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TWO–STEP MINIMIZATION APPROACH TO SOBOLEV–TYPE INEQUALITY WITH BOUNDED POTENTIAL IN 1D

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Abstract. We present a new method to determine the best constant of the Sobolev-type embedding in one dimension with a norm including a bounded inhomogeneous potential term. This problem is closely connected to the Green function of the Schrödinger operator with inhomogeneous potential. A minimization problem of a Rayleigh-type quotient in a Sobolev space gives the best constant of the Sobolev embedding. We decompose the minimization problem into two sub-minimization problems and show that the Green function provides the minimizer of the first minimization problem. Then, it enables us to derive a new precise estimate of the best constant and function for inhomogeneous bounded potential cases. As applications, we give some examples of the inhomogeneous potential whose best constant and function of the Sobolev-type embedding are explicitly determined.

1. Introduction

In this paper, $L^p(\mathbb{R})$ and $W^{1,p}(\mathbb{R})$ denote the real-valued Lebesgue and Sobolev spaces, and we set $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$. For every $1 \leqslant p \leqslant \infty$ and every function $u \in W^{1,p}(\mathbb{R})$, it is well known to hold the following Sobolev-type inequality:

$$\|u\|_\infty \leqslant C\|u\|_{W^{1,p}(\mathbb{R})}, \tag{1}$$

where $\|u\|_\infty = \|u\|_{L^\infty(\mathbb{R})}$, and $C = C(p) > 0$ is a constant independent of u; see [2, Theorem 8.8] for example. Indeed, we can choose $C(p) = p^{1/p} \leqslant e^{1/e}$ from an elementary calculation. We consider a generalization for the case of $p = 2$ and discuss the corresponding best constant. The Sobolev-type inequality of the form $\|u\|_{L^q(\Omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)}$ has been intensively studied even in higher dimensions with $\Omega \subset \mathbb{R}^n$ under various settings in connection with applications in PDE problems, for example, [3, 4, 6, 12]. In particular, the best constant for the case with an inhomogeneous potential is closely connected with the corresponding Schrödinger operator. However, there has not been much precise analysis of the best constant of the Sobolev-type inequality when there is a general inhomogeneous potential term.

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Let $V \in L^\infty(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} V(x) > 0$ for simplicity. We introduce a norm on $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$ such as

$$\|u\|_V^2 := \int_{-\infty}^{\infty} (|u'(x)|^2 + V(x)|u(x)|^2)\, dx,$$

which gives an equivalent norm to the usual $\|\cdot\|_{W^{1,2}(\mathbb{R})}$ from the assumptions on $V(x)$. We define a Rayleigh-type quotient

$$R(u;V) := \frac{\|u\|_V^2}{\|u\|_\infty^2}, \tag{2}$$

and consider a minimization problem of $R(u;V)$ for $u \in H^1(\mathbb{R})$. More precisely, we define

$$m(V) := \inf_{u\in H^1(\mathbb{R}), u\not\equiv 0} R(u;V), \tag{3}$$

$$M(V) := \left\{u \in H^1(\mathbb{R});\ m(V) = R(u;V),\ u \not\equiv 0\right\}. \tag{4}$$

We note that $m(V)^{-1/2}$ gives the best constant of the following Sobolev-type inequality

$$\|u\|_\infty \leqslant C\|u\|_V, \tag{5}$$

which is similar to (1). In this paper, we call $m(V)^{-1/2}$ the Sobolev best constant of the inequality (5).

Contrary to the precise and detailed analysis of the Sobolev best constant and function for the homogeneous case $V \equiv const.$, it is not easy to get the precise value of the Sobolev best constant and profile of the minimizer, which achieves the best constant for inhomogeneous potentials. However, the significance of inhomogeneous potential cases is increasing in connection with various applications. In this research, we focus on one-dimensional and inhomogeneous potential cases. This paper aims to develop a two-step minimization approach and establish new criteria to determine the precise value of the Sobolev best constant and function for (5). We clarify the influence of the potential V to $m(V)$ and the elements of $M(V)$, and construct concrete examples using the obtained new criteria.

The main idea that we employed in our analysis is the reduction of the minimization problem in the whole $H^1(\mathbb{R})$ to two-step minimizations in K_a and in $a \in \mathbb{R}$, where $K_a := \{u \in H^1(\mathbb{R});\ u(a) = \|u\|_\infty = 1\}$. In the first minimization in K_a, the unique minimizer u_a coincides with the Green function for the Schrödinger operator $-\frac{d^2}{dx^2} + V(x)$ with the potential function $V(x)$. This fact closely links the Sobolev best constant problem to the estimate of the Green function and gives one of our strong motivations. In the second minimization step among the parameter a, we give a necessary and sufficient condition for the global minimizer regarding the fundamental solutions of the linear ODE $-\frac{d^2}{dx^2}u + V(x)u = 0$.

The outline of the paper is as follows. In Section 2, we decompose the minimization problem (3) to the two sub-minimization problems and prove the equivalency between the original minimization problem and our two-step minimization problem. In Section 3, we first establish precise properties of the minimizer of the first minimization problem in K_a and a relation to the Green function. Moreover, by extending the right and left tails of the minimizer, we construct the fundamental solutions of $-\frac{d^2}{dx^2}u+V(x)u=0$ and give upper and lower estimates for them. Section 4 is devoted to the second minimization step. In Theorem 4.1, we establish some useful variational formulae for the minimum value obtained in the first minimization step. Section 5 establishes our main theorem, which gives three new necessary and sufficient conditions for the local minimality in the second minimization step. As an application, we construct a nontrivial example of the bounded inhomogeneous potential for which the Sobolev best constant and function are explicitly specified.

2. Two-step minimization approach

Throughout of this paper, we suppose

$$V \in L^\infty(\mathbb{R}), \quad 0 < v_0 \leqslant V(x) \leqslant v_1 \text{ a.e. } \; x \in \mathbb{R}, \tag{6}$$

for some constants v_0 and v_1. Then, for $u,\ v \in H^1(\mathbb{R})$, we define

$$(u,v)_V := \int_{\mathbb{R}} \big(u'(x)v'(x)+V(x)u(x)v(x)\big)\,dx, \quad \|u\|_V := (u,u)_V^{\frac{1}{2}}, \quad I(u;V) := \|u\|_V^2,$$

where we abbreviate $\frac{du}{dx}$ as u'. We remark that $(u,v)_V$ defines an inner product on $H^1(\mathbb{R})$, and the corresponding norm $\|u\|_V$ is equivalent to the norm of $H^1(\mathbb{R})$.

In this paper, without loss of generality, we suppose that $u \in H^1(\mathbb{R})$ (or more generally, $u \in W^{1,p}_{loc}(\mathbb{R})$ for $p \in [1,\infty]$) always satisfies $u \in C^0(\mathbb{R})$, since an element of the function space $W^{1,p}_{loc}(\mathbb{R})$ has a continuous representation (Theorem 8.8 of [2]). We also remark that $u \in H^1(\mathbb{R})$ satisfies $u \in L^\infty(\mathbb{R})$ and $\lim_{|x|\to\infty} u(x) = 0$ (Theorem 8.8 and Corollary 8.9 of [2]). These facts imply that, for any $u \in H^1(\mathbb{R})$, there exists $a \in \mathbb{R}$ such that $u(a) = \|u\|_\infty$ or $u(a) = -\|u\|_\infty$ holds. In our argument below, we assume the minimizer u satisfies $u(a) = \|u\|_\infty = 1$ without loss of generality, since $R(\alpha u;V) = R(u;V)$ for $\alpha \neq 0$ and $u \in H^1(\mathbb{R}) \setminus \{0\}$.

