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# NUMERICAL RADIUS INEQUALITIES FOR THE WEIGHTED SUMS OF HILBERT SPACE OPERATORS

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*Abstract.* Using the generalized Young inequality, operator convexity and positive operator matrix, we extend and refine some numerical radius inequalities for the weighted sums of Hilbert space operators. Precisely, for $r,t \in [1,2]$, if $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$ and $p_k \geqslant 0$ with $\sum_{k=1}^n p_k = P_n$, then

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^n p_k T_k V_k\right) \leqslant \frac{1}{2^{1/t}} w^{\frac{2}{t}}\left(\frac{1}{P_n}\sum_{k=1}^n p_k\left(|T_k^*|^{2rt} + i|V_k|^{2rt}\right)\right) - \inf_{\|x\|=1}\phi(x),$$

where

$$\phi(x) = \frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k|T_k^*|^{2r}\right)x,x\right\rangle - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k|V_k|^{2r}\right)x,x\right\rangle\right)^2.$$

## 1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space with usual inner product $\langle\cdot,\cdot\rangle$ and $\mathscr{B}(\mathscr{H})$ be $C^*$-algebra of all bounded linear operators on $\mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, we denote by $|T|$ the absolute value operator of $T$, that is $|T| = (T^*T)^{\frac{1}{2}}$, where $T^*$ is the adjoint operator of $T$, see [15]. Recall that a function $f : J \longrightarrow \mathbb{R}$ is convex if

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y),$$

for all $\alpha \in [0,1]$ and all $x,y \in J$. It is well known [21] that a continuous convex function $f$ in a real interval $I \subseteq \mathbb{R}$ can satisfy the property

$$f\left(\frac{1}{P_n}\sum_{k=1}^n p_k a_k\right) \leqslant \frac{1}{P_n}\sum_{k=1}^n p_k f(a_k), \tag{1.1}$$

where $a_k \in I$, $1 \leqslant k \leqslant n$ are given data points and $p_1, p_2, \dots, p_n$ are a set of non-negative real numbers constrained by $\sum_{k=1}^n p_k = P_n$. Moreover, if $f$ is concave, then the inequality (1.1) is reversed.

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For any $T \in \mathscr{B}(\mathscr{H})$, the numerical range of $T$ is defined as $W(T) = \{\langle Tx, x\rangle : \|x\| = 1, x \in \mathscr{H}\}$, which is a nonempty, bounded and convex subset of the complex plane $\mathbb{C}$ (see [23]). The numerical radius $w(T)$ and the spectral radius $\rho(T)$ are defined respectively as

$$w(T) = \sup\{|z| : z \in W(T)\} \text{ and } \rho(T) = \sup\{|z| : z \in \sigma(T)\}$$

where $\sigma(T) = \{z \in \mathbb{C} : zI - T \text{ is not invertible}\}$ is the spectrum of $T$. Also, $\overline{W(T)}$ is a convex subset of the complex plane containing $\sigma(T)$ (see [11, Chapter 2]). We know that the numerical radius $w(\cdot)$ is a norm on $\mathscr{H}$, which is equivalent to the usual operator norm, $\|T\| = \sup\{\|Tx\| : \|x\| = 1, x \in \mathscr{H}\}$. A basic relation between the numerical radius and the norm of an operator is provided below. For $T \in \mathscr{B}(\mathscr{H})$,

$$\frac{1}{2}\|T\| \leqslant w(T) \leqslant \|T\|. \tag{1.2}$$

Clearly, the inequalities (1.2) are sharp, i.e., $w(T) = \|T\|$ if $T$ is normal and $w(T) = \frac{1}{2}\|T\|$ if $T^2 = 0$, see [12]. Furthermore, the norm $w(\cdot)$ satisfies self-adjoint property i.e., $w(T) = w(T^*)$, weakly unitary invariant property i.e., $w(U^*TU) = w(T)$, for every unitary $U \in \mathscr{B}(\mathscr{H})$. Although the numerical radius does not have the submultiplicative property, it satisfies $w(TV) \leqslant 4w(T)w(V)$, for $T, V \in \mathscr{B}(\mathscr{H})$. Similar to the usual operator norm, the numerical radius also satisfies the power inequality, which asserts that

$$w(T^n) \leqslant w^n(T) \text{ for } n = 1, 2, \cdots. \tag{1.3}$$

Over the years, various generalizations and refinements of the inequalities (1.2) have been dicussed in [6, 9, 14, 16, 17, 20]. In particular, the improvement of the second inequality in (1.2) has been established in [15, 16], Kittaneh proved that for any $T \in \mathscr{B}(\mathscr{H})$,

$$w(T) \leqslant \frac{1}{2}\||T| + |T^*|\| \leqslant \frac{1}{2}\left(\|T\| + \|T^2\|^{1/2}\right) \tag{1.4}$$

and

$$\frac{1}{4}\left\||T|^2 + |T^*|^2\right\| \leqslant w^2(T) \leqslant \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{1.5}$$

More about powers of the absolute value of operators, one can refer to the following inequality obtained by Bhunia et al. in [5, 7], which simultaneously generalized and improved the second inequality (1.2) and the second inequality (1.4) as following

$$w^2(T) \leqslant (\|\alpha|T| + (1-\alpha)|T^*|\|)\,\|T\|, \tag{1.6}$$

where $0 \leqslant \alpha \leqslant 1$ and

$$w^{2r}(T) \leqslant \frac{1}{4}\left\||T|^{2r} + |T^*|^{2r}\right\| + \frac{1}{2}w(|T|^r|T^*|^r), \tag{1.7}$$

for all $r \geqslant 1$. And then, El-Haddad [10] also refined and generalized the first inequality (1.4) and the second inequality (1.5) into

$$w^r(T) \leqslant \frac{1}{2}\left\| |T|^{2r\alpha} + |T^*|^{2r(1-\alpha)} \right\|, \tag{1.8}$$

$$w^r(T_1+T_2) \leqslant 2^{r-2}\left\| |T_1|^{2r\alpha} + |T_2|^{2r\alpha} + |T_1^*|^{2r(1-\alpha)} + |T_2^*|^{2r(1-\alpha)} \right\|, \tag{1.9}$$

where $0<\alpha<1,\ r\geqslant 1$. Recently, Moradi in [19] established the following inequalities which is stronger than (1.8), namely,

$$w(T_1+T_2) \leqslant \frac{1}{\sqrt{2}} w\big(|T_1|+|T_2|+i(|T_1^*|+|T_2^*|)\big). \tag{1.10}$$

Afterwards, the study of numerical radius was extended to the product of two bounded operators. For example, Dragomir et al. [8] proved that if $T,V \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$, then

$$w^r(TV) \leqslant \frac{1}{2}\left\| |T^*|^{2r} + |V|^{2r} \right\|. \tag{1.11}$$

Meanwhile, Hedarbeygi et al. [13] established a refinement of (1.11) and proved that if $T,V \in \mathscr{B}(\mathscr{H})$ and $r \geqslant 1$, then

$$w^{2r}(TV) \leqslant \frac{1}{2} w^r\left(|T^*|^2|V|^2\right) + \frac{1}{4}\left\| |T^*|^{4r} + |V|^{4r} \right\|. \tag{1.12}$$

Moreover, they also refined the right-hand side of (1.2) to show that, if $T_1,\ T_2,\ V_1,\ V_2 \in \mathscr{B}(\mathscr{H})$ and $r,\ s \geqslant 1$ then

$$w^{2r}\left(\frac{T_1V_1+T_2V_2}{2}\right) \leqslant \left\| \frac{|T_1^*|^{2r}+|T_2^*|^{2r}}{2} \right\|^{\frac{1}{r}} \left\| \frac{|V_1|^{2s}+|V_2|^{2s}}{2} \right\|^{\frac{1}{s}}. \tag{1.13}$$

Very recently, Bhunia and Pual [4] showed that if $T_k,\ V_k \in \mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$ and $f,\ g$ are nonnegative functions on $[0,\infty)$ which are continous and satisfy the relation $f(u)g(u)=u$ for all $u\in[0,\infty)$, it holds that

$$w^r\left(\sum_{k=1}^n T_kV_k\right) \leqslant \frac{n^{r-1}}{\sqrt{2}} w\left(\sum_{k=1}^n \rho^r(V_k)\big(f^{2r}(|T_k|)+ig^{2r}(|T_k^*|)\big)\right) \tag{1.14}$$

for $r \geqslant 1$.

