\leqslant\nu<0$, or $r\geqslant 0\geqslant\nu$, or $r\leqslant 0\leqslant\nu$ and $t\geqslant 0$,

$$T^{\nu}X+t\left(T^{\mu\nu}XT^{\nu(1-\mu)}+T^{\nu(1-\mu)}XT^{\mu\nu}\right)+XT^{\nu}=H\circ\left(T^{r}XT^{\nu-r}+T^{\nu-r}XT^{r}\right)$$

where

$$h_{ij}=\begin{cases}\dfrac{\lambda_i^{\nu}+t\left(\lambda_i^{\mu\nu}\lambda_j^{\nu(1-\mu)}+\lambda_i^{\nu(1-\mu)}\lambda_j^{\mu\nu}+\lambda_j^{\nu}\right)}{\lambda_i^{r}\lambda_j^{\nu-r}+\lambda_i^{\nu-r}\lambda_j^{r}}; & \text{when } i\neq j\\ 1+t; & \text{when } i=j\end{cases}.$$

As is shown in the proof of Theorem 4.4 in [22], $H\geqslant O$. Thus,

$$\begin{aligned}
&\omega\left(T^{\nu}X+t\left(T^{\mu\nu}XT^{\nu(1-\mu)}+T^{\nu(1-\mu)}XT^{\mu\nu}\right)+XT^{\nu}\right)\\
&=\omega\left(H\circ\left(T^{r}XT^{\nu-r}+T^{\nu-r}XT^{r}\right)\right)\\
&\leqslant(1+t)\,\omega\left(T^{r}XT^{\nu-r}+T^{\nu-r}XT^{r}\right)
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\omega\left(T^{\nu}X+t\left(T^{\mu\nu}XT^{\nu(1-\mu)}+T^{\nu(1-\mu)}XT^{\mu\nu}\right)+XT^{\nu}\right)\\
&\leqslant(1+t)\,\omega\left(T^{r}XT^{\nu-r}+T^{\nu-r}XT^{r}\right).
\end{aligned}$$

If we set $\mu=\frac{1}{2}$ and replace t by $\frac{t}{2}$, in the above inequality, we obtain

$$2\omega\left(T^{\nu}X+tT^{\frac{\nu}{2}}XT^{\frac{\nu}{2}}+XT^{\nu}\right)\leqslant(2+t)\,\omega\left(T^{r}XT^{\nu-r}+T^{\nu-r}XT^{r}\right). \tag{2.1}$$

In particular, the case $\nu=2$ gives,

$$2\omega\left(T^{2}X+tTXT+XT^{2}\right)\leqslant(2+t)\,\omega\left(T^{2}X+XT^{2}\right). \tag{2.2}$$

On the other hand, one can see that for any $-2<t\leqslant 2$ and $1\leqslant 2r\leqslant 3$,

$$(2+t)\,\omega\left(T^{r}XT^{2-r}+T^{2-r}XT^{r}\right)\leqslant 2\omega\left(T^{2}X+tTXT+XT^{2}\right).$$

Indeed, this inequality is a consequence of the observation that (see Lemma 1.1)

$$(2+t)\left(T^{r}XT^{2-r}+T^{2-r}XT^{r}\right)=K\circ(T^{2}X+tTXT+XT^{3}),$$

where

$$k_{ij}=(2+t)\left(\frac{\lambda_i^{r}\lambda_j^{2-r}+\lambda_i^{2-r}\lambda_j^{r}}{\lambda_i^{2}+t\lambda_i\lambda_j+\lambda_j^{2}}\right).$$

The matrix K is positive definite by [26, Theorem 6]. Thus, when $r=1$, we have

$$(2+t)\,\omega(TXT)\leqslant\omega\left(T^{2}X+tTXT+XT^{2}\right). \tag{2.3}$$

Combining inequalities (2.2) and (2.3), we get

$$\begin{aligned}(2+t)\,\omega(TXT)&\leqslant\omega\left(T^{2}X+tTXT+XT^{2}\right)\\&\leqslant\frac{2+t}{2}\omega\left(T^{2}X+XT^{2}\right).\end{aligned} \tag{2.4}$$