For $a \in \mathbb{R}$, we set

$$K_a := \{u \in H^1(\mathbb{R});\ u(a) = \|u\|_\infty = 1\},$$

and define

$$F(a) := \inf_{u \in K_a} I(u;V).$$

The following proposition establishes a decomposition principle for the minimization problem (3). It gives a foundation for the two-step minimization method and enables us to capture the detail of the inhomogeneous potential V.

PROPOSITION 2.1. (decomposition principle) *Let V satisfy (6) and set $m(V)$ as (3). Then, we have*

$$m(V) = \inf_{a \in \mathbb{R}} F(a). \tag{7}$$

Proof. We define $\tilde{m}(V) := \inf_{a\in\mathbb{R}} F(a)$. Let $\{u_n\}_{n=1}^{\infty}$ be a minimizing sequence to the infimum of (3). Without loss of generality, we can assume $u_n \in K_{a_n}$ for some $a_n \in \mathbb{R}$ and $m(V) = \lim_{n\to\infty} I(u_n;V)$. Taking the limit as $n\to\infty$ in $\tilde{m}(V) \leqslant F(a_n) \leqslant I(u_n;V)$, we obtain $\tilde{m}(V) \leqslant m(V)$.

On the other hand, we choose $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\{u_{n,k}\}_{k=1}^{\infty} \subset K_{a_n}$ such that

$$\tilde{m}(V) = \lim_{n\to\infty} F(a_n), \quad F(a_n) = \lim_{k\to\infty} I(u_{n,k};V).$$

Since $m(V) \leqslant I(u_{n,k};V)$, taking the limit as $k\to\infty$, we have $m(V) \leqslant F(a_n)$. Then, taking the limit as $n\to\infty$, we obtain $m(V) \leqslant \tilde{m}(V)$. □

3. First minimization step

We consider the first minimization step:

$$F(a) := \inf_{u\in K_a} \|u\|_V^2. \tag{8}$$

THEOREM 3.1. *We suppose the condition* (6) *and fix* $a \in \mathbb{R}$. *Then there exists a unique minimizer* $u_a \in K_a$ *to* (8), *that is,*

$$u_a = \arg\min_{u\in K_a} \|u\|_V^2, \quad F(a) = \min_{u\in K_a} \|u\|_V^2 = \|u_a\|_V^2. \tag{9}$$

Furthermore, u_a *satisfies*

$$u_a \in W^{2,\infty}(\mathbb{R}\backslash\{a\}) \quad and \quad u_a''(x) = V(x)u_a(x) \ (a.e. \ \ x\in\mathbb{R}\backslash\{a\}), \tag{10}$$

and the following properties:

1) *For* $V(x) \equiv V$ *(constant):* $u_a(x) = e^{-\sqrt{V}|x-a|}$.

2) *For general* $V(x)$ *satisfying* (6)*:*

$$e^{-\sqrt{v_1}|x-a|} \leqslant u_a(x) \leqslant e^{-\sqrt{v_0}|x-a|} \quad (x\in\mathbb{R}), \tag{11}$$

$$\frac{v_0}{\sqrt{v_1}} e^{-\sqrt{v_1}|x-a|} \leqslant \operatorname{sgn}(a-x)u_a'(x) \leqslant \frac{v_1}{\sqrt{v_0}} e^{-\sqrt{v_0}|x-a|} \quad (x\in\mathbb{R}\backslash\{a\}). \tag{12}$$

Proof. We define $L_a := \{u \in H^1(\mathbb{R});\ u(a) = 1\}$. Since L_a is a closed affine subspace of $H^1(\mathbb{R})$, there exists a unique $u_a \in L_a$ defined by $u_a := \arg\min_{u\in L_a} \|u\|_V^2$ and it is equivalent to the orthogonality condition (see Theorem 5.2 and Corollary 5.4 of [2], for example):

$$u_a \in L_a \quad \text{and} \quad (u_a, v)_V = 0 \ \text{ for all } \ v \in H^1(\mathbb{R}) \ \text{ with } \ v(a) = 0. \tag{13}$$

It implies that $u_a'' = Vu_a$ in $\mathscr{D}'(\mathbb{R}\backslash\{a\})$ holds. As (11) is proven for u_a below, we will find that u_a belongs to K_a and satisfies (9).

1) Suppose that V is a positive constant function. Then, a solution of the linear ODE $u''(x) = Vu(x)$ on $\mathbb{R}$ is written in the form:

$$u(x) = c_1 e^{\sqrt{V}x} + c_2 e^{-\sqrt{V}x} \quad (c_1, c_2 \in \mathbb{R}). \tag{14}$$

Hence, in this case, the formula $u_a(x) = e^{-\sqrt{V}|x-a|}$ is immediately derived from (14).

2) Next, we prove (11) for a general potential V with (6). We define $\tilde{u}_a(x) := e^{-\sqrt{v_1}|x-a|}$ and

$$w(x) := \max\left(u_a(x), \tilde{u}_a(x)\right) \qquad (x \in \mathbb{R}), \tag{15}$$

$$\omega := \{x \in \mathbb{R};\ u_a(x) < \tilde{u}_a(x)\} : \text{open}. \tag{16}$$

From Theorem A.1 in Chapter II of [7], w satisfies $w \in L_a$ and

$$w(x) = \tilde{u}_a(x), \quad w'(x) = \tilde{u}_a'(x) \quad \text{a.e.} \quad x \in \omega,$$
$$w(x) = u_a(x), \quad w'(x) = u_a'(x) \quad \text{a.e.} \quad x \in \mathbb{R}\backslash\omega.$$

We set $v := u_a - w$, then $v \in H^1(\mathbb{R})$, $v(a) = 0$, and $v \leqslant 0$ holds.

$$\begin{aligned}\|u_a\|_V^2 - \|w\|_V^2 &= (u_a, v)_V + (w, v)_V = (w, v)_V \\ &= \int_\omega \left(w'(x)v'(x) + V(x)w(x)v(x)\right) dx \\ &= \int_\omega \left(\tilde{u}_a'(x)v'(x) + V(x)\tilde{u}_a(x)v(x)\right) dx.\end{aligned}$$

Since ω is an open set of $\mathbb{R}$ and $a \notin \omega$, it is decomposed to finite or countable connected components: $\omega = \cup_{\lambda\in\Lambda}\omega_\lambda$ where $\omega_\lambda = (p_\lambda, q_\lambda)$ is an open interval, and Λ is a finite or countable set. We remark that $v(p_\lambda) = 0$ holds for $p_\lambda \neq -\infty$, and $v(q_\lambda) = 0$ for $q_\lambda \neq \infty$.

$$\int_{\omega_\lambda} \tilde{u}_a'(x)v'(x)\,dx = \left[\tilde{u}_a'(x)v(x)\right]_{p_\lambda}^{q_\lambda} - \int_{\omega_\lambda} \tilde{u}_a''(x)v(x)\,dx = -\int_{\omega_\lambda} v_1\tilde{u}_a(x)v(x)\,dx.$$

This equality is true even when $p_\lambda = -\infty$ or $q_\lambda = \infty$. So, it implies

$$\begin{aligned}\int_\omega \tilde{u}_a'(x)v'(x)\,dx &= \sum_{\lambda\in\Lambda}\int_{\omega_\lambda} \tilde{u}_a'(x)v'(x)\,dx = -\sum_{\lambda\in\Lambda}\int_{\omega_\lambda} v_1\tilde{u}_a(x)v(x)\,dx \\ &= -\int_\omega v_1\tilde{u}_a(x)v(x)\,dx.\end{aligned}$$

Hence, we obtain

$$\|u_a\|_V^2 - \|w\|_V^2 = \int_\omega (V(x) - v_1)\tilde{u}_a(x)v(x)\,dx \geqslant 0.$$

Since u_a is the unique minimizer of (9) and $w \in L_a$, $w(x) = u_a(x)$ holds for $x \in \mathbb{R}$. In other words, $\tilde{u}_a(x) \leqslant u_a(x)$ holds for $x \in \mathbb{R}$. Another inequality of (11) is also shown similarly. From (11), we conclude that $u_a \in K_a$ and (9) holds.