These operators in (1.13) and (1.14) can be viewed as special cases of the weighted sums of operators denoted as $\frac{1}{P_n}\sum_{k=1}^n p_kT_k$ on $\mathscr{H}$ with $\sum_{k=1}^n p_k = P_n$. Motivated by the above conclusions for numerical radius inequalities especially from [4, 5, 10, 13], this naturally leads us to ask a more general question.

*How can we represent the numerical radius inequalities for the weighted sums of operators* $\frac{1}{P_n}\sum_{k=1}^n p_kT_k$ *on* $\mathscr{H}$ *with* $\sum_{k=1}^n p_k = P_n$*?*

The main aim of this work is to develop several numerical radius inequalities for the weighted sums of bounded linear operators. It is organized as follows: we collect a few results that are required to state and prove the results in the subsequent sections in Section 2. Various numerical radius inequalities for the weighted sums of operators on the Hilbert spaces are established by utilizing $2\times 2$ operator matrices in Section 3, and then these bounds are shown as the generalization and improvement on the existing bounds in (1.6), (1.7), (1.10), (1.12) and (1.13). Meanwhile, some examples are provided to illustrate our results. In Section 4, we further prove some numerical radius inequalities for the weighted sums of product operators by using two non-negative continuous functions on $[0,\infty)$, which generalize and improve the the existing inequalities (1.2), (1.8), (1.9) and (1.14).

## 2. Preliminaries

To prove our generalized numerical radius inequalities, we need several well-known lemmas. The first lemma follows from the spectral theorem for positive operators and Jensen's inequality.

LEMMA 2.1. [22, p. 20] *If $A\in\mathscr{B}(\mathscr{H})$ is positive operator, i.e, $T\geqslant 0$. Then for all $x\in\mathscr{H}$ with $\|x\|=1$, we have*
*(a) $\langle Tx,x\rangle^r\leqslant\langle T^rx,x\rangle$ for $r\geqslant 1$,*
*(b) $\langle T^rx,x\rangle\leqslant\langle Tx,x\rangle^r$ for $r\in[0,1]$.*

Manasrah and Kittaneh [1] obtained the following result which is a generalization of the scalar Young inequality.

LEMMA 2.2. *Let $a,b>0$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then for $m=1,2,\ldots$, we have*

$$(a^{\frac{1}{p}}b^{\frac{1}{q}})^m+r_0^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2\leqslant\left(\frac{a^t}{p}+\frac{b^t}{q}\right)^{\frac{m}{t}},\quad t\geqslant 1 \tag{2.1}$$

*where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$. In particular, if $p=q=2$, then*

$$(\sqrt{ab})^m+\frac{1}{2^m}(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2\leqslant 2^{-\frac{m}{t}}\left(a^t+b^t\right)^{\frac{m}{t}}. \tag{2.2}$$

*For $m=1$ and $p=q=2$, we have*

$$\sqrt{ab}+\frac{1}{2}(\sqrt{a}-\sqrt{b})^2\leqslant 2^{-\frac{1}{t}}\left(a^t+b^t\right)^{\frac{1}{t}}. \tag{2.3}$$

We note that (2.2) may fail for the case $m\notin\mathbb{N}^*$. For example, let $a=4$, $b=1$, $t=1$, $m=\frac{3}{2}$, the inequality (2.2) is reversed. Third lemma involves $2\times 2$ positive operator matrix.

LEMMA 2.3. [18, Lemma 1] *Let* $A, B, C \in \mathscr{B}(\mathscr{H})$ *with* $A, B \geqslant 0$. *Then the operator matrix*

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

*is positive, if and only if*

$$|\langle Cx, y\rangle|^2 \leqslant \langle Ax, x\rangle \langle By, y\rangle,$$

*for all* $x, y \in \mathscr{H}$.

The fourth lemma is known as Buzano's inequality.

LEMMA 2.4. [3] *Let* $x, y, z \in \mathscr{H}$, *where* $\|z\| = 1$. *Then*

$$|\langle x, z\rangle \langle z, y\rangle| \leqslant \frac{\|x\|\|y\| + |\langle x, y\rangle|}{2}.$$

The last lemma is an extension of both the generalized mixed Schwarz inequality and the generalized Reid inequality.

LEMMA 2.5. [18, Theorem 5] *Let* $T, V \in \mathscr{B}(\mathscr{H})$ *be such that* $|T|V = V^*|T|$, *and let* $f$, $g$ *be nonnegative functions on* $[0, \infty)$ *which are continous and satisfy the relation* $f(u)g(u) = u$ *for all* $u \in [0, \infty)$, *then*

$$|\langle TVx, y\rangle| \leqslant \rho(V)\|f(|T|)x\|\|g(|T^*|)y\|,$$

*for any vectors* $x, y \in \mathscr{H}$.

## 3. numerical radius inequalities for the weighted sums of operators

In this part, we will build several generic numerical radius inequalities for the weighted sums of operators, which generalize and improve some existing numerical radius inequalities. The proofs of our main results depend on the operator convexity of the function $f(t) = t^l$, $l \in [1, 2]$, (see [2]).

THEOREM 3.1. *Let* $A_k, B_k, C_k \in \mathscr{B}(\mathscr{H})$ $(k = 1, \ldots, n)$ *with* $A_k, B_k \geqslant 0$ *and such that the operator matrices*

$$\begin{pmatrix} A_k & C_k^* \\ C_k & B_k \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

*are positive, then for* $r, s \in [1, 2]$ *and* $p_k \geqslant 0$ *with* $\sum_{k=1}^n p_k = P_n$, *we have*

$$w^2\left(\frac{1}{P_n}\sum_{k=1}^n p_k C_k\right) \leqslant \left\|\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right\|^{1/r} \left\|\frac{1}{P_n}\sum_{k=1}^n p_k B_k^s\right\|^{1/s}.$$

*In particular,*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kC_k\right) \leqslant \left\|\frac{1}{P_n}\sum_{k=1}^{n} p_kA_k^r\right\| \left\|\frac{1}{P_n}\sum_{k=1}^{n} p_kB_k^r\right\|.$$

*Proof.* Let $x \in \mathscr{H}$ with $\|x\| = 1$. Then by Lemma 2.3 we have

$$|\langle C_kx,x\rangle| \leqslant \langle A_kx,x\rangle^{1/2}\langle B_kx,x\rangle^{1/2},$$

for $k = 1,\dots,n$. If we multiply by $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$ and sum, then by the weighted Cauchy-Bunyakovsky-Schwarz discrete inequality, we have

$$\begin{aligned}\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kC_k\right)x,x\right\rangle\right|^2 &\leqslant \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k|\langle C_kx,x\rangle|\right)^2\\ &\leqslant \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k\langle A_kx,x\rangle^{1/2}\langle B_kx,x\rangle^{1/2}\right)^2\\ &\leqslant \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kA_k\right)x,x\right\rangle\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kB_k\right)x,x\right\rangle.\end{aligned}$$

Utilizing the arithmetic-geometric mean inequality and then the operator convexity of the function $f(t) = t^l,\ l \in [1,2]$ in inequality (1.1), we have

$$\begin{aligned}&\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kA_k\right)x,x\right\rangle\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kB_k\right)x,x\right\rangle\\ &\leqslant \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kA_k\right)^r x,x\right\rangle^{1/r}\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kB_k\right)^s x,x\right\rangle^{1/s}\\ &\leqslant \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kA_k^r\right)x,x\right\rangle^{1/r}\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kB_k^s\right)x,x\right\rangle^{1/s}.\end{aligned}$$

Consequently, we get

$$\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kC_k\right)x,x\right\rangle\right|^2 \leqslant \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kA_k^r\right)x,x\right\rangle^{1/r}\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_kB_k^s\right)x,x\right\rangle^{1/s}. \tag{3.1}$$

Taking the supremum over all unit vectors $x \in \mathscr{H}$, we deduce the desired result. □

Letting $p_k = \frac{1}{n}$ for $k = 1,\dots,n$ in Theorem 3.1, we get the following result.