Now, if we assume T is an arbitrary matrix with the Cartesian decomposition $T=U|T|$, we get from (2.4) that

$$\begin{aligned}(2+t)\,\omega(TXT^{*})&=(2+t)\,\omega(U|T|X|T|U^{*})\\&=(2+t)\,\omega(|T|X|T|)\\&\leqslant\omega\left(|T|^{2}X+t|T|X|T|+X|T|^{2}\right)\\&\leqslant\frac{2+t}{2}\omega\left(|T|^{2}X+X|T|^{2}\right).\end{aligned}$$

This completes the proof. □

REMARK 2.4. Assume that $T=U|T|$ is the polar decomposition of T. The second inequality in Theorem 2.4 can be written in the following form

$$\omega(TXT^{*})\leqslant\frac{1}{2+t}\omega\left(T|T|XU^{*}+UX|T|T^{*}+tTXT^{*}\right), \tag{2.5}$$

due to

$$\begin{aligned}\omega(TXT^*) &\leqslant \frac{1}{2+t}\omega\left(|T|^2X+X|T|^2+t|T|X|T|\right)\\ &= \frac{1}{2+t}\omega\left(U\left(|T||T|X+X|T||T|+t|T|X|T|\right)U^*\right)\\ &= \frac{1}{2+t}\omega\left(T|T|XU^*+UX|T|T^*+tTXT^*\right).\end{aligned}$$

Notice that (2.5) is a generalization and refinement of [20, Lemma 2.1].

3. Norm bounds

In this section, we present some bounds for the operator norm of certain operators.

THEOREM 3.1. *Let A, B, and C be operators in $\mathscr{B}(\mathscr{H})$, where A and B are positive. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator in $\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$, then*

$$\|BCA+AC^*B\| \leqslant \frac{\|A^2+B^2\|}{2}\left\|\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\right\|.$$

Proof. Let x,y not both equal to zero, and let $z=\frac{\begin{bmatrix} x \\ y \end{bmatrix}}{\sqrt{\|x\|^2+\|y\|^2}}$. Then, z is a unit vector in $\mathscr{H}\oplus\mathscr{H}$. On the other hand, notice that [15, Lemma 1]

$$\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant O \Leftrightarrow |\langle Cx,y\rangle| \leqslant \sqrt{\langle Ax,x\rangle\langle By,y\rangle};\ (\forall x,y\in\mathscr{H}). \tag{3.1}$$

Therefore,

$$\begin{aligned}\frac{4\Re\langle Cx,y\rangle}{\|x\|^2+\|y\|^2} &\leqslant \frac{2|\langle Cx,y\rangle|+2\Re\langle Cx,y\rangle}{\|x\|^2+\|y\|^2} \quad (\text{since } \Re a\leqslant|a| \text{ for any } a\in\mathbb{C})\\ &\leqslant \frac{2\sqrt{\langle Ax,x\rangle\langle By,y\rangle}+2\Re\langle Cx,y\rangle}{\|x\|^2+\|y\|^2} \quad (\text{by } (3.1))\\ &\leqslant \frac{\langle Ax,x\rangle+\langle By,y\rangle+2\Re\langle Cx,y\rangle}{\|x\|^2+\|y\|^2}\\ &\qquad (\text{by the arithmetic-geometric mean inequality})\\ &= \frac{\left\langle \begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle}{\|x\|^2+\|y\|^2}\\ &= \left\langle \begin{bmatrix} A & C^* \\ C & B \end{bmatrix} z,z\right\rangle\\ &\leqslant \left\|\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}\right\|.\end{aligned}$$

Thus,

$$\Re\langle Cx,y\rangle \leqslant \left(\frac{\|x\|^2+\|y\|^2}{4}\right)\left\|\begin{bmatrix} A & C^* \\ C & B\end{bmatrix}\right\| \tag{3.2}$$

for all $x,y\in\mathscr{H}$. Now, replacing x and y by Ax and Bx, in (3.2), we infer that

$$\begin{aligned}\Re\langle BCAx,x\rangle &= \Re\langle CAx,Bx\rangle \\ &\leqslant \left(\frac{\|Ax\|^2+\|Bx\|^2}{4}\right)\left\|\begin{bmatrix} A & C^* \\ C & B\end{bmatrix}\right\| \\ &= \frac{\left\langle \left(A^2+B^2\right)x,x\right\rangle}{4}\left\|\begin{bmatrix} A & C^* \\ C & B\end{bmatrix}\right\| \\ &\leqslant \frac{\left\|A^2+B^2\right\|}{4}\left\|\begin{bmatrix} A & C^* \\ C & B\end{bmatrix}\right\|\end{aligned}$$

i.e.,

$$\Re\langle BCAx,x\rangle \leqslant \frac{\left\|A^2+B^2\right\|}{4}\left\|\begin{bmatrix} A & C^* \\ C & B\end{bmatrix}\right\|.$$