We next prove (12). Let us consider the left interval $(-\infty,a)$. From $u_a''=Vu_a\in L^\infty(-\infty,a)$, $u_a'\in C^0(-\infty,a)$ holds. Since $u_a'\in L^2(-\infty,a)$, there exists a sequence $\{r_n\}_{n=1}^\infty\subset(-\infty,a)$ such that $r_n\leqslant -n$ and $|u_a'(r_n)|\leqslant\frac{1}{n}$. Then, for $x<a$, we have

$$u_a'(x)=u_a'(r_n)+\int_{r_n}^x u_a''(t)\,dt=u_a'(r_n)+\int_{r_n}^x V(t)u_a(t)\,dt.$$

Taking the limit as $n\to\infty$,

$$u_a'(x)=\int_{-\infty}^x V(t)u_a(t)\,dt \quad \text{for } x\in(-\infty,a)$$

holds and, using (11), we obtain

$$u_a'(x)\geqslant v_0\int_{-\infty}^x e^{-\sqrt{v_1}|t-a|}\,dt=\frac{v_0}{\sqrt{v_1}}e^{-\sqrt{v_1}|x-a|},$$
$$u_a'(x)\leqslant v_1\int_{-\infty}^x e^{-\sqrt{v_0}|t-a|}\,dt=\frac{v_1}{\sqrt{v_0}}e^{-\sqrt{v_0}|x-a|}.$$

Similarly, for $x\in(a,\infty)$, we can derive the inequalities:

$$-\frac{v_1}{\sqrt{v_0}}e^{-\sqrt{v_0}|x-a|}\leqslant u_a'(x)\leqslant-\frac{v_0}{\sqrt{v_1}}e^{-\sqrt{v_1}|x-a|}.$$

Finally, $u_a\in W^{2,\infty}(\mathbb{R}\backslash\{a\})$ follows from $u_a''=Vu_a$ in $(-\infty,a)\cup(a,\infty)$ and (12). □

From Theorem 3.1, $u_a'(x)$ exists for $x\neq a$, and $u_a'(a)$ does not exist. However, the left and right derivatives at $x=a$ exist. We denote the right and left derivatives of $u_a(x)$ at $x=a$ by

$$u_a'(a_\pm):=\lim_{h\to\pm0}\frac{u_a(a+h)-u(a)}{h}.$$

PROPOSITION 3.2. *The Green function for the differential operator* $L:=-\frac{d^2}{dx^2}+V$ *is given by*

$$G(x,y)=\frac{u_y(x)}{u_y'(y_-)-u_y'(y_+)}=\frac{u_x(y)}{u_x'(x_-)-u_x'(x_+)} \quad (x,y\in\mathbb{R}). \tag{17}$$

Proof. For $x,y\in\mathbb{R}$, we define

$$g_y(x):=\frac{u_y(x)}{u_y'(y_-)-u_y'(y_+)}.$$

Then, $g_y\in H^1(\mathbb{R})$ holds. For all $v\in H^1(\mathbb{R})$, we have

$$\begin{aligned}(g_y,v)_V&=\int_{-\infty}^y\big(g_y'(x)v'(x)+V(x)g_y(x)v(x)\big)\,dx+\int_y^\infty\big(g_y'(x)v'(x)+V(x)g_y(x)v(x)\big)\,dx\\
&=\big[g_y'v\big]_{-\infty}^y+\int_{-\infty}^y\big(-g_y''(x)+V(x)g_y(x)\big)v(x)dx\\
&\quad+\big[g_y'v\big]_y^\infty+\int_y^\infty\big(-g_y''(x)+V(x)g_y(x)\big)v(x)dx\\
&=g_y'(y_-)v(y)-g_y'(y_+)v(y)=v(y)={}_{H^{-1}(\mathbb{R})}\langle\delta_y,v\rangle_{H^1(\mathbb{R})},\end{aligned}$$

where $\delta_y \in H^{-1}(\mathbb{R})$ denotes the Dirac delta distribution at y. This equality implies that $Lg_y = \delta_y$ in $H^{-1}(\mathbb{R})$. Furthermore, we obtain the second equality of (17) from the following equality for $y,z \in \mathbb{R}$:

$$g_z(y) = {}_{H^{-1}(\mathbb{R})}\left\langle \delta_y, g_z \right\rangle_{H^1(\mathbb{R})} = (g_y, g_z)_V = (g_z, g_y)_V = {}_{H^{-1}(\mathbb{R})}\left\langle \delta_z, g_y \right\rangle_{H^1(\mathbb{R})} = g_y(z). \quad \square$$

PROPOSITION 3.3. (comparison principle of u_a) *For $V, \tilde{V} \in L^\infty(\mathbb{R})$ with $0 < v_0 \leqslant V(x) \leqslant \tilde{V}(x)$, we set $u_a := \arg\min_{u \in K_a} \|u\|_V^2$ and $\tilde{u}_a := \arg\min_{u \in K_a} \|u\|_{\tilde{V}}^2$, then $u_a(x) \geqslant \tilde{u}_a(x)$ holds for $a \in \mathbb{R}$ and $x \in \mathbb{R}$.*

Proof. We define $w(x)$ and ω by (15) and (16). Then, in a similar way to the proof of Theorem 3.1, we can derive

$$\|u_a\|_V^2 - \|w\|_V^2 = \int_\omega (V(x) - \tilde{V}(x))\tilde{u}_a(x)(u_a(x) - w(x))\,dx \geqslant 0.$$

Since u_a is the unique minimizer of (9) and $w \in K_a$, $w(x) = u_a(x)$ holds for $x \in \mathbb{R}$. In other words, $\tilde{u}_a(x) \leqslant u_a(x)$ holds for $x \in \mathbb{R}$. $\square$

LEMMA 3.4. *Suppose (6). For $a \leqslant b$, u_a and u_b defined by (9) satisfy*

$$u_a(x) = \frac{u_b(x)}{u_b(a)} \quad (x \leqslant a), \quad u_b(x) = \frac{u_a(x)}{u_a(b)} \quad (x \geqslant a).$$

Proof. We define

$$\tilde{u}_a(x) := \begin{cases} \dfrac{u_b(x)}{u_b(a)} & (x \leqslant a), \\ u_a(x) & (x > a). \end{cases}$$

Then, $\tilde{u}_a \in K_a$ and, for $v \in H^1(\mathbb{R})$ with $v(a) = 0$, it satisfies

$$\begin{aligned}(\tilde{u}_a, v)_V &= \frac{1}{u_b(a)} \int_{-\infty}^a \left(u_b'(x)v'(x) + V(x)u_b(x)v(x)\right) dx \\ &\quad + \int_{-\infty}^a \left(u_a'(x)v'(x) + V(x)u_a(x)v(x)\right) dx \\ &= \frac{1}{u_b(a)} \int_{-\infty}^a \left(-u_b''(x) + V(x)u_b(x)\right) v(x)\,dx \\ &\quad + \int_{-\infty}^a \left(-u_a''(x) + V(x)u_a(x)\right) v(x)\,dx = 0.\end{aligned}$$

Since $\tilde{u}_a$ satisfies the orthogonality condition (13), $\tilde{u}_a = u_a$ holds. In other words, $u_a(x) = u_b(x)/u_b(a)$ holds for $x \leqslant a$. The other relation, $u_b(x) = u_a(x)/u_a(b)$ for $x \geqslant a$, is similarly proven. $\square$