COROLLARY 3.2. *Let $A_k, B_k, C_k \in \mathscr{B}(\mathscr{H})$ $(k = 1,\dots,n)$ with $A_k, B_k \geqslant 0$ and such that the operator matrices $\begin{pmatrix} A_k & C_k^* \\ C_k & B_k \end{pmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$ are positive, then for $r,s \in$*

$[1,2]$,

$$w^2\left(\frac{1}{n}\sum_{k=1}^{n}C_k\right)\leqslant\left\|\frac{1}{n}\sum_{k=1}^{n}A_k^r\right\|^{1/r}\left\|\frac{1}{n}\sum_{k=1}^{n}B_k^s\right\|^{1/s}.$$

By Theorem 3.1, we obtain the following corollary, which is a generalized form of the well-known inequality (1.13).

COROLLARY 3.3. *Let* $T_k, V_k\in\mathscr{B}(\mathscr{H})$ $(k=1,\ldots,n)$, *then for* $r, s\in[1,2]$ *and* $p_k\geqslant 0$ *with* $\sum_{k=1}^{n}p_k=P_n$, *we have*

$$w^2\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kT_kV_k\right)\leqslant\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k|T_k^*|^{2r}\right\|^{1/r}\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k|V_k|^{2s}\right\|^{1/s}.$$

*In particular,*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kT_kV_k\right)\leqslant\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k|T_k^*|^{2r}\right\|\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k|V_k|^{2r}\right\|.$$

*Proof.* Let $T_k, V_k\in\mathscr{B}(\mathscr{H})$ with $k=1,\ldots,n$. The operator matrices

$$\begin{pmatrix}T_kT_k^* & T_kV_k\\ V_k^*T_k^* & V_k^*V_k\end{pmatrix}\in\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$$

are positive, due to $|\langle V_k^*T_k^*x,y\rangle|^2\leqslant\langle|T_k^*|^2x,x\rangle\langle|V_k|^2x,x\rangle$, for all $x=y\in\mathscr{H}$, together with Lemma 2.3. Then letting $A_k=|T_k^*|^2$, $B_k=|V_k|^2$ and $C_k=V_k^*T_k^*$ in Theorem 3.1, we get the desired results. $\square$

THEOREM 3.4. *Let* $A_k, B_k, C_k\in\mathscr{B}(\mathscr{H})$ $(k=1,\ldots,n)$ *with* $A_k, B_k\geqslant 0$ *and such that the operator matrices*

$$\begin{pmatrix}A_k & C_k^*\\ C_k & B_k\end{pmatrix}\in\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$$

*are positive, then for* $r,t\in[1,2]$ *and* $p_k\geqslant 0$ *with* $\sum_{k=1}^{n}p_k=P_n$, *we have*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kC_k\right)\leqslant\frac{1}{2^{1/t}}w^{\frac{2}{t}}\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k\left(A_k^{rt}+iB_k^{rt}\right)\right)-\inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x)=\frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)x,x\right\rangle-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)x,x\right\rangle\right)^2.$$

*Proof.* By utilizing $s=r$ in inequality (3.1), we have

$$\begin{aligned}
&\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kC_k\right)x,x\right\rangle\right|^{2r}\\
&\leqslant\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)x,x\right\rangle\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)x,x\right\rangle\\
&\leqslant\left(\frac{\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)x,x\right\rangle^t+\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)x,x\right\rangle^t}{2}\right)^{\frac{2}{t}}\\
&\quad-\frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)x,x\right\rangle-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)x,x\right\rangle\right)^2\\
&\hspace{8cm}\text{(by inequality (2.2) when } m=2)\\
&\leqslant\left(\frac{\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)^t x,x\right\rangle+\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)^t x,x\right\rangle}{2}\right)^{\frac{2}{t}}-\inf_{\|x\|=1}\phi(x)\\
&\hspace{8cm}\text{(by Lemma 2.1 (a))}\\
&\leqslant\left(\frac{\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^{rt}\right)x,x\right\rangle+\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^{rt}\right)x,x\right\rangle}{2}\right)^{\frac{2}{t}}-\inf_{\|x\|=1}\phi(x)\\
&\hspace{8cm}\text{(by inequality (1.1))}\\
&\leqslant\left(\frac{\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^{rt}\right)x,x\right\rangle^2+\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^{rt}\right)x,x\right\rangle^2}{2}\right)^{\frac{1}{t}}-\inf_{\|x\|=1}\phi(x)\\
&\hspace{8cm}\text{(by convexity of } u^2)\\
&\leqslant\frac{1}{2^{1/t}}\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^{rt}\right)x,x\right\rangle+i\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^{rt}\right)x,x\right\rangle\right|^{\frac{2}{t}}-\inf_{\|x\|=1}\phi(x)\\
&=\frac{1}{2^{1/t}}\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k(A_k^{rt}+iB_k^{rt})\right)x,x\right\rangle\right|^{\frac{2}{t}}-\inf_{\|x\|=1}\phi(x).
\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

REMARK 3.5. In Theorem 3.4, it is obvious that

$$\inf_{\|x\|=1}\phi(x)>0\Leftrightarrow 0\notin\overline{W\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k(A_k^r-B_k^r)\right)}.$$

Choosing $t = 1$ in Theorem 3.4, we get the following result.

COROLLARY 3.6. *Let $A_k, B_k, C_k \in \mathscr{B}(\mathscr{H})$ $(k = 1,\dots,n)$ with $A_k, B_k \geqslant 0$ and such that the operator matrices*

$$\begin{pmatrix} A_k & C_k^* \\ C_k & B_k \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

*are positive, then for $r \in [1,2]$ and $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k C_k\right) \leqslant \frac{1}{2} w^2\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k\left(A_k^r + iB_k^r\right)\right) - \inf_{\|x\|=1} \phi(x),$$

*where*

$$\phi(x) = \frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^r\right)x, x\right\rangle - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^r\right)x, x\right\rangle\right)^2.$$

Choosing $n = 1$ in Corollary 3.6, we get the following result.

COROLLARY 3.7. *Let $A, B, C \in \mathscr{B}(\mathscr{H})$ with $A, B \geqslant 0$ and such that the operator matrice*

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

*is positive, then for $r \in [1,2]$,*

$$w^{2r}(C) \leqslant \frac{1}{2} w^2\left(A^r + iB^r\right) - \inf_{\|x\|=1} \phi(x),$$

*where*

$$\phi(x) = \frac{1}{4}\left(\langle A^r x, x\rangle - \langle B^r x, x\rangle\right)^2.$$

EXAMPLE 3.8. Let $r = 1$ and $A = \operatorname{diag}(5,1)$, $B = \operatorname{diag}(2,0)$ in Corollary 3.7. Then it is clear that $W(A - B) = [1,3]$ and

$$4\{\phi(x) : x \in \mathscr{H}, \|x\| = 1\} \subseteq \left[W(A-B)\right]^2 = [1,9].$$

So $\inf_{\|x\|=1} \phi(x) \geqslant \frac{1}{4}$.