So,

$$\|\Re(BCA)\| \leqslant \frac{\left\|A^2+B^2\right\|}{4}\left\|\begin{bmatrix} A & C^* \\ C & B\end{bmatrix}\right\|$$

as desired. □

COROLLARY 3.1. *Let $A,B\in\mathscr{B}(\mathscr{H})$, where A is self-adjoint, $B\geqslant O$, and $\pm A\leqslant B$. Then*

$$\|BAB\| \leqslant \frac{\|B\|^2}{2}\max\left(\|B+A\|,\|B-A\|\right).$$

In particular,

$$\|\,|A|\;A\;|A|\,\| \leqslant \frac{\|A\|^2}{2}\max\left(\|\,|A|+A\,\|,\|\,|A|-A\,\|\right).$$

Proof. Let $X=\begin{bmatrix} B & A \\ A & B\end{bmatrix}$. The matrix $\begin{bmatrix} B & A \\ A & B\end{bmatrix}$ is unitarily equivalent to $\begin{bmatrix} B+A & O \\ O & B-A\end{bmatrix}$, indeed,

$$U\begin{bmatrix} B & A \\ A & B\end{bmatrix}U^* = \begin{bmatrix} B+A & O \\ O & B-A\end{bmatrix};\ U=\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I\end{bmatrix}.$$

Meanwhile, $\pm A\leqslant B$, so it follows that X is positive. Thus, by Theorem 3.1, we have

$$\begin{aligned}\|BAB\| &\leqslant \frac{\left\|B^2\right\|}{2}\left\|\begin{bmatrix} B & A \\ A & B\end{bmatrix}\right\| \\ &= \frac{\left\|B^2\right\|}{2}\left\|\begin{bmatrix} B+A & O \\ O & B-A\end{bmatrix}\right\| \\ &= \frac{\left\|B^2\right\|}{2}\max\left(\|B+A\|,\|B-A\|\right),\end{aligned}$$

as desired.

To prove the second inequality, notice that if A is a self-adjoint operator, then $\pm A \leqslant |A|$. Now, the result follows from the first inequality. □

COROLLARY 3.2. *Let A, B, and C be operators in $\mathscr{B}(\mathscr{H})$, where A and B are positive. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is a positive operator in $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, then*

$$\|(A+B)(\Re C)(A+B)\| \leqslant \frac{\|A+B\|^2}{4} \max\left(\|A+B+2\Re C\|, \|A+B-2\Re C\|\right).$$

Proof. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \geqslant O$, then $\begin{bmatrix} B & C \\ C^* & A \end{bmatrix} \geqslant O$. So, $\begin{bmatrix} A+B & 2\Re C \\ 2\Re C & A+B \end{bmatrix} \geqslant O$. Now, applying Theorem 3.1, we get

$$\begin{aligned}
\|(A+B)(\Re C)(A+B)\| &\leqslant \frac{\|A+B\|^2}{4} \left\| \begin{bmatrix} A+B & 2\Re C \\ 2\Re C & A+B \end{bmatrix} \right\| \\
&= \frac{\|A+B\|^2}{4} \left\| \begin{bmatrix} A+B+2\Re C & O \\ O & A+B-2\Re C \end{bmatrix} \right\| \\
&= \frac{\|A+B\|^2}{4} \max\left(\|A+B+2\Re C\|, \|A+B-2\Re C\|\right)
\end{aligned}$$

as desired. □

COROLLARY 3.3. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be positive operators. Then*

$$\|AB+BA\| \leqslant \frac{\left\|A^{\frac{4}{3}}+B^{\frac{4}{3}}\right\| \left\|A^{\frac{2}{3}}+B^{\frac{2}{3}}\right\|}{2}.$$