From Lemma 3.4, we have

$$u_a'(a_-) = \frac{u_b'(a)}{u_b(a)} \quad (a < b), \qquad u_a'(a_+) = \frac{u_b'(a)}{u_b(a)} \quad (b < a).$$

These expressions of $u_a'(a_\pm)$ also imply that

$$[a \to u_a'(a_\pm)] \in W^{1,\infty}_{loc}(\mathbb{R}). \tag{18}$$

THEOREM 3.5. *Under the condition (6), $\varphi_\pm \in W^{2,\infty}_{loc}(\mathbb{R})$ uniquely exists such that*

$$\begin{cases} \varphi_\pm''(x) = V(x)\varphi_\pm(x) & (a.e.\ x \in \mathbb{R}), \\ \varphi_\pm(0) = 1, \\ \lim\limits_{x\to\pm\infty} \varphi_\pm(x) = 0. \end{cases} \tag{19}$$

Furthermore, they satisfy the following inequalities:

$$\sqrt{\frac{v_0}{v_1}} e^{-\max(\sqrt{v_0}x, \sqrt{v_1}x)} \leqslant \varphi_+(x) \leqslant \sqrt{\frac{v_1}{v_0}} e^{-\min(\sqrt{v_0}x, \sqrt{v_1}x)} \quad (x \in \mathbb{R}), \tag{20}$$

$$\sqrt{\frac{v_0}{v_1}} e^{\min(\sqrt{v_0}x, \sqrt{v_1}x)} \leqslant \varphi_-(x) \leqslant \sqrt{\frac{v_1}{v_0}} e^{\max(\sqrt{v_0}x, \sqrt{v_1}x)} \quad (x \in \mathbb{R}). \tag{21}$$

Moreover,

$$u_a(x) = \begin{cases} \dfrac{\varphi_-(x)}{\varphi_-(a)} & (x < a) \\ \dfrac{\varphi_+(x)}{\varphi_+(a)} & (x \geqslant a) \end{cases} \tag{22}$$

holds for $a \in \mathbb{R}$.

Proof. We set $m_\pm := u_0'(0_\pm)$. We consider the following initial value problem of linear ordinary differential equation:

$$\begin{cases} \varphi''(x) = V(x)\varphi(x) \quad \text{a.e.} \quad x \in \mathbb{R}, \\ \varphi(0) = 1,\ \varphi'(0) = m_\pm. \end{cases} \tag{23}$$

Since the coefficient V may not be continuous, we consider a mild solution for (23):

$$\varphi \in C^1(\mathbb{R}), \quad \varphi(0) = 1, \quad \varphi'(x) = m_\pm + \int_0^x V(t)\varphi(t)\,dt \ \ (x \in \mathbb{R}). \tag{24}$$

It is well known that there exists a unique mild solution $\varphi_\pm \in C^1(\mathbb{R})$ of (24) as it is a linear ODE with bounded coefficient $V \in L^\infty(\mathbb{R})$. Moreover, from (24), $\varphi_\pm \in W^{2,\infty}_{loc}(\mathbb{R})$ also holds.

We can show that the function defined by the right-hand side of (22) satisfies the condition (13) for all $a \in \mathbb{R}$, similar to the proof of Lemma 3.4. So, we conclude (22). Then, from the estimate (11), we find that $\varphi_\pm$ satisfy the initial value problems of (19).

Next, we prove the estimate (21). Since $\varphi_-(x) = u_0(x)$ for $x \leqslant 0$, (21) for $x \leqslant 0$ follows from (11). For $x > 0$, we first remark that $\varphi_-(x) > 0$ holds from (24). If

$\varphi_-(x_0)=0$ and $\varphi_-(x)>0$ for $0<x<x_0$, then there should be an $x_1\in(0,x_0)$ such that $\varphi_-'(x_1)<0$, but it is impossible from (24). We define $y(x):=\varphi_-'(x)+\sqrt{v_0}\varphi_-(x)$ for $x\geqslant 0$. Then, we have

$$\begin{aligned} y'(x) &= \varphi_-''(x)+\sqrt{v_0}\varphi_-'(x)=\sqrt{v_0}\varphi_-'(x)+V(x)\varphi_-(x)\\ &\geqslant \sqrt{v_0}\varphi_-'(x)+v_0\varphi_-(x)=\sqrt{v_0}y(x). \end{aligned}$$

Solving this differential inequality, we obtain $y(x)\geqslant y(0)e^{\sqrt{v_0}x}$. Since $y(0)=m_-+\sqrt{v_0}$, this is equivalent to

$$\varphi_-'(x)+\sqrt{v_0}\varphi_-(x)\geqslant(m_-+\sqrt{v_0})e^{\sqrt{v_0}x}.$$

We solve this differential inequality as follows:

$$\begin{aligned} \left(e^{\sqrt{v_0}x}\varphi_-(x)\right)' &= e^{\sqrt{v_0}x}\left(\varphi_-'(x)+\sqrt{v_0}\varphi_-(x)\right)\geqslant(m_-+\sqrt{v_0})e^{2\sqrt{v_0}x}\\ &= \left(\frac{m_-+\sqrt{v_0}}{2\sqrt{v_0}}e^{2\sqrt{v_0}x}\right)'. \end{aligned}$$

Integrating the above inequality on the interval $(0,x)$, we have

$$e^{\sqrt{v_0}x}\varphi_-(x)-1\geqslant\frac{m_-+\sqrt{v_0}}{2\sqrt{v_0}}\left(e^{2\sqrt{v_0}x}-1\right)\geqslant\sqrt{\frac{v_0}{v_1}}\left(e^{2\sqrt{v_0}x}-1\right), \tag{25}$$

where the last inequality follows from $m_-\geqslant v_0/\sqrt{v_1}$ by (12). From (25), we get

$$\varphi_-(x)\geqslant\sqrt{\frac{v_0}{v_1}}e^{\sqrt{v_0}x}+\left(1-\sqrt{\frac{v_0}{v_1}}\right)e^{-\sqrt{v_0}x}\geqslant\sqrt{\frac{v_0}{v_1}}e^{\sqrt{v_0}x},$$

which gives the lower bound estimate in (21) for $x>0$.

For the upper bound estimate in (21), we define $y(x):=\varphi_-'(x)+\sqrt{v_1}\varphi_-(x)$ for $x\geqslant 0$. Then, we have

$$\begin{aligned} y'(x) &= \varphi_-''(x)+\sqrt{v_1}\varphi_-'(x)=\sqrt{v_1}\varphi_-'(x)+V(x)\varphi_-(x)\\ &\leqslant \sqrt{v_1}\varphi_-'(x)+v_1\varphi_-(x)=\sqrt{v_1}y(x). \end{aligned}$$

Solving this differential inequality, we obtain $y(x)\leqslant y(0)e^{\sqrt{v_1}x}$. Since $y(0)=m_-+\sqrt{v_1}$, this is equivalent to

$$\varphi_-'(x)+\sqrt{v_1}\varphi_-(x)\leqslant(m_-+\sqrt{v_1})e^{\sqrt{v_1}x}.$$

We solve this differential inequality as follows:

$$\begin{aligned} \left(e^{\sqrt{v_1}x}\varphi_-(x)\right)' &= e^{\sqrt{v_1}x}\left(\varphi_-'(x)+\sqrt{v_1}\varphi_-(x)\right)\\ &\leqslant(m_-+\sqrt{v_1})e^{2\sqrt{v_1}x}=\left(\frac{m_-+\sqrt{v_1}}{2\sqrt{v_1}}e^{2\sqrt{v_1}x}\right)'. \end{aligned}$$

Integrating the above inequality on the interval $(0,x)$, we have

$$e^{\sqrt{v_1}x}\varphi_-(x)-1\leqslant\frac{m_-+\sqrt{v_1}}{2\sqrt{v_1}}\left(e^{2\sqrt{v_1}x}-1\right)\leqslant\sqrt{\frac{v_1}{v_0}}\left(e^{2\sqrt{v_1}x}-1\right), \tag{26}$$

where the last inequlity follows from $m_-\leqslant v_1/\sqrt{v_0}$ by (12). From (26), we get

$$\varphi_-(x)\leqslant\sqrt{\frac{v_1}{v_0}}e^{\sqrt{v_1}x}+\left(1-\sqrt{\frac{v_1}{v_0}}\right)e^{-\sqrt{v_1}x}\leqslant\sqrt{\frac{v_1}{v_0}}e^{\sqrt{v_1}x},$$

which gives the upper bound estimate in (21) for $x>0$. The estimate (20) can be proven smilarly, and we omit it.