REMARK 3.9. Let $r = 1$ in Corollary 3.7, we can obtain

$$w^2(C) \leqslant \frac{1}{2} w^2(A + iB) - \inf_{\|x\|=1} \phi(x), \tag{3.2}$$

where

$$\phi(x) = \frac{1}{4}\left(\langle Ax,x\rangle - \langle Bx,x\rangle\right)^2.$$

Setting $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$\begin{aligned}
|\langle Ax,x\rangle + i\langle Bx,x\rangle|^2 &= \langle Ax,x\rangle^2 + \langle Bx,x\rangle^2 \\
&\leqslant \left\langle A^2x,x\right\rangle + \left\langle B^2x,x\right\rangle \quad \text{(by Lemma 2.1 (a))} \\
&= \left\langle (A^2+B^2)x,x\right\rangle.
\end{aligned}$$

Then taking the supremum over all unit vectors in $\mathscr{H}$, we have $w^2(A+iB) \leqslant \|A^2 + B^2\|$. This together with (3.2) generalize and improve on the inequality in [5], namely,

$$w^2(C) \leqslant \frac{1}{2}\|A^2+B^2\|.$$

Letting $C_k = T_k,\ A_k = |T_k|$ and $B_k = |T_k^*|\ \ (k = 1,\dots,n)$ in Corollary 3.6, the operator matrices

$$\begin{pmatrix} |T_k| & T_k^* \\ T_k & |T_k^*| \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

are positive, due to $|\langle T_kx,y\rangle|^2 \leqslant \langle |T_k|x,x\rangle\langle |T_k^*|x,x\rangle$, for all $x = y \in \mathscr{H}$, and Lemma 2.3, we get the following result.

COROLLARY 3.10. *Let* $T_k \in \mathscr{B}(\mathscr{H})\ \ (k=1,\dots,n)$, *then for* $r \in [1,2]$ *and* $p_k \geqslant 0$ *with* $\sum_{k=1}^n p_k = P_n$, *we have*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^n p_kT_k\right) \leqslant \frac{1}{2}w^2\left(\frac{1}{P_n}\sum_{k=1}^n p_k\left(|T_k|^r + i|T_k^*|^r\right)\right) - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x) = \frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k|T_k|^r\right)x,x\right\rangle - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k|T_k^*|^r\right)x,x\right\rangle\right)^2.$$

REMARK 3.11. Setting $r = 1,\ n = 2$ and $p_1 = p_2 = 1$ in Corollary 3.10, we obtain the following inequality, which improves the bound (1.10)

$$w^2(T_1+T_2) \leqslant \frac{1}{2}w^2\left(|T_1| + |T_2| + i(|T_1^*| + |T_2^*|)\right) - \inf_{\|x\|=1}\phi(x)$$

where

$$\phi(x) = \frac{1}{16}\left(\langle(|T_1|+|T_2|)x,x\rangle - \langle(|T_1^*|+|T_2^*|)x,x\rangle\right)^2.$$

THEOREM 3.12. *Let $A_k, B_k, C_k \in \mathscr{B}(\mathscr{H})$ $(k = 1,\dots,n)$ with $A_k, B_k \geqslant 0$ and such that the operator matrices*

$$\begin{pmatrix} A_k & C_k^* \\ C_k & B_k \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

*are positive, then for $r,t \in [1,2]$ and $p_k \geqslant 0$ with $\sum_{k=1}^n p_k = P_n$, we have*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^n p_k C_k\right) \leqslant \frac{1}{2} w\left(\left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^r\right)\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right)\right) + \frac{1}{2^{1+1/t}}\left\|\frac{1}{P_n}\sum_{k=1}^n p_k (A_k^{2rt} + B_k^{2rt})\right\|^{\frac{1}{t}} - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x) = \frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}} - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

*Proof.* Let $\|x\| = 1$, by applying $s = r$ in inequality (3.1), we have

$$\begin{aligned}
&\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k C_k\right)x,x\right\rangle\right|^{2r} \\
&\leqslant \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right)x,x\right\rangle\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^r\right)x,x\right\rangle \\
&\leqslant \frac{1}{2}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right)x, \left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^r\right)x\right\rangle\right. \\
&\quad \left. + \left\|\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right)x\right\|\left\|\left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^r\right)x\right\|\right) \qquad \text{(by Lemma 2.4)} \\
&= \frac{1}{2}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right)x, \left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^r\right)x\right\rangle\right. \\
&\quad \left. + \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right)^2 x,x\right\rangle^{\frac{1}{2}}\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^r\right)^2 x,x\right\rangle^{\frac{1}{2}}\right) \\
&\leqslant \frac{1}{2}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^r\right)x, \left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^r\right)x\right\rangle\right. \\
&\quad \left. + \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k A_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}}\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k B_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}}\right) \qquad \text{(by inequality (1.1))}
\end{aligned}$$

$$\leqslant \frac{1}{2}\left\langle \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^r\right)\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^r\right)x,x\right\rangle$$
$$+\frac{1}{2}\left(\frac{\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^{2r}\right)x,x\right\rangle^t+\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^{2r}\right)x,x\right\rangle^t}{2}\right)^{\frac{1}{t}}$$
$$-\frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}}-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}}\right)^2$$

(by inequality (2.3))

$$\leqslant \frac{1}{2}\left\langle \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^r\right)\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^r\right)x,x\right\rangle$$
$$+\frac{1}{2}\left(\frac{\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^{2r}\right)^t x,x\right\rangle+\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^{2r}\right)^t x,x\right\rangle}{2}\right)^{\frac{1}{t}}-\inf_{\|x\|=1}\phi(x)$$

(by Lemma 2.1 (a))

$$\leqslant \frac{1}{2}\left\langle \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^r\right)\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^r\right)x,x\right\rangle$$
$$+\frac{1}{2}\left(\frac{\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^{2rt}\right)x,x\right\rangle+\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^{2rt}\right)x,x\right\rangle}{2}\right)^{\frac{1}{t}}-\inf_{\|x\|=1}\phi(x)$$

(by inequality (1.1)).

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

REMARK 3.13. In Theorem 3.12, it is obvious that

$$\inf_{\|x\|=1}\phi(x)=0 \Leftrightarrow 0\in \overline{W\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k(A_k^{2r}-B_k^{2r})\right)}.$$

Choosing $t=1$ in Theorem 3.12, we get the following result.

COROLLARY 3.14. *Let $A_k, B_k, C_k \in \mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$ with $A_k, B_k \geqslant 0$ and such that the operator matrices*

$$\begin{pmatrix} A_k & C_k^* \\ C_k & B_k \end{pmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$$

*are positive, then for $r\in[1,2]$ and $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k C_k\right)\leqslant \frac{1}{2}w\left(\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k B_k^r\right)\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k A_k^r\right)\right)$$
$$+\frac{1}{4}\left\|\frac{1}{P_n}\sum_{k=1}^{n} p_k(A_k^{2r}+B_k^{2r})\right\|-\inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x)=\frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}}-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^{2r}\right)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Letting $n=1$ in Corollary 3.14, we get the following result, which generalizes and improves on the inequality in [5], namely,

$$w^2(C)\leqslant\frac{1}{2}w(AB)+\frac{1}{4}\|A^2+B^2\|.$$

COROLLARY 3.15. *Let* $A,B,C\in\mathscr{B}(\mathscr{H})$ *with* $A,B\geqslant 0$ *and such that the operator matrice*

$$\begin{pmatrix}A & C^*\\ C & B\end{pmatrix}\in\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$$

*is positive, then for* $r\in[1,2]$,

$$w^{2r}(C)\leqslant\frac{1}{2}w^2(A^rB^r)+\frac{1}{4}\left\|A^{2r}+B^{2r}\right\|-\inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x)=\frac{1}{4}\left(\left\langle A^{2r}x,x\right\rangle^{\frac{1}{2}}-\left\langle B^{2r}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Letting $C_k=T_k$, $A_k=|T_k|$ and $B_k=|T_k^*|$ in Corollary 3.14, we get the following result.