Proof. Let $X = \begin{bmatrix} A^{\frac{1}{2}} & B^{\frac{1}{2}} \\ O & O \end{bmatrix}$. Then $X^*X = \begin{bmatrix} A & A^{\frac{1}{2}}B^{\frac{1}{2}} \\ B^{\frac{1}{2}}A^{\frac{1}{2}} & B \end{bmatrix} \geqslant O$. Now, by employing Theorem 3.1, we get

$$\left\|B^{\frac{3}{2}}A^{\frac{3}{2}} + A^{\frac{3}{2}}B^{\frac{3}{2}}\right\| \leqslant \frac{\|A^2+B^2\|}{2} \left\| \begin{bmatrix} A & A^{\frac{1}{2}}B^{\frac{1}{2}} \\ B^{\frac{1}{2}}A^{\frac{1}{2}} & B \end{bmatrix} \right\|.$$

Notice that

$$\begin{aligned}
\left\| \begin{bmatrix} A & A^{\frac{1}{2}}B^{\frac{1}{2}} \\ B^{\frac{1}{2}}A^{\frac{1}{2}} & B \end{bmatrix} \right\| &= \left\| \begin{bmatrix} A^{\frac{1}{2}} & O \\ B^{\frac{1}{2}} & O \end{bmatrix} \begin{bmatrix} A^{\frac{1}{2}} & B^{\frac{1}{2}} \\ O & O \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} A^{\frac{1}{2}} & B^{\frac{1}{2}} \\ O & O \end{bmatrix} \begin{bmatrix} A^{\frac{1}{2}} & O \\ B^{\frac{1}{2}} & O \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} A+B & O \\ O & O \end{bmatrix} \right\| \\
&= \|A+B\|,
\end{aligned}$$

i.e.,

$$\left\|A^{\frac{3}{2}}B^{\frac{3}{2}}+B^{\frac{3}{2}}A^{\frac{3}{2}}\right\| \leqslant \frac{\left\|A^2+B^2\right\|\|A+B\|}{2}.$$

Replacing A and B by $A^{\frac{2}{3}}$ and $B^{\frac{2}{3}}$, we deduce the desired result. □

If A and B are arbitrary, then letting $X=\begin{bmatrix} A & B \\ O & O\end{bmatrix}$ and use the positivity of the matrix $X^*X=\begin{bmatrix} |A|^2 & A^*B \\ B^*A & |B|^2\end{bmatrix}$. The same arguments imply that

$$\left\||A|^2A^*B|B|^2+|B|^2B^*A|A|^2\right\| \leqslant \frac{\left\||A|^4+|B|^4\right\|\left\||A|^2+|B|^2\right\|}{2}.$$

REMARK 3.1. Let $T=U|T|$ be the polar decomposition of T. Replacing $A=|T|^t$ and $B=|T|^{1-t}$ in Corollary 3.3 with $0\leqslant t\leqslant 1$.

$$\begin{aligned}
\left\||T|^t|T|^{1-t}+|T|^{1-t}|T|^t\right\| &= \left\|U\left(|T|^t|T|^{1-t}+|T|^{1-t}|T|^t\right)U^*\right\| \\
&= \left\|U|T|^t|T|^{1-t}U^*+U|T|^{1-t}|T|^tU^*\right\| \\
&= 2\|U|T|U^*\| \\
&= 2\|\,|T^*|\,\| \quad \text{(by [7, p. 58])} \\
&= 2\|T\|.
\end{aligned}$$

Thus,

$$\|T\| \leqslant \frac{\left\||T|^{\frac{4}{3}t}+|T|^{\frac{4}{3}(1-t)}\right\|\left\||T|^{\frac{2}{3}t}+|T|^{\frac{2}{3}(1-t)}\right\|}{4};\ (0\leqslant t\leqslant 1)$$

for any $T\in\mathscr{B}(\mathscr{H})$. The equality holds when $t=\frac{1}{2}$. Indeed, in this case, we obtain $\|T\|\leqslant\left\||T|^{\frac{2}{3}}\right\|\left\||T|^{\frac{1}{3}}\right\|$, but the right side is equal to $\|T\|$ (remember, if $X\in\mathscr{B}(\mathscr{H})$, and if f is a non-negative increasing function on $[0,\infty)$, then $\|f(|X|)\|=f(\|X\|)$).

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(Received August 28, 2024)

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