Lastly, we prove the uniqueness of the boundary value problems (19). We prove it for φ_-, since we can prove it for φ_+ similarly. Let φ_-^1 and φ_-^2 be two solutions of (19). Setting $\varphi_0:=\varphi_-^1-\varphi_-^2$, it satisfies

$$\begin{cases}\varphi_0''(x)=V(x)\varphi_0(x) & \text{(a.e. } x\in\mathbb{R}),\\ \varphi_0(0)=0, \\ \lim_{x\to-\infty}\varphi_0(x)=0.\end{cases}$$

Since a pair of φ_+ and φ_- consists a basis of the solution space of the second order linear ODE $\varphi''+V(x)\varphi=0$, there exist $c_\pm\in\mathbb{R}$ such that $\varphi_0(x)=c_+\varphi_+(x)+c_-\varphi_-(x)$ holds for a.e. $x\in\mathbb{R}$. The condition $\lim_{x\to-\infty}\varphi_0(x)=0$ implies that $c_+=0$ and $c_-=c_-\varphi_-(0)=\varphi_0(0)=0$ follows. Hence, the solution of (19) is unique. □

4. Second minimization step

Based on (7), this section considers the second minimization problem $\inf_{a\in\mathbb{R}}F(a)$. Thanks to Theorem 3.5 we established, surprisingly we are able to represent the derivatives of the function $F(a)$ in terms of the information of the singularity of the minimizer u_a; the right and left derivatives at $x=a$ of u_a, as shown in the next theorem. Some

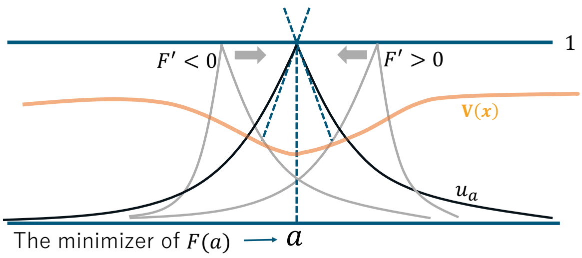

Figure 1: *Profiles of the minimizers $u_a(x)$ for three different values of a for an inhomogeneous $V(x)$. The left, middle, and right curves are the cases with $F'(a)<0$, $F'(a)=0$, and $F'(a)>0$, respectively. The middle one is the minimizer of $F(a)$.*

typical profiles of the minimizer $u_a(x)$ for some a are illustrated in Figure 1. The representations of $F(a)$, $F'(a)$, and $F''(a)$ given in the next theorem enable us to handle the second minimization step.

THEOREM 4.1. *We suppose* (6) *and define* $F(a)$ *by* (8)*. Then,* $F \in W^{2,\infty}_{loc}(\mathbb{R})$ *and the following estimates hold:*

$$F(a) = u_a'(a_-) - u_a'(a_+) \quad (a \in \mathbb{R}), \tag{27}$$

$$F'(a) = |u_a'(a_+)|^2 - |u_a'(a_-)|^2 \quad (a \in \mathbb{R}), \tag{28}$$

$$F''(a) = 2F(a)\left(|u_a'(a_+)|^2 + u_a'(a_+)u_a'(a_-) + |u_a'(a_-)|^2 - V(a)\right) \quad (a.e.\ a \in \mathbb{R}), \tag{29}$$

where $u_a'(a_+)$ *and* $u_a'(a_-)$ *are the right and left derivatives of* $u_a(x)$ *at* $x = a$*, respectively. Moreover, if* V *is continuous on* $\mathbb{R}$*,* $F \in C^2(\mathbb{R})$ *holds.*

Proof. From Theorem 3.5, we have

$$\begin{aligned}
F(a) &= \int_{-\infty}^{a} \left(|u_a'(x)|^2 + V(x)|u_a(x)|^2\right) dx + \int_a^{\infty} \left(|u_a'(x)|^2 + V(x)|u_a(x)|^2\right) dx \\
&= \int_{-\infty}^{a} \left(|u_a'(x)|^2 + u_a''(x)u_a(x)\right) dx + \int_a^{\infty} \left(|u_a'(x)|^2 + u_a''(x)u_a(x)\right) dx \\
&= \int_{-\infty}^{a} \left(u_a(x)u_a'(x)\right)' dx + \int_a^{\infty} \left(u_a(x)u_a'(x)\right)' dx \\
&= \Big[u_a(x)u_a'(x)\Big]_{-\infty}^{a} + \Big[u_a(x)u_a'(x)\Big]_a^{\infty} \\
&= u_a(a)u_a'(a_-) - u_a(a)u_a'(a_+) = u_a'(a_-) - u_a'(a_+).
\end{aligned}$$

From this formula and (18), we also obtain $F \in W^{1,\infty}_{loc}(\mathbb{R})$. Using Theorem 3.5, we obtain

$$\frac{d}{da}\left(u_a'(a_\pm)\right) = \frac{d}{da}\left(\frac{\varphi_\pm'(a)}{\varphi_\pm(a)}\right) = \frac{\varphi_\pm''(a)\varphi_\pm(a) - |\varphi_\pm'(a)|^2}{|\varphi_\pm(a)|^2} = V(a) - |u_a'(a_\pm)|^2,$$

and (28) follows. We also have

$$F'(a) = \left(u_a'(a_+) - u_a'(a_-)\right)\left(u_a'(a_+) + u_a'(a_-)\right) = -F(a)\left(u_a'(a_+) + u_a'(a_-)\right).$$

This formula implies $F \in W^{2,\infty}_{loc}(\mathbb{R})$. Then, differentiating $F'(a)$, we obtain

$$\begin{aligned}
F''(a) &= -F'(a)\left(u_a'(a_+) + u_a'(a_-)\right) - F(a)\left(2V(a) - |u_a'(a_+)|^2 - |u_a'(a_-)|^2\right) \\
&= 2F(a)\left(|u_a'(a_+)|^2 + u_a'(a_+)u_a'(a_-) + |u_a'(a_-)|^2 - V(a)\right),
\end{aligned} \tag{30}$$

for a.e. $a \in \mathbb{R}$. Moreover, if $V \in C^0(\mathbb{R})$, $F \in C^2(\mathbb{R})$ holds from (18) and (30). □

THEOREM 4.2. *Under the condition* (6)*, it holds that*

$$2\frac{v_0}{\sqrt{v_1}} \leqslant m(V) \leqslant 2\frac{v_1}{\sqrt{v_0}}.$$

Proof. From the estimate (12), we have $v_0/\sqrt{v_1} \leqslant \mp u_a'(a_\pm) \leqslant v_1/\sqrt{v_0}$ for $a \in \mathbb{R}$. Then, from (27), $2v_0/\sqrt{v_1} \leqslant F(a) \leqslant 2v_1/\sqrt{v_0}$ holds. Since $m(V) = \inf_{a\in\mathbb{R}} F(a)$, we conclude the assertion of the theorem. □

When $V \in C^0(\mathbb{R})$, if $F(x)$ has a local minimum at $x = a$, then the following condition holds:

$$F'(a) = 0, \qquad F''(a) \geqslant 0. \tag{31}$$

So, we define

$$N(V) := \left\{a \in \mathbb{R}\,;\, F'(a) = 0, F''(a) \geqslant 0\right\}.$$

Then, as a consequence of Theorem 4.2, we obtain the following theorem.