COROLLARY 3.16. *Let* $T_k\in\mathscr{B}(\mathscr{H})$ $(k=1,\ldots,n)$ *and* $r\in[1,2]$, $p_k\geqslant 0$ *with* $\sum_{k=1}^{n}p_k=P_n$, *we have*

$$\begin{aligned}w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kT_k\right)&\leqslant\frac{1}{2}w\left(\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k|T_k^*|^r\right)\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k|T_k|^r\right)\right)\\&\quad+\frac{1}{4}\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k(|T_k|^{2r}+|T_k^*|^{2r})\right\|-\inf_{\|x\|=1}\phi(x),\end{aligned}$$

*where*

$$\phi(x)=\frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k|T_k|^{2r}\right)x,x\right\rangle^{\frac{1}{2}}-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_k|T_k^*|^{2r}\right)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

REMARK 3.17. Considering $n=1$ in Corollary 3.16, we get the following inequality

$$w^{2r}(T) \leqslant \frac{1}{2}w(|T|^r|T^*|^r) + \frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\| - \inf_{\|x\|=1}\phi(x),$$

where

$$\phi(x) = \frac{1}{4}\left(\left\langle |T|^{2r}x,x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2r}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Apparently, it follows that the above inequality is an improvement of the inequality (1.7).

THEOREM 3.18. *Let* $A_k, B_k, C_k \in \mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$ *with* $A_k, B_k \geqslant 0$ *and such that the operator matrices*

$$\begin{pmatrix} A_k & C_k^* \\ C_k & B_k \end{pmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$$

*are positive, then for* $r,t\in[1,2]$, $\alpha\in(0,1]$ *and* $p_k\geqslant 0$ *with* $\sum_{k=1}^n p_k = P_n$, *we have*

$$\begin{aligned} w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^n p_kC_k\right) \leqslant \left(\left\|\frac{1}{P_n}\sum_{k=1}^n p_k\big(\alpha A_k^{rt}+(1-\alpha)B_k^{rt}\big)\right\|^{\frac{1}{t}} - \inf_{\|x\|=1}\phi(x)\right) \\ \times \|A\|^{r(1-\alpha)}\|B\|^{r\alpha}, \end{aligned}$$

*where* $\|A\| = \max_{1\leqslant k\leqslant n}\|A_k\|$, $\|B\| = \max_{1\leqslant k\leqslant n}\|B_k\|$,

$$\phi(x) = \min\{\alpha,1-\alpha\}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_kA_k^r\right)x,x\right\rangle^{\frac{1}{2}} - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_kB_k^r\right)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

*Proof.* By applying $s=r$ in inequality (3.1), we have

$$\begin{aligned} &\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_kC_k\right)x,x\right\rangle\right|^{2r} \\ &\leqslant \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_kA_k^r\right)x,x\right\rangle\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_kB_k^r\right)x,x\right\rangle \\ &= \left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_kA_k^r\right)x,x\right\rangle^{1-\alpha+\alpha}\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_kB_k^r\right)x,x\right\rangle^{\alpha+1-\alpha} \\ &\leqslant \left\|\frac{1}{P_n}\sum_{k=1}^n p_kA_k^r\right\|^{1-\alpha}\left\|\frac{1}{P_n}\sum_{k=1}^n p_kB_k^r\right\|^{\alpha} \\ &\quad\times\left(\left(\alpha\left\langle\Big(\frac{1}{P_n}\sum_{k=1}^n p_kA_k^r\Big)x,x\right\rangle^t + (1-\alpha)\left\langle\Big(\frac{1}{P_n}\sum_{k=1}^n p_kB_k^r\Big)x,x\right\rangle^t\right)^{\frac{1}{t}}\right. \end{aligned}$$

$$-\min\{\alpha,\,1-\alpha\}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)x,x\right\rangle^{\frac{1}{2}}-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)x,x\right\rangle^{\frac{1}{2}}\right)^2\Bigg)$$

(by inequality (2.1))

$$\leqslant \|A\|^{r(1-\alpha)}\|B\|^{r\alpha}\Bigg(\left(\alpha\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)^t x,x\right\rangle+(1-\alpha)\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)^t x,x\right\rangle\right)^{\frac{1}{t}}$$

$$-\inf_{\|x\|=1}\phi(x)\Bigg)$$

(by Lemma 2.1 (a))

$$\leqslant \|A\|^{r(1-\alpha)}\|B\|^{r\alpha}\Bigg(\left(\alpha\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^{rt}\right)x,x\right\rangle+(1-\alpha)\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^{rt}\right)x,x\right\rangle\right)^{\frac{1}{t}}$$

$$-\inf_{\|x\|=1}\phi(x)\Bigg)$$

(by inequality (1.1)).

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. $\square$

Choosing $t=1$ in Theorem 3.18, we get the following result.

COROLLARY 3.19. *Let $A_k, B_k, C_k \in \mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$ with $A_k, B_k \geqslant 0$ and such that the operator matrices*

$$\begin{pmatrix} A_k & C_k^* \\ C_k & B_k \end{pmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$$

*are positive, then for $r\in[1,2]$ and $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kC_k\right)\leqslant\left(\left\|\frac{1}{P_n}\sum_{k=1}^{n}p_k\big(\alpha A_k^r+(1-\alpha)B_k^r\big)\right\|-\inf_{\|x\|=1}\phi(x)\right)\|A\|^{r(1-\alpha)}\|B\|^{r\alpha},$$

*where $\|A\|=\max_{1\leqslant k\leqslant n}\|A_k\|$, $\|B\|=\max_{1\leqslant k\leqslant n}\|B_k\|$,*

$$\phi(x)=\min\{\alpha,1-\alpha\}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kA_k^r\right)x,x\right\rangle^{\frac{1}{2}}-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n}p_kB_k^r\right)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Letting $n=1$ in Corollary 3.19, we get the following result, which generalizes and improves on the inequality in [5], namely,

$$w^2(C)\leqslant\|\alpha A+(1-\alpha)B\|\|A\|^{1-\alpha}\|B\|^{\alpha},\ \text{for all }\alpha\in[0,1].$$

COROLLARY 3.20. *Let $A, B, C \in \mathscr{B}(\mathscr{H})$ with $A, B \geqslant 0$ and such that the operator matrice*

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

*is positive, then for $r \in [1,2]$,*

$$w^{2r}(C) \leqslant \left( \|\alpha A^r + (1-\alpha)B^r\| - \inf_{\|x\|=1} \phi(x) \right) \|A\|^{r(1-\alpha)} \|B\|^{r\alpha},$$

*where*

$$\phi(x) = \min\{\alpha, 1-\alpha\} \left( \langle A^r x, x \rangle^{\frac{1}{2}} - \langle B^r x, x \rangle^{\frac{1}{2}} \right)^2.$$

Letting $C_k = T_k,\ A_k = |T_k|$ and $B_k = |T_k^*|$ in Corollary 3.19, we get the following result.

COROLLARY 3.21. *Let $T_k \in \mathscr{B}(\mathscr{H})$ $(k = 1, \ldots, n)$ and for $r \in [1,2],\ p_k \geqslant 0$ with $\sum_{k=1}^n p_k = P_n$, we have*

$$w^{2r}\left( \frac{1}{P_n} \sum_{k=1}^n p_k T_k \right) \leqslant \left( \left\| \frac{1}{P_n} \sum_{k=1}^n p_k \left( \alpha |T_k|^r + (1-\alpha)|T_k^*|^r \right) \right\| - \inf_{\|x\|=1} \phi(x) \right) \|T\|^r,$$

*where $\|T\| = \max_{1 \leqslant k \leqslant n} \|T_k\|,\ \|T^*\| = \max_{1 \leqslant k \leqslant n} \|T_k^*\|$,*

$$\phi(x) = \min\{\alpha, 1-\alpha\} \left( \left\langle \left( \frac{1}{P_n} \sum_{k=1}^n p_k |T_k|^r \right) x, x \right\rangle^{\frac{1}{2}} - \left\langle \left( \frac{1}{P_n} \sum_{k=1}^n p_k |T_k^*|^r \right) x, x \right\rangle^{\frac{1}{2}} \right)^2.$$

REMARK 3.22. Choosing $n = r = 1$ in Corollary 3.21, we deduce a refinement of the bound in (1.6),

$$w^2(T) \leqslant \left( \|\alpha |T| + (1-\alpha)|T^*|\| - \inf_{\|x\|=1} \phi(x) \right) \|T\|,$$

where

$$\phi(x) = \min\{\alpha, 1-\alpha\} \left( \langle |T| x, x \rangle^{\frac{1}{2}} - \langle |T^*| x, x \rangle^{\frac{1}{2}} \right)^2.$$

Replacing $A_k = |T_k^*|^2,\ B_k = |V_k|^2$ and $C_k = V_k^* T_k^*$ in Theorems 3.4, 3.12 and 3.18, we obtain several upper bounds for the weighted sums of product operators $w\left( \frac{1}{P_n} \sum_{k=1}^n p_k T_k V_k \right)$.

PROPOSITION 3.23. *Let $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$, then for $r,t \in [1,2]$ and $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have the following bounds:*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \frac{1}{2^{1/t}} w^{\frac{2}{t}}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k\left(|T_k^*|^{2rt} + i|V_k|^{2rt}\right)\right) - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x) = \frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |T_k^*|^{2r}\right)x,x\right\rangle - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |V_k|^{2r}\right)x,x\right\rangle\right)^2.$$

*Moreover,*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \frac{1}{2} w\left(\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |V_k|^{2r}\right)\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |T_k^*|^{2r}\right)\right)$$
$$+ \frac{1}{2^{1+1/t}}\left\|\frac{1}{P_n}\sum_{k=1}^{n} p_k(|T_k^*|^{4rt} + |V_k|^{4rt})\right\|^{\frac{1}{t}} - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x) = \frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |T_k^*|^{4r}\right)x,x\right\rangle^{\frac{1}{2}} - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |V_k|^{4r}\right)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

*And for $\alpha \in (0,1]$,*

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \left(\left\|\frac{1}{P_n}\sum_{k=1}^{n} p_k\left(\alpha|T_k^*|^{2rt} + (1-\alpha)|V_k|^{2rt}\right)\right\|^{\frac{1}{t}} - \inf_{\|x\|=1}\phi(x)\right)$$
$$\times \|T\|^{2r(1-\alpha)}\|V\|^{2r\alpha},$$

*where $\|T\| = \max_{1\leqslant k\leqslant n}\|T_k\|$, $\|V\| = \max_{1\leqslant k\leqslant n}\|V_k\|$,*

$$\phi(x) = \min\{\alpha, 1-\alpha\}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |T_k^*|^{2r}\right)x,x\right\rangle^{\frac{1}{2}} - \left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k |V_k|^{2r}\right)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Letting $n = t = 1$ in Proposition 3.23, we obtain the following results.

COROLLARY 3.24. *Let $T, V \in \mathscr{B}(\mathscr{H})$ and for $r \in [1,2]$, we have the following bounds:*

$$w^{2r}(TV) \leqslant \frac{1}{2} w^2\left(|T^*|^{2r} + i|V|^{2r}\right) - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x)=\frac{1}{4}\left(\langle |T^*|^{2r}x,x\rangle-\langle |V|^{2r}x,x\rangle\right)^2.$$

*Moreover,*

$$w^{2r}(TV)\leqslant\frac{1}{2}w\left(|V|^{2r}|T^*|^{2r}\right)+\frac{1}{4}\left\||T^*|^{4r}+|V|^{4r}\right\|-\inf_{\|x\|=1}\phi(x), \tag{3.3}$$

*where*

$$\phi(x)=\frac{1}{4}\left(\langle |T^*|^{4r}x,x\rangle^{\frac{1}{2}}-\langle |V|^{4r}x,x\rangle^{\frac{1}{2}}\right)^2.$$

*And for* $\alpha\in(0,1]$,

$$w^{2r}(TV)\leqslant\left(\left\|\alpha|T^*|^{2r}+(1-\alpha)|V|^{2r}\right\|-\inf_{\|x\|=1}\phi(x)\right)\|T\|^{2r(1-\alpha)}\|V\|^{2r\alpha},$$

*where*

$$\phi(x)=\min\{\alpha,1-\alpha\}\left(\langle |T^*|^{2r}x,x\rangle^{\frac{1}{2}}-\langle |V|^{2r}x,x\rangle^{\frac{1}{2}}\right)^2.$$

REMARK 3.25. Notice that the inequality (3.3) provides an improvement for the inequality (1.12).

## 4. Numerical radius inequalities for the weighted sums of product operators

Still for the weighted sums of product operators, we continue to present several upper bounds involving the numerical radius and spectral radius. These can achieve some nice generalizations and refinements of the classical inequality.

THEOREM 4.1. *Let* $T_k,V_k\in\mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$ *be such that* $|T_k|V_k=V_k^*|T_k|$, *and* $p,q\geqslant 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *and let* $f$, $g$ *be nonnegative functions on* $[0,\infty)$ *which are continous and satisfy the relation* $f(u)g(u)=u$ *for all* $u\in[0,\infty)$. *Then for* $p_k\geqslant 0$ *with* $\sum_{k=1}^n p_k=P_n$, *we have*

$$w^2\left(\frac{1}{P_n}\sum_{k=1}^n p_kT_kV_k\right)\leqslant\frac{\sqrt{2}}{\mu}w\left(\frac{1}{P_n}\sum_{k=1}^n p_k\rho^2(V_k)\left(f^{2p}(|T_k|)+ig^{2q}(|T_k^*|)\right)\right)-\inf_{\|x\|=1}\phi(x),$$

*where* $\mu=\min\{p,q\}$,

$$\phi(x)=\left[\min\left\{\frac{1}{p},\frac{1}{q}\right\}\right]^2\frac{1}{P_n}\sum_{k=1}^n p_k\rho^2(V_k)\left(\langle f^{2p}(|T_k|)x,x\rangle^{\frac{1}{2}}-\langle g^{2q}(|T_k^*|)x,x\rangle^{\frac{1}{2}}\right)^2.$$

*Proof.* Let $x \in \mathscr{H}$ with $\|x\| = 1$. Then we have

$$
\begin{aligned}
&\left|\left\langle \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) x, x\right\rangle\right|^2 \\
&\leqslant \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \left|\langle T_k V_k x, x\rangle\right|\right)^2 \\
&\leqslant \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho(V_k)\|f(|T_k|)x\|\|g(|T_k^*|)x\|\right)^2 \qquad \text{(by Lemma 2.5)} \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left(\left\langle f^2(|T_k|)x, x\right\rangle^{\frac{1}{2}}\left\langle g^2(|T_k^*|)x, x\right\rangle^{\frac{1}{2}}\right)^2 \quad \text{(by inequality (1.1))} \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left(\langle f^{2p}(|T_k|)x, x\rangle^{\frac{1}{2p}}\langle g^{2q}(|T_k^*|)x, x\rangle^{\frac{1}{2q}}\right)^2 \text{(by Lemma 2.1 (b))} \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left(\frac{\left\langle f^{2p}(|T_k|)x, x\right\rangle^{\frac{t}{2}}}{p} + \frac{\left\langle g^{2q}(|T_k^*|)x, x\right\rangle^{\frac{t}{2}}}{q}\right)^{\frac{2}{t}} \\
&\quad - \left[\min\left\{\frac{1}{p}, \frac{1}{q}\right\}\right]^2 \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left(\left\langle f^{2p}(|T_k|)x, x\right\rangle^{\frac{1}{2}} - \left\langle g^{2q}(|T_k^*|)x, x\right\rangle^{\frac{1}{2}}\right)^2 \\
&\qquad\qquad \text{(by inequality (2.1))} \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left(\frac{\langle f^{2p}(|T_k|)x, x\rangle}{p} + \frac{\langle g^{2q}(|T_k^*|)x, x\rangle}{q}\right) - \inf_{\|x\|=1}\phi(x) \\
&\qquad\qquad \text{(by convexity of } u^{\frac{2}{t}}\text{, where } 2 \geqslant t) \\
&\leqslant \frac{\sqrt{2}}{\mu}\left|\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left\langle\left(f^{2p}(|T_k|) + i g^{2q}(|T_k^*|)\right)x, x\right\rangle\right| - \inf_{\|x\|=1}\phi(x) \\
&\qquad\qquad \text{(as } |a+b| \leqslant \sqrt{2}|a+ib| \text{ for all } a, b \in \mathbb{R}).
\end{aligned}
$$

Now, the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

REMARK 4.2. In Theorem 4.1, $\inf_{\|x\|=1}\phi(x) > 0$ if and only if

$$0 \notin \overline{W\left(f^{2p}(|T_k|) - g^{2q}(|T_k^*|)\right)}$$

for all $k = 1, \ldots, n$.