THEOREM 4.3. *We suppose* (6) *and* $V \in C^0(\mathbb{R})$. *Then,*

$$m(V) = \min\left(\liminf_{|a|\to\infty} F(a), \inf_{a\in N(V)} F(a)\right), \tag{32}$$

$$M(V) = \{cu_a\,;\, c \in \mathbb{R}\backslash\{0\},\, F(a) = m(V)\}. \tag{33}$$

In particular, $M(V) = \emptyset$ *holds if and only if*

$$\liminf_{|a|\to\infty} F(a) < F(b) \quad \textit{for all} \quad b \in N(V). \tag{34}$$

Proof. Since $V \in C^0(\mathbb{R})$, $F \in C^2(\mathbb{R})$ follows from Theorem 4.1. Then, (32) holds from Proposition 2.1. We set

$$\tilde{M}(V) := \{cu_a\,;\, c \in \mathbb{R}\backslash\{0\},\, F(a) = m(V)\}.$$

If $M(V) \neq \emptyset$, for any $u \in M(V)$, we choose $a \in \mathbb{R}$ such that $|u(a)| = \|u\|_\infty$ and define $\tilde{u}(x) := u(x)/u(a)$. Since $\tilde{u} \in K_a$, we obtain

$$m(V) \leqslant F(a) \leqslant I(\tilde{u};V) = R(\tilde{u};V) = R(u;V) = m(V).$$

Hence, $I(\tilde{u};V) = F(a) = m(V)$ follows. From Theorem 3.1, we have $\tilde{u} = u_a$ and it implies $M(V) = \tilde{M}(V)$. It is also obvious that $\tilde{M}(V) = \emptyset$ if $M(V) = \emptyset$. So, we obtain (33).

When $M(V) = \emptyset$, since $m(V) = \liminf_{|a|\to\infty} F(a)$ and $m(V) < F(b)$ for all $b \in N(V)$, (34) holds. Conversely, (34) also implies $M(V) = \emptyset$, since $m(V) \leqslant \liminf_{|a|\to\infty} F(a)$. □

We immediately obtain characterizations of $m(V)$ and $M(V)$ for the case of constant potential as a corollary of Theorem 4.3.

COROLLARY 4.4. *Let* V *be a positive constant:* $V > 0$. *Then it holds that*

$$m(V) = 2\sqrt{V}, \quad M(V) = \{cu_a;\, c \in \mathbb{R}\backslash\{0\},\, a \in \mathbb{R}\}, \tag{35}$$

where $u_a(x) = e^{-\sqrt{V}|x-a|}$.

Proof. Since $v_0 = v_1 = V$, from Theorem 3.1 and Theorem 4.2, we obtain $u_a(x) = e^{-\sqrt{V}|x-a|}$ and $m(V) = 2\sqrt{V}$, respectively. As $F(a) = \|u_a\|_V^2$ is constant, $N(V) = \mathbb{R}$ and (35) follow from Theorem 4.3. $\square$

We also give characterizations of $m(V)$ and $M(V)$ for the case of nondecreasing potential as follows.

THEOREM 4.5. *We suppose* (6) *and* V *is a non-decreasing function with*

$$\lim_{x\to-\infty} V(x) = v_0 > 0, \quad \lim_{x\to\infty} V(x) = v_1.$$

Then, it holds that

$$m(V) = \lim_{a\to-\infty} F(a) = 2\sqrt{v_0}.$$

Furthermore, if $v_0 < v_1$, *then* F *is a strictly increasing function and* $M(V) = \emptyset$.

Proof. For $u \in H^1(\mathbb{R})\setminus\{0\}$, since $V(x) \geqslant v_0$, we have $R(u;V) \geqslant R(u;v_0) \geqslant m(v_0)$. Taking the infimum of $R(u;V)$ with respect to u, we obtain that $m(V) \geqslant m(v_0)$.

We put $v_a(x) := e^{-\sqrt{v_0}|x-a|}$ for $a \in \mathbb{R}$. We also obtain

$$\begin{aligned} m(v_0) &\leqslant m(V) \leqslant F(a) \leqslant I(v_a;V) \\ &= I(v_a;v_0) + (I(v_a;V) - I(v_a;v_0)) \\ &= m(v_0) + \int_{-\infty}^{\infty} (V(x) - v_0) e^{-2\sqrt{v_0}|x-a|} dx. \end{aligned}$$

Using the following estimate for $R > 0$ and $a \in \mathbb{R}$,

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} (V(x) - v_0) e^{-2\sqrt{v_0}|x-a|} dx \\ &= \int_{-\infty}^{a+R} (V(x) - v_0) e^{-2\sqrt{v_0}|x-a|} dx + \int_{a+R}^{\infty} (V(x) - v_0) e^{-2\sqrt{v_0}|x-a|} dx \\ &\leqslant \sup_{x\leqslant a+R} (V(x) - v_0) \int_{-\infty}^{\infty} e^{-2\sqrt{v_0}|x-a|} dx + (v_1 - v_0) \int_{a+R}^{\infty} e^{-2\sqrt{v_0}|x-a|} dx \\ &= \frac{1}{\sqrt{v_0}} \sup_{x\leqslant a+R} (V(x) - v_0) + \frac{v_1 - v_0}{2\sqrt{v_0}} e^{-2\sqrt{v_0}R}, \end{aligned}$$

we obtain

$$\lim_{a\to-\infty} \int_{-\infty}^{\infty} (V(x) - v_0) e^{-2\sqrt{v_0}|x-a|} dx = 0.$$

and $m(v_0) \leqslant m(V) \leqslant \lim_{a\to-\infty} F(a) = m(v_0)$. Hence, it holds that

$$m(V) = \lim_{a\to-\infty} F(a) = m(v_0) = 2\sqrt{v_0}.$$

Furthermore, for $b < a$, we set $u_a \in K_a$ by (9) and define $w_b \in K_b$ by $w_b(x) := u_a(x - b + a)$. Then, if $v_0 < v_1$, we have

$$F(b) - F(a) \leqslant I(w_b;V) - I(u_a;V) = \int_{-\infty}^{\infty} (V(x+b-a) - V(x)) |u_a(x)|^2 dx < 0.$$

This estimate implies that $M(V) = \emptyset$, if $v_0 < v_1$. $\square$

5. A representation formula of the potential and its application

We study the condition (31) to investigate $m(V)$ and $M(V)$ in detail. Surprisingly enough, we are able to express the *given* potential V in terms of $\varphi_\pm$, where $\varphi_\pm$ are the functions defined by Theorem 3.5. In this section, we observe this fact and use it to construct an example of V, for which $m(V)$ and $M(V)$ are obtained explicitly. We set

$$h_\pm(x) := \varphi_\pm(x)^2, \quad \ell_\pm(x) := \log\varphi_\pm(x),$$
$$H_+(x) := \int_{-\infty}^{x} \frac{1}{h_+(\xi)}d\xi, \quad H_-(x) := \int_{x}^{\infty} \frac{1}{h_-(\xi)}d\xi,$$

where $H_\pm(x)$ are well defined from the estimate (20) and (21). Then, we have the following proposition.