Choosing $p = q = 2$ in Theorem 4.1 we get the following corollary.

COROLLARY 4.3. *Let $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k = 1, \ldots, n)$ be such that $|T_k|V_k = V_k^*|T_k|$, and let $f$, $g$ be nonnegative functions on $[0,\infty)$ which are continous and satisfy the re-*

*lation* $f(u)g(u) = u$ *for all* $u \in [0,\infty)$. *Then for* $p_k \geqslant 0$ *with* $\sum_{k=1}^{n} p_k = P_n$, *we have*

$$w^2\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \frac{\sqrt{2}}{2} w\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left(f^4(|T_k|) + ig^4(|T_k^*|)\right)\right) - \inf_{\|x\|=1} \phi(x),$$

*where*

$$\phi(x) = \frac{2^{-2}}{P_n}\sum_{k=1}^{n} p_k \rho^2(V_k)\left(\left\langle f^4(|T_k|)x,x\right\rangle^{\frac{1}{2}} - \left\langle g^4(|T_k^*|)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

For $p_k = 1,\ k = 1,\ldots,n$ in Corollary 4.3, we get the following numerical radius inequality, which refines the inequality (1.14).

COROLLARY 4.4. *Let* $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k = 1,\ldots,n)$ *be such that* $|T_k|V_k = V_k^*|T_k|$, *and let* $f$, $g$ *be nonnegative functions on* $[0,\infty)$ *which are continous and satisfy the relation* $f(u)g(u) = u$ *for all* $u \in [0,\infty)$. *Then we have*

$$w^2\left(\sum_{k=1}^{n} T_k V_k\right) \leqslant \frac{n}{\sqrt{2}} w\left(\sum_{k=1}^{n} \rho^2(V_k)\left(f^4(|T_k|) + ig^4(|T_k^*|)\right)\right) - \inf_{\|x\|=1} \phi(x),$$

*where*

$$\phi(x) = \frac{n}{4}\sum_{k=1}^{n} \rho^2(V_k)\left(\left\langle f^4(|T_k|)x,x\right\rangle^{\frac{1}{2}} - \left\langle g^4(|T_k^*|)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Letting $n = 1$ in Corollary 4.4, the following result is true.

COROLLARY 4.5. *Let* $T, V \in \mathscr{B}(\mathscr{H})$ *be such that* $|T|V = V^*|T|$ *and let* $f$, $g$ *be nonnegative functions on* $[0,\infty)$ *which are continous and satisfy the relation* $f(u)g(u) = u$ *for all* $u \in [0,\infty)$. *Then*

$$w^2(TV) \leqslant \frac{\rho^2(V)}{\sqrt{2}} w\left(f^4(|T|) + ig^4(|T^*|)\right) - \inf_{\|x\|=1} \phi(x),$$

*where*

$$\phi(x) = \frac{\rho^2(V)}{4}\left(\left\langle f^4(|T|)x,x\right\rangle^{\frac{1}{2}} - \left\langle g^4(|T^*|)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Letting $f(u) = u^{\alpha}$, $g(u) = u^{1-\alpha}$ with $0 \leqslant \alpha \leqslant 1$ in Corollary 4.5, we get the following assertion.

COROLLARY 4.6. *Let* $T, V \in \mathscr{B}(\mathscr{H})$ *be such that* $|T|V = V^*|T|$. *Then*

$$w^2(TV) \leqslant \frac{\rho^2(V)}{\sqrt{2}} w\left(|T|^{4\alpha} + i|T^*|^{4(1-\alpha)}\right) - \inf_{\|x\|=1} \phi(x),$$

*where* $0 \leqslant \alpha \leqslant 1$ *and*

$$\phi(x) = \frac{\rho^2(V)}{4}\left(\left\langle |T|^{4\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{4(1-\alpha)}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Considering $V_k = I$ for $k = 1, \dots, n$ in Theorem 4.1, we have the following inequality for the weighted sums of operators.

COROLLARY 4.7. *Let $T_k \in \mathscr{B}(\mathscr{H})$ $(k = 1, \dots, n)$ and $p, q \geqslant 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $f$, $g$ be nonnegative functions on $[0, \infty)$ which are continous and satisfy the relation $f(u)g(u) = u$ for all $u \in [0, \infty)$. Then for $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^2\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k\right) \leqslant \frac{\sqrt{2}}{\mu} w\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k\left(f^{2p}(|T_k|) + i g^{2q}(|T_k^*|)\right)\right) - \inf_{\|x\|=1} \phi(x),$$

*where $\mu = \min\{p, q\}$,*

$$\phi(x) = \left[\min\left\{\frac{1}{p}, \frac{1}{q}\right\}\right]^2 \frac{1}{P_n}\sum_{k=1}^{n} p_k \left(\left\langle f^{2p}(|T_k|)x, x\right\rangle^{\frac{1}{2}} - \left\langle g^{2q}(|T_k^*|)x, x\right\rangle^{\frac{1}{2}}\right)^2.$$

REMARK 4.8. In particular, letting $n = 1$, $p = q = 2$ and $f(u) = g(u) = \sqrt{u}$ in Corollary 4.7, we have the following inequality

$$w^2(T) \leqslant \frac{1}{\sqrt{2}} w\left(|T|^2 + i|T^*|^2\right) - \inf_{\|x\|=1} \phi(x),$$

where

$$\phi(x) = \frac{1}{4}\left(\left\langle |T|^2 x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^2 x, x\right\rangle^{\frac{1}{2}}\right)^2.$$

It is easy to verify that $\frac{1}{\sqrt{2}} w\left(|T|^2 + i|T^*|^2\right) \leqslant \|T\|^2$. Therefore, we would like to remark that Corollary 4.6 improves the classical bound (1.2).

The next result reads as follows and operator concavity of the function $f(t) = t^l$, $l \in (0, 1]$, (see [2]) is used in the proof.

THEOREM 4.9. *Let $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k = 1, \dots, n)$ be such that $|T_k|V_k = V_k^*|T_k|$, and $f$, $g$ be nonnegative functions on $[0, \infty)$ which are continous and satisfy the relation $f(u)g(u) = u$ for all $u \in [0, \infty)$. Then for $r \in [1, 2]$, $t \geqslant 1$ and $m \leqslant \min\{r, t\}$, $m \in \mathbb{N}^*$, $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^r\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \frac{1}{2^{m/t}} w^{\frac{m}{t}}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^{\frac{rt}{m}}(V_k)\left(f^{\frac{2rt}{m}}(|T_k|) + g^{\frac{2rt}{m}}(|T_k^*|)\right)\right) - \inf_{\|x\|=1} \phi(x),$$

*where*

$$\phi(x) = 2^{-m}\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\left\langle f^{\frac{2r}{m}}(|T_k|)x, x\right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}}(|T_k^*|)x, x\right\rangle^{\frac{m}{2}}\right)^2.$$

*Proof.* Let $x \in \mathscr{H}$ be a unit vector. Then we have

$$\begin{aligned}
&\left|\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right)x,x\right\rangle\right|^r \\
&\leqslant \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \left|\langle T_k V_k x,x\rangle\right|\right)^r \\
&\leqslant \left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho(V_k)\|f(|T_k|)x\|\|g(|T_k^*|)x\|\right)^r \qquad \text{(by Lemma 2.5)} \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\langle f^2(|T_k|)x,x\rangle^{\frac{r}{2}}\langle g^2(|T_k^*|)x,x\rangle^{\frac{r}{2}} \qquad \text{(by inequality (1.1))} \\
&= \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\langle f^2(|T_k|)x,x\rangle^{\frac{r}{2m}}\langle g^2(|T_k^*|)x,x\rangle^{\frac{r}{2m}}\right)^m \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\langle f^{\frac{2r}{m}}(|T_k|)x,x\rangle^{\frac{1}{2}}\langle g^{\frac{2r}{m}}(|T_k^*|)x,x\rangle^{\frac{1}{2}}\right)^m \text{(by Lemma 2.1 (a))} \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\frac{\langle f^{\frac{2r}{m}}(|T_k|)x,x\rangle^t+\langle g^{\frac{2r}{m}}(|T_k^*|)x,x\rangle^t}{2}\right)^{\frac{m}{t}} \\
&\quad -2^{-m}\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\left\langle f^{\frac{2r}{m}}(|T_k|)x,x\right\rangle^{\frac{m}{2}}-\left\langle g^{\frac{2r}{m}}(|T_k^*|)x,x\right\rangle^{\frac{m}{2}}\right)^2 \\
&\qquad\qquad \text{(by inequality (2.2))} \\
&\leqslant \frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\frac{\langle f^{\frac{2rt}{m}}(|T_k|)x,x\rangle+\langle g^{\frac{2rt}{m}}(|T_k^*|)x,x\rangle}{2}\right)^{\frac{m}{t}} - \inf_{\|x\|=1}\phi(x) \\
&\qquad\qquad \text{(by Lemma 2.1 (a))} \\
&\leqslant \frac{1}{2^{m/t}}\left\langle\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^{\frac{rt}{m}}(V_k)\left(f^{\frac{2rt}{m}}(|T_k|)+g^{\frac{2rt}{m}}(|T_k^*|)\right)\right)x,x\right\rangle^{\frac{m}{t}} - \inf_{\|x\|=1}\phi(x) \\
&\qquad\qquad \text{(by operator concavity of } u^{\frac{m}{t}}\text{)}.
\end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors in $\mathscr{H}$. □

REMARK 4.10. In Theorem 4.9, it is obvious that

$$\inf_{\|x\|=1}\phi(x) > 0 \Leftrightarrow 0 \notin \bigcap_{k=1}^{n} \overline{W\left(f^{\frac{2r}{m}}(|T_k|)-g^{\frac{2r}{m}}(|T_k^*|)\right)}.$$

Letting $m = 1$ in Theorem 4.9 we get the following corollary.

COROLLARY 4.11. *Let $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k = 1,\ldots,n)$ be such that $|T_k|V_k = V_k^*|T_k|$, and $f$, $g$ be nonnegative functions on $[0,\infty)$ which are continous and satisfy* *the relation $f(u)g(u) = u$ for all $u \in [0,\infty)$. Then for all $r \in [1,2]$, $t \geqslant 1$ and $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^r\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \frac{1}{2^{1/t}} w^{\frac{1}{t}}\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^{rt}(V_k)\left(f^{2rt}(|T_k|)+g^{2rt}(|T_k^*|)\right)\right) - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x) = 2^{-1}\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\left\langle f^{2r}(|T_k|)x,x\right\rangle^{\frac{1}{2}} - \left\langle g^{2r}(|T_k^*|)x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

Letting $t = m$ in Theorem 4.9 we get the following result.

COROLLARY 4.12. *Let $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k = 1,\dots,n)$ be such that $|T_k|V_k = V_k^*|T_k|$, and $f$, $g$ be nonnegative functions on $[0,\infty)$ which are continous and satisfy the relation $f(u)g(u) = u$ for all $u \in [0,\infty)$. Then for $r \in [1,2]$, $m \leqslant r, m \in \mathbb{N}^*$ and $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^r\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \frac{1}{2} w\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^{r}(V_k)\left(f^{2r}(|T_k|)+g^{2r}(|T_k^*|)\right)\right) - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x) = 2^{-m}\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^r(V_k)\left(\left\langle f^{\frac{2r}{m}}(|T_k|)x,x\right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}}(|T_k^*|)x,x\right\rangle^{\frac{m}{2}}\right)^2.$$

Next example is provided to demonstrate Corollary 4.12.

EXAMPLE 4.13. Letting $n = r = m = 1$ and $f(u) = u$, $g(u) = 1$ for all $u \in [0,\infty)$, $T = \mathrm{diag}(2,2)$ and $V = \mathrm{diag}(1,1)$ in Corollary 4.12, we have

$$\begin{aligned}\phi(x) &= 2^{-m}\frac{1}{P_n}\sum_{k=1}^{n} p_k r \rho^r(V_k)\left(\left\langle f^{\frac{2r}{m}}(|T_k|)x,x\right\rangle^{\frac{m}{2}} - \left\langle g^{\frac{2r}{m}}(|T_k^*|)x,x\right\rangle^{\frac{m}{2}}\right)^2 \\ &= \frac{1}{2}\left(\left\langle f^2(|T|)x,x\right\rangle^{\frac{1}{2}} - \langle Ix,x\rangle^{\frac{1}{2}}\right)^2 \\ &= \frac{1}{2}.\end{aligned}$$

Putting $f(u) = u^{\alpha}$ and $g(u) = u^{1-\alpha}$, $0 \leqslant \alpha \leqslant 1$ in Corollary 4.12, we get the following assertion.

COROLLARY 4.14. *Let $T_k, V_k \in \mathscr{B}(\mathscr{H})$ $(k = 1,\dots,n)$ be such that $|T_k|V_k = V_k^*|T_k|$. Then for $r \in [1,2]$, $m \leqslant r, m \in \mathbb{N}^*$ and $p_k \geqslant 0$ with $\sum_{k=1}^{n} p_k = P_n$, we have*

$$w^r\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k T_k V_k\right) \leqslant \frac{1}{2} w\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k \rho^{r}(V_k)\left(|T_k|^{2r\alpha}+|T_k^*|^{2r(1-\alpha)}\right)\right) - \inf_{\|x\|=1}\phi(x),$$

*where*

$$\phi(x) = 2^{-m}\frac{1}{P_n}\sum_{k=1}^{n} p_k r\rho^r(V_k)\left(\left\langle |T_k|^{\frac{2r\alpha}{m}}x,x\right\rangle^{\frac{m}{2}} - \left\langle |T_k^*|^{\frac{2r(1-\alpha)}{m}}x,x\right\rangle^{\frac{m}{2}}\right)^2.$$

REMARK 4.15. Setting $V=I$ and $n=m=1$ in Corollary 4.14, we have the following inequality

$$w^r(T) \leqslant \frac{1}{2}w\left(|T|^{2r\alpha}+|T^*|^{2r(1-\alpha)}\right) - \inf_{\|x\|=1}\phi(x), \tag{4.1}$$

where $0\leqslant\alpha\leqslant 1,\ r\in[1,2]$ and

$$\phi(x) = \frac{1}{2}\left(\left\langle |T|^{2r\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2r(1-\alpha)}x,x\right\rangle^{\frac{1}{2}}\right)^2.$$

One can notice that (4.1) is a refinement of the inequality (1.8). Furthermore, considering $n=2,\ m=1$ and $V_1=V_2=I,\ p_1=p_2=1$ in Corollary 4.14, we obtain

$$w^r(T_1+T_2) \leqslant 2^{r-2}w\left(|T_1|^{2r\alpha}+|T_2|^{2r\alpha}+|T_1^*|^{2r(1-\alpha)}+|T_2^*|^{2r(1-\alpha)}\right) - \inf_{\|x\|=1}\phi(x), \tag{4.2}$$

where $0\leqslant\alpha\leqslant 1,\ r\in[1,2]$ and

$$\begin{aligned}\phi(x) =& \frac{1}{4}\left(\left\langle |T_1|^{2r\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle |T_1^*|^{2r(1-\alpha)}x,x\right\rangle^{\frac{1}{2}}\right)^2\\ &+\frac{1}{4}\left(\left\langle |T_2|^{2r\alpha}x,x\right\rangle^{\frac{1}{2}} - \left\langle |T_2^*|^{2r(1-\alpha)}x,x\right\rangle^{\frac{1}{2}}\right)^2.\end{aligned}$$

Hence, it is easy to see that the inequality (4.2) is stronger than the inequality (1.9).

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