PROPOSITION 5.1. *We suppose* (6) *and set* $W := \varphi_-'(0) - \varphi_+'(0) > 0$. *Then, the following equalities hold:*

$$u_x'(x_\pm) = \frac{\varphi_\pm'(x)}{\varphi_\pm(x)} = \ell_\pm'(x) \quad (x\in\mathbb{R}), \tag{36}$$
$$\varphi_\mp(x) = W\varphi_\pm(x)H_\pm(x) \quad (x\in\mathbb{R}), \tag{37}$$
$$V(x) = \ell_\pm''(x) + \ell_\pm'(x)^2 \quad (a.e.\ x\in\mathbb{R}), \tag{38}$$
$$F(x) = \ell_-'(x) - \ell_+'(x) \quad (x\in\mathbb{R}), \tag{39}$$
$$F'(x) = -F(x)(\ell_+'(x)+\ell_-'(x)) = -\frac{WF(x)}{\varphi_+(x)\varphi_-(x)}(h_\pm'(x)H_\pm(x)\pm 1) \quad (x\in\mathbb{R}), \tag{40}$$
$$F''(x) = -2F'(x)\ell_\mp'(x) - 2F(x)\ell_\pm''(x) \quad (a.e.\ x\in\mathbb{R}). \tag{41}$$

Proof. The equality (36) immediately follows from (22) and the definition of $\ell_\pm$. Then, (39) and the first equality of (40) follow from Theorem 4.1 and (36). Also, from (36), we have

$$\ell_\pm''(x) = \frac{\varphi_\pm''(x)\varphi_\pm(x) - \varphi_\pm'(x)^2}{\varphi_\pm(x)^2} = V(x) - \ell_\pm'(x)^2,$$

for a.e. $x\in\mathbb{R}$, which implies the formula (38).

We define the Wronskian, $W := \varphi_+(x)\varphi_-'(x) - \varphi_+'(x)\varphi_-(x)$. Then, since

$$W'(x) := \varphi_+(x)\varphi_-''(x) - \varphi_+''(x)\varphi_-(x) = \varphi_+(x)(V(x)\varphi_-(x)) - (V(x)\varphi_+(x))\varphi_-(x) = 0,$$

holds for a.e. $x\in\mathbb{R}$, we obtain that W is a constant and $W = \varphi_-'(0) - \varphi_+'(0) > 0$ holds. Then, we obtain

$$\frac{d}{dx}\left(\frac{\varphi_\mp(x)}{\varphi_\pm(x)}\right) = \frac{\varphi_\mp'(x)\varphi_\pm(x) - \varphi_\mp(x)\varphi_\pm'(x)}{\varphi_\pm(x)^2} = \frac{\pm W}{h_\pm(x)} = WH_\pm'(x).$$

Integrating this equality, we obtain $\varphi_{\mp}(x)/\varphi_{\pm}(x) = WH_{\pm}(x)$, since $\lim_{x\to\mp\infty} H_{\pm}(x) = 0$ and $\lim_{x\to\mp\infty} \varphi_{\mp}(x)/\varphi_{\pm}(x) = 0$. Hence, (37) holds.

From (37), we obtain $\varphi_{+}(x)\varphi_{-}(x) = Wh_{\pm}(x)H_{\pm}(x)$ and

$$\frac{d}{dx}\big(\varphi_{+}(x)\varphi_{-}(x)\big) = W\big(h'_{\pm}(x)H_{\pm}(x) + h_{\pm}(x)H'_{\pm}(x)\big) = W\big(h'_{\pm}(x)H_{\pm}(x) \pm 1\big), \tag{42}$$

where we used the relation $H'_{\pm}(x) = \pm h_{\pm}(x)^{-1}$. On the other hand, we have

$$\begin{aligned}\frac{d}{dx}\big(\varphi_{+}(x)\varphi_{-}(x)\big) &= \varphi'_{+}(x)\varphi_{-}(x) + \varphi_{+}(x)\varphi'_{-}(x)\\ &= \varphi_{+}(x)\varphi_{-}(x)\left(\frac{\varphi'_{+}(x)}{\varphi_{+}(x)} + \frac{\varphi'_{-}(x)}{\varphi_{-}(x)}\right)\\ &= \varphi_{+}(x)\varphi_{-}(x)\big(\ell'_{+}(x) + \ell'_{-}(x)\big).\end{aligned} \tag{43}$$

From (42) and (43), we obtain

$$\ell'_{+}(x) + \ell'_{-}(x) = \frac{W\,(h'_{\pm}(x)H_{\pm}(x) \pm 1)}{\varphi_{+}(x)\varphi_{-}(x)},$$

which implies the second equality of (40).

Lastly, we show (41). Differentiating (39), we have $F'(x) = \ell''_{-}(x) - \ell''_{+}(x)$ for a.e. $x \in \mathbb{R}$. So, it is equivalent to

$$\ell''_{+}(x) + \ell''_{-}(x) = 2\ell''_{\pm}(x) \pm F'(x) \quad (\text{a.e.} \quad x \in \mathbb{R}). \tag{44}$$

Hence, from the first equality of (40) and (44), we obtain (41) as follows:

$$\begin{aligned}F''(x) &= -F'(x)(\ell'_{+}(x) + \ell'_{-}(x)) - F(x)(\ell''_{+}(x) + \ell''_{-}(x))\\ &= -F'(x)(\ell'_{+}(x) + \ell'_{-}(x)) - F(x)(2\ell''_{\pm}(x) \pm F'(x))\\ &= -F'(x)(\ell'_{+}(x) + \ell'_{-}(x) \pm F(x)) - 2F(x)\ell''_{\pm}(x)\\ &= -2F'(x)\ell'_{\mp}(x) - 2F(x)\ell''_{\pm}(x). \quad \square\end{aligned}$$

Thus, the necessary and sufficient condition for satisfying condition (31) is obtained as the following theorem.

THEOREM 5.2. *We suppose* (6) *and* $V \in C^0(\mathbb{R})$. *Then, each of the following conditions is equivalent to* (31)*:*

1. $-\ell'_{+}(a) = \ell'_{-}(a) \geqslant \sqrt{V(a)}$.
2. $h'_{+}(a)H_{+}(a) = -1$ *and* $\ell''_{+}(a) \leqslant 0$.
3. $h'_{-}(a)H_{-}(a) = 1$ *and* $\ell''_{-}(a) \leqslant 0$.

Proof. From (29) and (36), we obtain

$$F''(x) = 2F(x)\ell'_+(x)(\ell'_+(x) + \ell'_-(x)) + 2F(x)(\ell'_-(x)^2 - V(x)). \tag{45}$$

From (45) and the first equality of (40), we conclude that condition 1 is equivalent to (31). The equivalency of condition 2 or 3 to (31) is shown from (40) and (41). □

Using the new criterion which we obtained in Theorem 5.2, we give a nontrivial example of the bounded inhomogeneous potential for which the Sobolev best constant and function are explicitly specified.

EXAMPLE 5.3. For $A > 0$ and $B > 0$, we set

$$\varphi_+(x) := \frac{Ae^{-Bx}}{\sqrt{x^2 + A^2}}. \tag{46}$$

Then, $\varphi_+(x) > 0$ and we have

$$\begin{aligned}
\ell_+(x) &:= \log \varphi_+(x) = \log A - Bx - \frac{1}{2}\log(x^2 + A^2), \\
\ell'_+(x) &= -B - \frac{x}{x^2 + A^2} = -\frac{B(x^2 + A^2) + x}{x^2 + A^2}, \\
\ell''_+(x) &= -\frac{(x^2 + A^2) - x(2x)}{(x^2 + A^2)^2} = \frac{x^2 - A^2}{(x^2 + A^2)^2}.
\end{aligned} \tag{47}$$

Then, from (38), we obtain

$$V(x) := \ell''_+(x) + \ell'_+(x)^2 = \frac{(B(x^2 + A^2) + x)^2 + x^2 - A^2}{(x^2 + A^2)^2},$$

and $V \in L^\infty(\mathbb{R})$ holds. We choose A and B as V satisfies (6). Since

$$\begin{aligned}
V(x) &= B^2 + \frac{2Bx}{x^2 + A^2} + \frac{2x^2 - A^2}{(x^2 + A^2)^2} \\
&\geqslant B^2 + \min_{x\in\mathbb{R}} \frac{2Bx}{x^2 + A^2} + \min_{x\in\mathbb{R}} \frac{2x^2 - A^2}{(x^2 + A^2)^2} \\
&= B^2 - \frac{B}{A} - \frac{1}{A^2} = \frac{(AB)^2 - AB - 1}{A^2},
\end{aligned}$$

if $AB > (1 + \sqrt{5})/2$, then $V(x) \geqslant (A^2B^2 - AB - 1)A^{-2} > 0$ holds. It is easy to check that φ_+ defined by (46) satisfies the condition (19) for the above $V(x)$.

Since

$$\begin{aligned}
h_+(x) &= \frac{A^2 e^{-2Bx}}{x^2 + A^2}, \quad h'_+(x) = -\frac{2A^2\{B(x^2 + A^2) + x\}e^{-2Bx}}{(x^2 + A^2)^2}, \\
H_+(x) &= \frac{1}{A^2}\int_{-\infty}^{x} (x^2 + A^2)e^{2Bx}\,dx = \frac{(2B^2x^2 - 2Bx + 2A^2B^2 + 1)e^{2Bx}}{4A^2B^3},
\end{aligned}$$

using the formula (37), we have

$$
\begin{aligned}
\varphi_-(x) &= W\varphi_+(x)H_+(x) \\
&= \frac{W(2B^2x^2-2Bx+2A^2B^2+1)e^{Bx}}{4AB^3\sqrt{x^2+A^2}} \\
&= \frac{A(2B^2x^2-2Bx+2A^2B^2+1)e^{Bx}}{(2A^2B^2+1)\sqrt{x^2+A^2}},
\end{aligned}
$$

where the last equality follows from the condition $\varphi_-(0)=1$. Then, we have

$$
\begin{aligned}
\ell_-(x) &= \log\frac{A}{(2A^2B^2+1)} + \log(2B^2x^2-2Bx+2A^2B^2+1) - \frac{1}{2}\log(x^2+A^2) + Bx, \\
\ell_-'(x) &= \frac{4B^2x-2B}{2B^2x^2-2Bx+2A^2B^2+1} - \frac{x}{x^2+A^2} + B.
\end{aligned}
$$

From (39), we obtain

$$
\lim_{|a|\to\infty} F(a) = \lim_{|a|\to\infty}\left(\ell_-'(a)-\ell_+'(a)\right) = 2B,
$$

since $\ell_\pm'(x)\to\mp B$ as $|x|\to\infty$.

Let us check the condition 2 of Theorem 5.2. From (47), the condition $\ell_+''(a)\leqslant 0$ is equivalent to $|a|\leqslant A$. Using the identity:

$$
\begin{aligned}
h_+'(x)H_+(x) &= -\frac{\{B(x^2+A^2)+x\}(2B^2x^2-2Bx+2A^2B^2+1)}{2B^3(x^2+A^2)^2} \\
&= \frac{Bx^2-x-A^2B}{2B^3(x^2+A^2)^2} - 1,
\end{aligned}
$$

we obtain that the other condition $h_+'(a)H_+(a)=-1$ is equivalent to $Ba^2-a-A^2B=0$, namely

$$
a = a_1 := \frac{1-\sqrt{1+4A^2B^2}}{2B} \in (-A,0)
$$

or

$$
a = a_2 := \frac{1+\sqrt{1+4A^2B^2}}{2B} > A.
$$

Hence we have $N(V)=\{a_1\}$. We check that the condition (34) does not hold. Since

$$
\begin{aligned}
F(a_1) = -2\ell_+'(a_1) &= 2\left(B+\frac{a_1}{(a_1)^2+A^2}\right) \\
&= 2B\left(1-\frac{1}{\sqrt{1+4A^2B^2}}\right) \\
&< 2B = \lim_{|a|\to\infty} F(a),
\end{aligned}
$$

we conclude that

$$m(V) = 2B\left(1 - \frac{1}{\sqrt{1+4A^2B^2}}\right), \qquad M(V) = \{cu_{a_1};\ c \in \mathbb{R}\backslash\{0\}\}.$$

The profile of $V(x)$ and the positions of $x = a_1$ and $x = a_2$ are illustrated in Figure 2. The function $F(a)$ has a global minimum at $a = a_1$, which is close to the minimum point of $V(x)$ but not same.

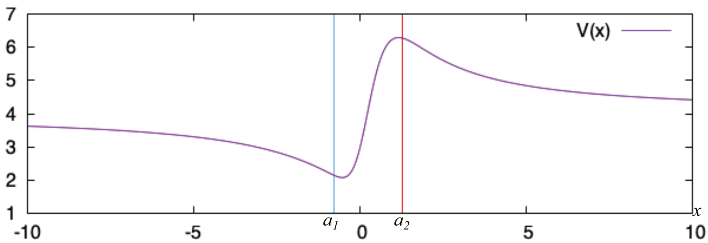

Figure 2: *The profile of the potential $V(x)$ of Example 5.3 with $A = 1$ and $B = 2$ and the values of a_1 and a_2.*

6. Conclusion

We developed a new variational approach to determine the best constant of the Sobolev-type embedding inequality in one dimension with a bounded inhomogeneous potential term. We adopted two-step minimization method. In the first minimization step, the Green function of the differential operator $-\frac{d^2}{dx^2} + V(x)$ was captured as the minimizer. We studied the fine properties of the minimizer and the fundamental solutions of the ODE $-\frac{d^2}{dx^2}u + V(x)u = 0$. It enabled us to derive new necessary and sufficient conditions for the local minimality in the second minimization step (Theorem 5.2). Furthermore, as applications of our estimates, we constructed some concrete examples of the inhomogeneous potential $V(x)$ for which the best constant and function of the Sobolev-type embedding is exactly identified.

Since our approach provides a new tool to study the fine properties of the best constant and function of the Sobolev-type embedding with an inhomogeneous potential term, further extension of our strategies is expected in future work. For example, the case the potential $V(x)$ is a periodic function is interested in connection with the periodic crystal lattice, e.g., Kronig-Penney potential [9]. Extention of the Bloch theorem [8] for general periodic potentials in our framework is also a challenging and worthy topic.

Another interesting application is the stable stationary state of a biological population model in an inhomogeneous environment. In a population model where u is the population density, the carrying capacity $m > 0$ is often set and the condition $0 \leqslant u \leqslant m$

is imposed. The stationary state is determined by the inhomogeneity of the environment expressed by the potential term $V(x)$, the effect of diffusion, and the limitation of the carrying capacity. When $m = 1$, the stationary state corresponds to the minimizer in $M(V)$ considered in this paper. The analysis of the stationary state of the population density determined by the balance between the diffusion and inhomogeneous carrying capacity has been investigated using the diffusive logistic equation, for example, in [5, 10, 11].

On the other hand, the H^1-Sobolev function is bounded only in the one-dimensional case, so the Sobolev inequality framework does not lend itself to multidimensional extension. However, it is expected to be useful for variational problems under the L^∞-constraints mentioned above, especially for applications to multidimensional population models. In fact, we successfully extended our approach to the L^∞-constraint one-dimensional variational problems with indefinite potentials [1]. We are also currently planning to discuss applications of our method to multidimensional population models in our forthcoming paper. Our approach in this paper is the first step toward that end.

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